SIMULTANEOUS TRIANGULARIZATION OF OPERATORS ON A BANACH SPACE

A. KATAVOLOS AND H. RADJAVI

1. *Introduction*

A collection $\mathscr{S}$ of operators on a finite-dimensional space is said to be *simultaneously triangularizable* if there is a basis of the space with respect to which the matrices of all members of $\mathscr{S}$ are upper triangular. Equivalently, there exists a maximal totally ordered set of subspaces which are invariant under all members of $\mathscr{S}$. More generally, a collection $\mathscr{S}$ of bounded operators on a Banach space X is called simultaneously triangularizable if there is a maximal totally ordered complete set of (closed) subspaces of X (a maximal *nest*) which are $\mathscr{S}$-invariant. Several authors have recently worked on the problem of simultaneous triangularizability.

In this paper, we obtain some simple necessary and sufficient conditions for simultaneous triangularizability of a (not necessarily norm-closed) algebra of compact operators on a Banach space. This improves on results of **[5, 7]**, and allows us to apply a theorem of Aupetit **[1]** to the present context. We also extend to Banach-space operators a result of **[10]** on simultaneous triangularizability of semigroups of trace-class operators on a Hilbert space. Using our first theorem, we are able to present a proof which is considerably shorter, even in the Hilbert-space case treated in **[10]**. We include a result on triangularizability of certain algebras of Hilbert-space operators that 'approximate' maximal abelian von Neumann algebras.

2. *Preliminaries*

Throughout, X will denote a Banach space, $\mathscr{B}(X)$ (respectively $\mathscr{K}(X)$) will be the Banach algebra of all bounded (respectively compact) operators on X. A set $\mathscr{S} \subseteq \mathscr{B}(X)$ will be called *transitive* if it has no (non-trivial) invariant closed subspaces. A *nest* $\mathscr{N}$ will be a totally ordered set of closed subspaces of X, closed under the formation of arbitrary unions and intersections. For $A \in \mathscr{B}(X)$, $\sigma(A)$ will denote its spectrum in $\mathscr{B}(X)$, $\rho(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$ will be its spectral radius, and $\delta(A) = \sup\{|\lambda-\mu| : \lambda, \mu \in \sigma(A)\}$ its spectral diameter. The following well-known results are crucial.

THE CONTINUITY THEOREM. *The spectral radius and the diameter are continuous functions on $\mathscr{K}(X)$.*

Received 27 January 1989.

1980 *Mathematics Subject Classification* (1985 *Revision*) 47D30.

The first author is on sabbatical leave from the University of Athens, Athens, Greece; the research of the second author was supported by NSERC.

(This can be easily deduced from the fact that, if the set of (non-empty) compact subsets of the plane is equipped with the Hausdorff metric, the function $A \to \sigma(A)$ is continuous on compact operators [**1**, Corollary 1.1.7].)

LOMONOSOV'S THEOREM [**6**]. *If $\mathscr{A}$ is a (non-zero) transitive subalgebra of $\mathscr{K}(X)$, then $\mathscr{K}(X)\mathscr{F} \subseteq \bar{\mathscr{A}}$*, where $\mathscr{F}$ is a total set of rank-one operators.

RINGROSE'S THEOREM [**12**]. (i) *A nest $\mathscr{N}$ in a Banach space X is maximal if, and only if, for all $N \in \mathscr{N}$, $\dim(N/N_-) \leqslant 1$, where N_- denotes the smallest member of $\mathscr{N}$ containing all $M \in \mathscr{N}$ such that $M \subsetneq N$.*

(ii) *If $A \in \mathscr{K}(X)$ leaves each member of a maximal nest $\mathscr{N}$ invariant, then its eigenvalues are, with the possible exception of zero, precisely its 'diagonal coefficients', namely the numbers $\lambda_N(A)$, $N \in \mathscr{N}$, where $\lambda_N(A)$ is the (scalar) operator induced by A on N/N_-.*

3. *Algebras of compact operators*

The following theorem is an improvement on a result of [**5**, Corollary 3.5].

THEOREM 1. *A subalgebra $\mathscr{A}$ of $\mathscr{K}(X)$ is simultaneously triangularizable if, and only if, $AB-BA$ is quasinilpotent for all $A, B \in \mathscr{A}$.*

Proof. Observe that if M, N are $\mathscr{A}$-invariant subspaces and $M \subseteq N$, then, if $\hat{A}$ denotes the operator on N/M induced by $A \in \mathscr{A}$, the mapping $A \to \hat{A}$ is a homomorphism of $\mathscr{A}$.

Suppose that there exists a maximal $\mathscr{A}$-invariant nest $\mathscr{N}$. Then, in the notation of Ringrose's theorem, we have, for all $A, B \in \mathscr{A}$,

$$\lambda_N(AB-BA) = \lambda_N(A)\,\lambda_N(B) - \lambda_N(B)\,\lambda_N(A) = 0$$

for all $N \in \mathscr{N}$. Thus Ringrose's theorem implies that $AB-BA$ has no non-zero eigenvalues.

Conversely, suppose that $AB-BA$ is quasinilpotent for all $A, B \in \mathscr{A}$. Consider a nest $\mathscr{N}$ of $\mathscr{A}$-invariant subspaces, maximal with respect to the property of being $\mathscr{A}$-invariant (such a nest exists, by Zorn's lemma; *a priori*, it may be trivial). We claim that $\mathscr{N}$ is a maximal nest.

For any $N \in \mathscr{N}$, the induced algebra $\hat{\mathscr{A}}$ on N/N_- is transitive, by maximality. Thus by Lomonosov's theorem, given any $A, B \in \mathscr{K}(N/N_-)\mathscr{F}$, we may find sequences $\{A_n\}$, $\{B_n\}$ in $\mathscr{A}$ such that

$$A = \lim_n \hat{A}_n, \qquad B = \lim_n \hat{B}_n.$$

But $A_n B_n - B_n A_n$ is quasinilpotent by hypothesis; hence so is $\hat{A}_n \hat{B}_n - \hat{B}_n \hat{A}_n$. Thus

$$\rho(AB-BA) = \lim_n \rho(\hat{A}_n \hat{B}_n - \hat{B}_n \hat{A}_n) = 0.$$

Since A, B are arbitrary in $\mathscr{K}(N/N_-)\mathscr{F}$, this is only possible if $\dim(N/N_-) \leqslant 1$. Thus $\mathscr{N}$ is a maximal nest by Ringrose's theorem.

It should be noted that in the above result no closure hypotheses are necessary on the algebra $\mathscr{A}$.

In [7] it was shown, using results of [5], that for a norm-closed algebra $\mathscr{A}$ of compact operators simultaneous triangularizability is equivalent to commutativity modulo the radical of $\mathscr{A}$. If $\mathscr{A}$ is not norm-closed, its (Jacobson) radical is in general smaller than the set of all $A \in \mathscr{A}$ such that AB is quasinilpotent for all $B \in \mathscr{A}$. For example, if $\mathscr{A}$ is the (abelian) algebra generated by the Volterra (integration) operator on $L^2[0, 1]$, then every operator in $\mathscr{A}$ is quasinilpotent, but

$$\operatorname{Rad}(\mathscr{A}) = \{A \in \mathscr{A} : 1 + AB \text{ is invertible in } \mathscr{A} + \mathbb{C}I \text{ for all } B \in \mathscr{A}\} = \{0\}.$$

This leads us to consider a larger 'radical', which turns out to be more appropriate for our purposes.

DEFINITION.

$$\operatorname{Rad}_1(\mathscr{A}) = \{A \in \mathscr{A} : AB \text{ is quasinilpotent for all } B \in \mathscr{A}\}.$$

It is not *a priori* obvious that $\operatorname{Rad}_1(\mathscr{A})$ is even a linear space. However, we can prove the following lemma for compact operators.

LEMMA. *If $\mathscr{A}$ is a subalgebra of $\mathscr{K}(X)$, then*
(i) $\operatorname{Rad}_1(\mathscr{A}) \subseteq \operatorname{Rad}(\bar{\mathscr{A}})$,
(ii) $\operatorname{Rad}_1(\mathscr{A})$ *is an ideal, (relatively) closed in* $\mathscr{A}$.

Proof. (i) Let $A \in \operatorname{Rad}_1(\mathscr{A})$. For any $B \in \bar{\mathscr{A}}$, there is a sequence $B_n \in \mathscr{A}$ converging to B. Since each AB_n is compact, $\rho(AB) = \lim_n \rho(AB_n) = 0$. Thus AB is quasinilpotent, and so $A \in \operatorname{Rad}(\bar{\mathscr{A}})$.

(ii) If $A_1, A_2 \in \operatorname{Rad}_1(\mathscr{A})$, then, by (i), $A_1, A_2 \in \operatorname{Rad}(\bar{\mathscr{A}})$. Thus for all $B \in \mathscr{A}$, $(A_1 + A_2)B$ is quasinilpotent, hence $A_1 + A_2 \in \operatorname{Rad}_1(\mathscr{A})$. The remaining properties of an ideal follow easily from the definition and the fact that RS is quasinilpotent if and only if SR is.

Finally, if $\{A_n\}$ is a sequence in $\operatorname{Rad}_1(\mathscr{A})$ converging to some $A \in \mathscr{A}$, then for any $B \in \mathscr{A}$, $\rho(AB) = \lim_n \rho(A_n B) = 0$, showing that $A \in \operatorname{Rad}_1(\mathscr{A})$. Thus $\operatorname{Rad}_1(\mathscr{A})$ is a closed ideal of $\mathscr{A}$.

EXAMPLES. (i) This lemma fails for non-compact operators. Indeed, an algebra $\mathscr{A}$, consisting of nilpotent operators on a Hilbert space, whose norm-closure is semisimple is constructed in [2]. Here $\operatorname{Rad}_1(\mathscr{A}) = \mathscr{A}$ while $\operatorname{Rad}(\bar{\mathscr{A}})$ is trivial.

(ii) In general, $\operatorname{Rad}_1(\mathscr{A})$ is not dense in $\operatorname{Rad}(\bar{\mathscr{A}})$, even in the commutative case. For example, let A, B be defined on $l^2(\mathbb{N})$ by

$$Ae_n = \begin{cases} e_1 + e_2, & n = 2, \\ \dfrac{1}{n} e_n, & n \neq 2, \end{cases} \qquad Be_n = \begin{cases} e_1, & n = 2, \\ 0, & n \neq 2, \end{cases}$$

where $\{e_n\}$ is the standard basis of $l^2(\mathbb{N})$. If p is any polynomial with $p(0) = 0$, all $p(1/n)$ are eigenvalues of $p(A)$ for $n \neq 2$. Thus the algebra $\mathscr{A}$ generated by A contains no non-zero quasinilpotents, and hence $\operatorname{Rad}_1(\mathscr{A}) = \{0\}$. But the sequence $\{k^{-1}A^k\}$ converges to the nilpotent operator B, hence $\operatorname{Rad}(\bar{\mathscr{A}}) \neq \{0\}$. (In an abelian Banach algebra, any quasinilpotent is in the radical.)

The following theorem is the appropriate extension of the result in [7] for non-closed algebras. It contains, and simplifies, the proofs of results in [5].

THEOREM 2. *A subalgebra $\mathscr{A}$ of $\mathscr{K}(X)$ is simultaneously triangularizable if and only if $\mathscr{A}/\mathrm{Rad}_1(\mathscr{A})$ is abelian. In this case $\mathrm{Rad}_1(\mathscr{A})$ contains all the quasinilpotent elements in $\mathscr{A}$.*

Proof. If $\mathscr{A}/\mathrm{Rad}_1(\mathscr{A})$ is abelian, then for every pair A, B in $\mathscr{A}$ the operator $AB-BA$ is quasinilpotent, and thus $\mathscr{A}$ is triangularizable by Theorem 1. Conversely, let $\mathscr{N}$ be a maximal nest of invariant subspaces for $\mathscr{A}$. If $A\in\mathscr{A}$ is quasinilpotent then, for all B in $\mathscr{A}$,

$$\lambda_N(AB)=\lambda_N(A)\,\lambda_N(B)=0$$

for each N in $\mathscr{N}$, so that AB is quasinilpotent by Ringrose's theorem; that is, $A\in\mathrm{Rad}_1(\mathscr{A})$. Thus

$$\mathrm{Rad}_1(\mathscr{A})=\{A\in\mathscr{A}: A \text{ is quasinilpotent}\}.$$

Now Theorem 1 implies that for all B, C in $\mathscr{A}$, $BC-CB$ is quasinilpotent, and hence in $\mathrm{Rad}_1(\mathscr{A})$. It follows that $\mathscr{A}/\mathrm{Rad}_1(\mathscr{A})$ is abelian.

The following general result on Banach algebras is due to Aupetit; when applied to the special case of compact operators it yields interesting necessary and sufficient conditions for simultaneous triangularizability in terms of the spectrum.

AUPETIT'S THEOREM [**1**]. *For a Banach algebra $\mathscr{A}$ the following conditions are equivalent:*

(i) *$\mathscr{A}/\mathrm{Rad}(\mathscr{A})$ is abelian;*
(ii) *there exists $a>0$ such that $\rho(A+B)\leqslant a(\rho(A)+\rho(B))$ for all $A, B\in\mathscr{A}$;*
(iii) *there exists $b>0$ such that $\rho(AB)\leqslant b\rho(A)\rho(B)$ for all $A, B\in\mathscr{A}$;*
(iv) *ρ is uniformly continuous on $\mathscr{A}$;*
(v) *there exists $c>0$ such that $|\rho(A)-\rho(B)|\leqslant c\|A-B\|$ for all $A, B\in\mathscr{A}$;*
(vi) *δ is uniformly continuous on $\mathscr{A}$;*
(vii) *there exists $d>0$ such that $|\delta(A)-\delta(B)|\leqslant d\|A-B\|$ for all $A, B\in\mathscr{A}$;*
(viii) *there exists $e>0$ such that $\delta(A+B)\leqslant e(\delta(A)+\delta(B))$ for all $A, B\in\mathscr{A}$.*

(For several other equivalent conditions see [**1**, p. 48].)

COROLLARY 3. *Let $\mathscr{A}$ be an arbitrary subalgebra of $\mathscr{K}(X)$. Then $\mathscr{A}$ is simultaneously triangularizable if and only if any of the conditions* (ii) *to* (viii) *in Aupetit's theorem hold. If any, hence all, of these conditions hold, then the constants a, b, c, d, e can all be taken to be equal to* 1.

Proof. We first show that $\mathscr{A}/\mathrm{Rad}_1(\mathscr{A})$ is abelian if and only if $\bar{\mathscr{A}}/\mathrm{Rad}(\bar{\mathscr{A}})$ is. Assuming the commutativity of the latter let A, B be in $\mathscr{A}$. Then $AB-BA\in\mathrm{Rad}(\bar{\mathscr{A}})$, so that $(AB-BA)\,C$ is quasinilpotent for all C in $\mathscr{A}$, and hence $AB-BA\in\mathrm{Rad}_1(\mathscr{A})$. Conversely, assume that $\mathscr{A}/\mathrm{Rad}_1(\mathscr{A})$ is abelian and let A, B, C be any members of $\bar{\mathscr{A}}$. Approximate them in $\mathscr{A}$ by A_n, B_n, C_n respectively. Then $A_nB_n-B_nA_n\in\mathrm{Rad}_1(\mathscr{A})$ implies that $(A_nB_n-B_nA_n)\,C_n$ is quasinilpotent and, hence, so is $(AB-BA)\,C$.

Now $\mathscr{A}$ is simultaneously triangularizable if and only if its closure is. (They have the same invariant subspaces.) Also, the conditions (ii) to (viii) in the above theorem hold in $\bar{\mathscr{A}}$ if they hold in $\mathscr{A}$ (by continuity of ρ and δ). Thus the equivalence of each of these conditions to simultaneous triangularizability follows from Theorem 2.

When $\mathscr{A}$ is simultaneously triangular, a simple calculation with the diagonal coefficients (and a possibly absent zero) shows that all the constants in the inequalities of the theorem can be taken to be 1.

4. '*Perturbed*' *self-adjoint algebras*

An analogue of Theorem 1 can be obtained for certain 'compact perturbations' of abelian self-adjoint algebras on a Hilbert space H, that is, subalgebras $\mathscr{A}$ of $\mathscr{B}(H)$ lying between $\mathscr{M}$ and $\mathscr{M}+\mathscr{K}(H)$, where $\mathscr{M}$ is a maximal abelian self-adjoint subalgebra of $\mathscr{B}(H)$. We can actually treat a somewhat larger class of algebras $\mathscr{A}$.

THEOREM 4. *Let $\mathscr{M}$ be a maximal abelian self-adjoint algebra of operators on the Hilbert space H and suppose that $\mathscr{M}_0$ is weak-operator-dense in $\mathscr{M}$. Let $\mathscr{A}$ be any algebra with*

$$\mathscr{M}_0 \subseteq \mathscr{A} \subseteq \mathscr{M}+\mathscr{K}(H).$$

Then $\mathscr{A}$ is simultaneously triangularizable if and only if $AB-BA$ is quasinilpotent for all A, B in $\mathscr{A}$.

Proof. By the commutativity of $\mathscr{M}$, every commutator $AB-BA$ in $\mathscr{A}$ is compact. Now if $\mathscr{A}$ is triangularizable with a nest $\mathscr{N}$, and if $N \ominus N_-$ is a one-dimensional 'gap' in $\mathscr{N}$, then, as in the proof of Theorem 1,

$$\lambda_N(AB-BA) = \lambda_N(A)\,\lambda_N(B)-\lambda_N(B)\,\lambda_N(A) = 0$$

(although A and B are not themselves necessarily compact). This shows that $AB-BA$ is quasinilpotent.

To prove the converse, we first show that $\mathscr{A}$ has a non-trivial invariant subspace. If $\mathscr{A}$ is commutative then, by the density of $\mathscr{M}_0$ in $\mathscr{M}$, we must have $\mathscr{A} \subseteq \mathscr{M}$ and thus $\mathscr{A}$ has reducing subspaces. Otherwise there are A_0 and B_0 in $\mathscr{A}$ with $A_0 B_0 - B_0 A_0 \neq 0$. Let $\mathscr{J}$ be the ideal of $\mathscr{A}$ generated by the compact operator $A_0 B_0 - B_0 A_0$. Then $\mathscr{J}$ is a subalgebra of $\mathscr{K}(H)$ and is thus triangularizable by Theorem 1. This implies that $\mathscr{A}$ is not transitive, because every non-zero ideal of a transitive algebra is transitive. (This is not hard to see; for a short proof see [9]. Triangularizability of an ideal does not in general imply that the algebra is triangularizable.)

Now let $\mathscr{N}$ be a totally ordered set of $\mathscr{A}$-invariant subspaces, maximal with respect to being invariant. We must show that $\mathscr{N}$ is a maximal nest. Let N be any member of $\mathscr{N}$ with $H_1 = N \ominus N_- \neq \{0\}$. Since N and N_- are both reducing for $\mathscr{M}_0$, so is H_1. Thus the restriction $\mathscr{M}_0^{(1)}$ of $\mathscr{M}_0$ to the subspace H_1 is weak-operator-dense in $\mathscr{M}^{(1)}$ and is contained in $\mathscr{A}^{(1)}$, where $\mathscr{M}^{(1)}$ and $\mathscr{A}^{(1)}$ are the respective compressions of $\mathscr{M}$ and $\mathscr{A}$ to H_1. Observe that $\mathscr{M}^{(1)}$ is a maximal abelian self-adjoint subalgebra of $\mathscr{B}(H_1)$ and that

$$\mathscr{M}_0^{(1)} \subseteq \mathscr{A}^{(1)} \subseteq \mathscr{M}^{(1)}+\mathscr{K}(H_1).$$

Since every commutator in $\mathscr{A}^{(1)}$ is the compression of a compact quasinilpotent member of $\mathscr{A}$ to H_1, it is itself quasinilpotent, and thus $\mathscr{A}_1$ satisfies the hypotheses of the theorem with H_1 in place of H. It follows that if the dimension of H_1 were greater than one, it would have a non-trivial invariant subspace by the preceding paragraph, contradicting the assumption on $\mathscr{N}$. We have shown that $\mathscr{N}$ is a maximal subspace nest.

REMARKS. (i) It would be very interesting to know what happens if in the statement of the above theorem the inclusion $\mathscr{M}_0 \subseteq \mathscr{A}$ is dropped and $\mathscr{A}$ is merely assumed to be contained in $\mathscr{M}+\mathscr{K}(H)$. Even in the following very special case the question is still unsettled. Let $\mathscr{A}$ be the algebra generated by a single operator A that is a compact perturbation of a normal operator. Then $\mathscr{A}$ satisfies the weakened hypotheses since it is commutative. It is not even known whether a compact perturbation of a self-adjoint operator necessarily possesses an invariant subspace.

(ii) Compare Theorem 4 to the result in [3] that if $\mathscr{A}$ is a weak-operator-closed subalgebra of $\mathscr{B}(H)$ containing a maximal abelian self-adjoint algebra, then a sufficient condition for simultaneous triangularizability is that $\mathscr{A}/\mathrm{Rad}(\mathscr{A})$ be abelian.

5. '*Traceable*' *operators*

In [**10**] it was shown that a (multiplicative) semigroup $\mathscr{S}$ of trace-class operators on a Hilbert space is simultaneously triangularizable if and only if trace is permutable on $\mathscr{S}$; that is, if $\mathrm{tr}(ABC) = \mathrm{tr}(BAC)$ for all A, B, C in $\mathscr{S}$.

In order to extend this result to arbitrary Banach spaces, it is necessary to have a suitable replacement for the ideal of trace-class operators. There seems to be no unique natural choice. There exist (uncountably!) many distinct classes of operators on a Banach space X which collapse to the trace class when X is a Hilbert space. For many of these, there exists no continuous linear extension of the trace of finite rank operators. Even when the trace of an operator is well defined, it often fails to equal the sum of the eigenvalues of the operator. (There exists an operator T on c_0 such that $T^2 = 0$, but its trace is (well defined and) equal to 1 [**8**].) However, König and others have shown that there exist ideals of compact operators on an arbitrary Banach space on which the trace is well defined and well behaved.

DEFINITIONS [**4, 8**]. (i) Let $T\in\mathscr{B}(X)$, $n\in\mathbb{N}$. The *nth approximation number* of T is defined by

$$a_n(T) = \inf\{\|T-F\| : F\in\mathscr{B}(\mathrm{X}), \mathrm{rank}(F) < n\}.$$

(ii) Let X, Y be Banach spaces. An operator $T\in\mathscr{B}(X, Y)$ is *absolutely* 2-*summing* if there exists $c > 0$ such that, for any finite sequence $x_1, \ldots, x_n$ in X and any linear functional f on X with $\|f\| = 1$,

$$\sum_{i=1}^{n} \|Tx_i\|^2 \leqslant c\sum_{i=1}^{n} |f(x_i)|^2.$$

Denote by $S_a^1(X)$ the class of all $T\in\mathscr{B}(X)$ such that $\sum_{n=1}^{\infty} a_n(T) < \infty$. Let $\Pi_2^2(X)$ stand for the class of all $T\in\mathscr{B}(X)$ which factor through absolutely 2-summing operators; that is, such that there is a Banach space Y and absolutely 2-summing operators $R\in\mathscr{B}(X, Y)$ and $S\in\mathscr{B}(Y, X)$ with $T = SR$.

It can be shown [**4, 8**] that $S_a^1(X)$ and $\Pi_2^2(X)$ are ideals in $\mathscr{K}(X)$, containing the finite rank operators. Moreover, König [**4**] proves that if $T\in S_a^1(X)$ or $T\in\Pi_2^2(X)$, then $\mathrm{tr}(T)$ is well defined as the continuous linear extension of the trace of finite rank operators (in appropriate topologies), and in fact

$$\mathrm{tr}(T) = \sum_{n=1}^{\infty} \lambda_n(T),$$

where the $\lambda_n(T)$ are the eigenvalues of T, repeated according to algebraic multiplicity, and the above sum converges absolutely.

Using these results we can give our triangularizability condition for certain (multiplicative) semigroups of operators.

THEOREM 5. *Let $\mathscr{S}$ be a semigroup contained either in S^1_a or in $\Pi^2_2(X)$. Then $\mathscr{S}$ is simultaneously triangularizable if and only if* $\operatorname{tr}(ABC) = \operatorname{tr}(ACB)$ *for all* $A, B, C \in \mathscr{S}$.

REMARK. In the Hilbert-space case **[10]** it suffices to assume that the above trace condition holds for all A in $\mathscr{S}$ and B, C in a generating set for $\mathscr{S}$.

Proof. Suppose that $\mathscr{S}$ is simultaneously triangularizable. Then, by Ringrose's theorem, the non-zero eigenvalues of any $A \in \mathscr{S}$ are its diagonal coefficients. Thus, for all $A, B, C \in \mathscr{S}$,

$$\lambda_n(ABC) = \lambda_n(A)\,\lambda_n(B)\,\lambda_n(C) = \lambda_n(ACB).$$

Hence, using König's theorem,

$$\operatorname{tr}(ABC) = \sum_n \lambda_n(ABC) = \sum_n \lambda_n(ACB) = \operatorname{tr}(ACB).$$

Conversely, suppose that the permutability condition of the theorem is satisfied. Let $\mathscr{A} \subseteq \mathscr{K}(X)$ be the linear space spanned by $\mathscr{S}$. Since $\mathscr{S}$ is a semigroup, $\mathscr{A}$ is an algebra. Linearity of the trace shows that for all A, B, C in $\mathscr{A}$, $\operatorname{tr}(ABC) = \operatorname{tr}(ACB)$, or

$$\operatorname{tr}(A(BC-CB)) = 0.$$

In particular, given any B, C in $\mathscr{S}$, we put $A = (BC-CB)^{n-1}$ in the above equation to get

$$\operatorname{tr}((BC-CB)^n) = 0$$

for $n \geqslant 2$ and hence for all $n \geqslant 1$ (the case $n = 1$ being true automatically). This implies, by König's theorem, that

$$\sum_{i=1}^{\infty} \lambda_i((BC-CB)^n) = \sum_{i=1}^{\infty} (\lambda_i(BC-CB))^n = 0.$$

But $\sum_{i=1}^{\infty} \lambda_i^n = 0$ for all n implies that all the λ_i are zero. (See the following lemma.) Thus $BC-CB$ is quasinilpotent for every pair B, C in $\mathscr{A}$ and, consequently, $\mathscr{A}$ is triangularizable by Theorem 1.

The following lemma is a special case of a lemma in [9]. For completeness we include it here. We thank M. Khalkhali for a discussion that led to this short proof.

LEMMA. *If $\sum_{i=1}^{\infty}|a_i| < \infty$ and $\sum_{i=1}^{\infty} a_i^n = 0$ for all $n \in \mathbb{N}$, then $a_i = 0$ for all i.*

Proof. Assume, to the contrary, that $a_1 \neq 0$. Dividing by a_1, we may suppose that

$$1 = |a_1| = |a_2| = \dots = |a_k| > |a_{k+1}| \geqslant \dots.$$

Thus $\lim_n \sum_{i=k+1}^{\infty} a_i^n = 0$, since the a_i are summable. But then

$$\lim_n \sum_{i=1}^{k} a_i^n = 0.$$

Choosing a subsequence $\{n_j\}$ of integers such that each $a_i^{n_j}$ converges, say to b_i, we must have $\sum_{i=1}^{k} b_i^m = 0$ for all $m \in \mathbb{N}$, since $a_i^{mn_j}$ converges to b_i^m. Letting p be the number of distinct b_i, we can assume, after a permutation, that $b_1, \ldots, b_p$ are distinct. Then we rewrite the equation $\sum_{i=1}^{k} b_i^m = 0$ as

$$\sum_{i=1}^{p} r_i b_i^m = 0, \qquad m \in \mathbb{N}.$$

Thus the non-zero vector $(r_1 b_1, \ldots, r_p b_p)$ is a solution of the homogeneous system

$$\sum_{i=1}^{p} b_i^n x_i = 0, \qquad n = 0, 1, \ldots, p-1.$$

It therefore follows that the Vandermonde matrix $((b_i^j))$ has zero determinant, or $\prod_{i \neq j}(b_i - b_j) = 0$, a contradiction.

References

1. B. Aupetit. *Propriétés spectrales des algèbres de Banach*, Lecture Notes in Mathematics 735 (Springer, Berlin, 1979).
2. D. Hadwin, E. Nordgren, M. Radjabalipour, H. Radjavi and P. Rosenthal, 'A nil algebra of bounded operators on Hilbert space with semisimple norm closure', *Integral Equations Operator Theory* 9 (1986) 739–743.
3. D. Hadwin, E. Nordgren, M. Radjabalipour, H. Radjavi and P. Rosenthal, 'On simultaneous triangularization of collections of operators', to appear.
4. H. König, *Eigenvalue distribution of compact operators*, Operator Theory: Advances and Applications 16 (Birkhauser, Basle, 1986).
5. C. Laurie, E. Nordgren, H. Radjavi and P. Rosenthal, 'On triangularization of algebras of operators', *J. Reine Angew. Math.* 327 (1981) 143–155.
6. V. J. Lomonosov, 'Invariant subspaces for operators commuting with compact operators', *Funct. Anal. Appl.* 7 (1973) 213–214.
7. G. J. Murphy, 'Triangularizable algebras of compact operators', *Proc. Amer. Math. Soc.* 84 (1982) 354–356.
8. A. Pietsch, *Operator ideals* (North-Holland, Amsterdam, 1980).
9. H. Radjavi, 'On the reduction and triangularization of semigroups of operators', *J. Operator Theory* 13 (1985) 63–71.
10. H. Radjavi, 'A trace condition equivalent to simultaneous triangularizability', *Canad. J. Math.* 38 (1986) 376–386.
11. C. E. Rickart, *General theory of Banach algebras* (Van Nostrand, New York, 1960).
12. J. R. Ringrose, *Compact nonselfadjoint operators* (Van Nostrand Reinhold, New York, 1971).

Dalhousie University
Halifax
Nova Scotia
Canada B3H 3J5