Interlacing polynomials, restricted invertibility and multi-paving

Summer School in Operator Theory

Part III

July 10, 2019

Paving conjecture

Anderson

For every $\epsilon > 0$ there exists $r = r(\epsilon) \in \mathbb{N}$ such that, for every zero diagonal complex $n \times n$ matrix A there exists a paving $\sigma_1 \cup \cdots \cup \sigma_r = [n]$ which satisfies

$$\|P_{\sigma_i} A P_{\sigma_i}\| \leqslant \epsilon \|A\|$$

for all $i = 1, \ldots, r$.

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This result implies a positive solution to the Kadison-Singer problem.

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Let $\epsilon > 0$ and let $A^{(1)}, \dots, A^{(k)} \in M_n(\mathbb{C})$ be zero diagonal Hermitian contractions. There exist $r \leqslant 18k/\epsilon^2$ and a partition $\sigma_1 \cup \cdots \cup \sigma_r = [n]$ such that

$$\lambda_{\max}(P_{\sigma_i} A^{(j)} P_{\sigma_i}) \leqslant \epsilon$$

for all $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant k$.

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The next theorem concerns simultaneous paving of a k-tuple of Hermitian matrices.

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By successive application of the Marcus-Spielman-Srivastava theorem one can prove a statement like this; however, the dependence of $r(\epsilon)$ on ϵ would be

$$r(\epsilon) \leqslant (6/\epsilon)^{2k}.$$

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Let $A^{(1)}, \dots, A^{(k)} \in M_n(\mathbb{C})$ be zero diagonal Hermitian contractions. For any $\epsilon > 0$ we may find a subset $\tau \subset [n]$ with $|\tau| \geqslant \epsilon^2 n/6k$ such that

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- We shall sketch the proof of this result.

Basic definitions

- For any $\sigma \subset [n]$ we write A_σ for the submatrix of A indexed by rows and columns with indices in σ, and A_{σ^*} for the matrix obtained if the rows and columns with indices in σ are removed.

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- If $\mathbf{A} = (A^{(1)}, \ldots, A^{(k)})$ is a k-tuple of matrices in $M_n(\mathbb{C})$ then their mixed determinant is

$$D(\mathbf{A}) = \sum_{\sigma_1 \cup \cdots \cup \sigma_k = [n]} \det(A^{(1)}_{\sigma_1}) \cdots \det(A^{(k)}_{\sigma_k}).$$

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- If $\mathbf{A} = (A^{(1)}, \ldots, A^{(k)})$ is a k-tuple of matrices in $M_n(\mathbb{C})$ then their mixed determinantal polynomial (MDP) is
$$\chi[\mathbf{A}](x) = \frac{1}{k^n} D\big(xI - A^{(1)}, \ldots, xI - A^{(k)}\big).$$

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$$\chi[\mathbf{A}_\sigma] = \chi[A_\sigma^{(1)}, \dots A_\sigma^{(k)}]$$

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- If $\Sigma = (\sigma_1, \dots, \sigma_r)$ is a partition of $[n]$, for any $A \in M_n(\mathbb{C})$ we define

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- Similarly, if $\Sigma = (\sigma_1, \dots, \sigma_r)$ is a partition of $[n]$ and $A^{(1)}, \dots, A^{(k)} \in M_n(\mathbb{C})$ we define

$$\chi[\mathbf{A}_\Sigma] = \prod_{i=1}^{r} \chi[\mathbf{A}_{\sigma_i}] = \prod_{i=1}^{r} \chi[A_{\sigma_i}^{(1)}, \dots, A_{\sigma_i}^{(k)}].$$

Formulas

1. If $A^{(1)}, \dots, A^{(k)} \in M_n(\mathbb{C})$ then

$$\chi[A^{(1)}, \dots, A^{(k)}] = \frac{1}{(k!)^n} \partial^{(k-1)[n]} \prod_{j=1}^{k} \det[Z - A^{(j)}] \Big|_{Z=xI}.$$

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2. Moreover, if $\sigma \subset [n]$ then

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3. Moreover, if $\Sigma = \{\sigma_1, \ldots, \sigma_r\}$ is a partition of $[n]$ then

$$\chi[\mathbf{A}_\Sigma] = \frac{1}{(k!)^{rn}} \left(\prod_{i=1}^{r} \partial_{(i)}^{k([n]\setminus\sigma_i)} \right) \prod_{i=1}^{r} \prod_{j=1}^{k} \det[Z_i - A^{(j)}] \Big|_{Z_1=\cdots=Z_r=xI}.$$

Joint control of the largest root

Theorem 1

Let $A^{(1)}, \ldots, A^{(k)}$ be zero diagonal Hermitian matrices. Then,

$$\max_{1 \leqslant i \leqslant k} \lambda_{\max}(\chi[A^{(i)}]) \leqslant k \cdot \lambda_{\max}(\chi[\mathbf{A}]),$$

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Proposition 2

Let $\mathbf{A} = (A^{(1)}, \ldots, A^{(k)})$ be a k-tuple of zero diagonal Hermitian matrices and B be a zero diagonal Hermitian matrix. Then,

$$\lambda_{\max}(\chi[\mathbf{A}, B]) \geqslant \lambda_{\max}(\chi[\mathbf{A}, 0]).$$

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- We show that

$$\lambda_{\text{max}}(\chi[\mathbf{A}, B]) \geqslant \lambda_{\text{max}}(\chi[\mathbf{A}, B_{(\{1\})}]).$$

  Here, $B_{(\sigma)}$ is the matrix that we obtain if we set the entries of the rows and columns of σ equal to 0.

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- In order to show this we define $B_t = D_t B D_t$ where $D_t = \mathrm{diag}(\sqrt{t}, 1, \ldots, 1)$ and the polynomial

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  This is real rooted for all $t \geqslant 0$. Then, we show that the function $f : [0, \infty) \to \mathbb{R}$ defined by

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- Applying this successively we see that
$$\lambda_{\max}(\chi[\mathbf{A}, B]) \geqslant \lambda_{\max}(\chi[\mathbf{A}, B_{(\{1,\dots,k\})}])$$
for all $k \leqslant n$. When $k = n$ we get the result.

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- By symmetry, the same holds for all $\lambda_{\max}(\chi[A^{(j)}])$.

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Let $A^{(1)}, \ldots, A^{(k)} \in M_n(\mathbb{C})$ be zero diagonal Hermitian contractions. For any $\epsilon > 0$ we may find a subset $\tau \subset [n]$ with $|\tau| \geqslant \epsilon^2 n/6k$ such that

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- Recall that A_{σ^*} denotes the submatrix of A with the rows and columns in σ removed, and χ_{σ^*} is the corresponding restricted characteristic polynomial.

Lemma 3

If $\mathbf{A} = (A^{(1)}, \ldots, A^{(k)})$ is a k-tuple of matrices in $M_n(\mathbb{C})$ then, for any $m \leqslant n$ we have

$$m! \sum_{|\sigma|=m} \chi[\mathbf{A}_{\sigma^*}] = \chi^{(m)}[\mathbf{A}].$$

Here, $f^{(m)}$ denotes the m-th derivative of f.

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Proposition 4

Let $\mathbf{A} = (A^{(1)}, \ldots, A^{(k)})$ be a k-tuple of Hermitian matrices in $M_n(\mathbb{C})$. For any $m \leqslant n$ there exists a subset $\sigma \subset [n]$ with $|\sigma| = m$ such that

$$\lambda_{\max}(\chi[\mathbf{A}_{\sigma^*}]) \leqslant \lambda_{\max}\Big(\sum_{|\sigma|=m} \chi[\mathbf{A}_{\sigma^*}]\Big) = \lambda_{\max}(\chi^{(m)}[\mathbf{A}]).$$

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- Fix $i \in [n]$. First we show that the polynomials $\chi[\mathbf{A}_{\{i\}^*}]$ have a common interlacing. To see this we check that every convex combination

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- By Lemma 3 we have

$$\lambda_{\max}\Big(\sum_{i=1}^{n} \chi^{(m-1)}[\mathbf{A}_{\{i\}^*}]\Big) = \lambda_{\max}(\chi^{(m)}[\mathbf{A}]),$$

therefore, for some $i_1 \in [n]$ we have

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- Repeating the same argument we find $i_2 \in [n] \setminus \{i_1\}$ such that

$$\begin{aligned} \lambda_{\max}(\chi^{(m-2)}[\mathbf{A}_{\{i_1,i_2\}^*}]) &\leqslant \lambda_{\max}\Big(\sum_{i \in [n]\setminus\{i_1\}} \chi^{(m-2)}[\mathbf{A}_{\{i_1,i\}^*}] \Big) \\ &= \lambda_{\max}(\chi^{(m-1)}[\mathbf{A}_{\{i_1\}^*}]) \leqslant \lambda_{\max}(\chi^{(m)}[\mathbf{A}]). \end{aligned}$$

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- After m steps we get $\sigma = \{i_1, \ldots, i_m\}$ with $|\sigma| = m$ so that

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- The first observation is that the characteristic polynomial of a zero diagonal $n \times n$ Hermitian matrix A is of the form

$$x^n - \frac{\mathrm{tr}(A^2)}{2} x^{n-2} + \text{lower order terms}.$$

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- Given $\mathbf{A} = (A^{(1)}, \dots, A^{(k)})$ we set $\mathrm{tr}(\mathbf{A}^2) = \sum_{i=1}^k \mathrm{tr}((A^{(i)})^2)$.
- Then, summing over all partitions $\Sigma = (\sigma_1, \dots, \sigma_k)$ of $[n]$ and noting that each pair of indices has probability $1/k^2$ of being in some σ_i, we finally get

$$q(x) = \chi[\mathbf{A}](x) = \mathbb{E}_{\sigma_1, \dots, \sigma_k} \prod_{i=1}^k \chi[A^{(i)}_{\sigma_i^*}](x) = x^n - \frac{\mathrm{tr}(\mathbf{A}^2)}{2k^2} x^{n-2} + \cdots$$

One can also check that all the roots of $q(x)$ are in $[-1, 1]$.

Root shrinking lemma

Let q be a real rooted polynomial of degree n with roots in $[-1,1]$. Assume that the sum of the roots is 0 and the average of the squares of the roots is α. For any $c \leqslant \frac{1}{1+\alpha}$ we have

$$\lambda_{\max}(q^{((1-c)n)}) \leqslant c(1-\alpha) + 2\sqrt{c(1-c)\alpha}.$$

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- We apply this lemma for the polynomial $q = \chi[\mathbf{A}]$. The average of the squares of the roots is

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Joint restricted invertibility

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for all $i \leqslant m$.
- Given $0 < \epsilon < 1$ we choose $c = \epsilon^2/6k$ and get the theorem with $\tau = [n] \setminus \sigma$.