introductory lecture

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the shift operator

H separable Hilbert space with orthonormal basis $\{e_n : n = 0, 1, 2 \ldots\}$.
An operator is a bounded linear map $H \to H$.
If T is an operator, the adjoint of T is an operator T^* which satisfies:

$$\langle T^* x, y \rangle = \langle x, Ty \rangle .$$

Definition

S is the operator on H defined by $Se_n = e_{n+1}$.

the shift operator

- The adjoint operator S^* satisfies $S^* e_n = e_{n-1}$ for $n = 1, 2, 3, \ldots$ and $S^* e_0 = 0$.
- the operator S is $1 - 1$ but not onto
- the operator S^* is onto but not $1 - 1$
- $S^* S = I$
- $SS^* = P$ where $P(e_n) = e_n$ for $e_n : n = 1, 2, 3, 4, \ldots$ and $P(e_0) = 0$.
- $\|Sx\| = \|x\|$ for every $x \in H$.
- $(S^*)^n x \to 0$ for every $x \in H$.

the shift operator

The matrix of S is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and the matrix of S^* is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

the shift operator

Definition

The spectrum of an operator T is the set

$$\{\lambda \in \mathbb{C} : T - \lambda I \;\text{ not }\; \text{invertible}\}$$

Example

If T is an operator on a finite dimensional space over $\mathbb{C}$, the spectrum is the set of eigenvalues.

the shift operator

Theorem

The spectrum of S is $\{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

proof Let $\lambda \in \mathbb{C}$, $|\lambda| < 1$.

$$S^*(\sum_{n=0}^{\infty} \lambda^n e_n) = \sum_{n=1}^{\infty} \lambda^n e_{n-1} = \sum_{n=0}^{\infty} \lambda^{n+1} e_n = \lambda \sum_{n=0}^{\infty} \lambda^n e_n.$$

λ is an eigenvalue of S^* and so, $\{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \subseteq \mathrm{sp}(S^*)$. Since $||S^*|| \leq 1$ we have $\mathrm{sp}(S^*) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. We obtain $\mathrm{sp}(S^*) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Hence

$$\mathrm{sp}(S) = \{\overline{\lambda} : \lambda \in \mathrm{sp}(S^*)\} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$$

.

the shift operator

Remark

S has no eigenvalues

0 is not an eigenvalue, since S is $1-1$.

If $\lambda \neq 0$ is an eigenvalue of S with eigenvector $\sum_{n=0}^{\infty} a_n e_n$ then:

$$S(\sum_{n=0}^{\infty} a_n e_n) = \lambda \left(\sum_{n=0}^{\infty} a_n e_n \right) \Leftrightarrow \sum_{n=0}^{\infty} a_n e_{n+1} = \sum_{n=0}^{\infty} \lambda a_n e_n$$

and we will have $\lambda a_0 = 0$ and $\lambda a_{n+1} = a_n$ for all n. Hence $a_n = 0$ for all n.

invariant subspaces

Definition

Let T be an operator on a Banach space X. A subspace V of X is invariant if $Tx \in V$ for all $x \in V$.

Example

If T is an operator on a finite dimensional space over $\mathbb{C}$, and v is an eigenvector of T, then the space $\{\mu v : \mu \in \mathbb{C}\}$ is an invariant subspace for T.

Question

Let X be a separable Banach space. Does every operator on X have a closed invariant subspace, different from $\{0\}$ and X?

invariant subspaces

Theorem (Enflo, 1975)

There exists an infinite dimensional separable Banach space X, and an operator T on X with no invariant closed subspace, different from $\{0\}$ and X.

Theorem (Argyros-Haydon, 2011)

There exists an infinite dimensional separable Banach space X, such that every operator on X has non trivial closed invariant subspace.

The answer to the following question is unknown.

Question

Let H be a separable Hilbert space (i.e. ℓ^2). Does every operator on H have a closed invariant subspace, different from $\{0\}$ and X?

the shift operator

We will describe the invariant subspaces of the shift. We will need another representation of the operator S.

Definition

$$L^2(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C} : \|f\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{ix})|^2 dx < +\infty\}$$

$L^2(\mathbb{T})$ is a Hilbert space for the scalar product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})\overline{g(e^{ix})}dx$$

and the family

$$\{\zeta_n : n \in \mathbb{Z}\} \quad \text{where} \quad \zeta_n(e^{ix}) = e^{inx}$$

is orthonormal: $\langle \zeta_n, \zeta_m\rangle = \delta_{nm}$.

the shift operator

For $f \in L^1(\mathbb{T})$ define

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})e^{-inx}dx, \qquad n \in \mathbb{Z}.$$

The map

$$\mathcal{F} : f \to (\hat{f}(n))_{n\in\mathbb{Z}}$$

is the Fourier transform and defines an isometry $L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$.

Definition

$$H^2(\mathbb{T}) = \{f \in L^2(\mathbb{T}) : \hat{f}(-k) = 0 \text{ for all } k = 1, 2, \ldots\}.$$

Let $T : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ be the map $Tf = \zeta_1 f$.
We have

$$S\mathcal{F} = \mathcal{F}T$$

and V is T-invariant iff $\mathcal{F}V$ is S-invariant.

the shift operator

A function $\phi \in H^2(\mathbb{T})$ with $|\phi(z)| = 1$ for almost all $z \in \mathbb{T}$ is called an *inner function*. If ϕ is an inner function the space

$$\phi H^2 = \{\phi f : f \in H^2(\mathbb{T})\}$$

is a closed subspace of $H^2(\mathbb{T})$ and is invariant under T.

Theorem (Beurling)

A closed nonzero subspace $E \subseteq H^2(\mathbb{T})$ is T-invariant if and only if there exists $\phi \in H^2(\mathbb{T})$ with $|\phi(z)| = 1$ for almost all $z \in \mathbb{T}$ such that $E = \phi H^2$. Moreover, ϕ is essentially unique in the sence that if $E = \psi H^2(\mathbb{T})$ where $|\psi| = 1$ a.e. then $\frac{\phi}{\psi}$ is (a.e. equal to) a constant (of modulus 1).

C^*-algebras

Definition

Let $\mathcal{A}$ be a Banach algebra. An involution on $\mathcal{A}$ is a map $a \to a^*$ on $\mathcal{A}$ s.t.

- $(a+b)^* = a^* + b^*$
- $(\lambda a)^* = \overline{\lambda} a^*, \lambda \in \mathbb{C}$
- $a^{**} = a$
- $(ab)^* = b^* a^*$

C^*-algebras

Definition

A C^*-algebra is a Banach algebra with an involution wich satisfies

$$\|a^*a\| = \|a\|^2.$$

C^*-algebras

Examples

- $\mathbb{C}$
  $\|z\| = |z|$
  $z^* = \bar{z}$
- $\mathcal{C}(X)$, for X compact.
  $\|g\| = \sup_{x \in X} |g(x)|$,
  $g^*(x) = \overline{g(x)}$
- $\mathcal{B}(H)$, for a Hilbert space H
  $\|T\| = \sup_{x \in H, \|x\| \leq 1} \|Tx\|$
  $\langle Tx, y \rangle = \langle x, T^* y \rangle$

C^*-algebras

If H is a Hilbert space, $\mathcal{B}(H)$ is the space of bounded linear operators on H.

Theorem

Let $\mathcal{A}$ be a C^*-algebra. Then $\mathcal{A}$ is isometrically isomorphic to a closed subalgebra of $\mathcal{B}(H)$ for some Hilbert space H.

the Kadison-Singer problem

Definition

Let $\mathcal{A}$ be a C^*-algebra with unit e. A state is a linear form f on $\mathcal{A}$ which is positive and satisfies $f(e) = 1$.

The set of states $S(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$ is a w^*-compact subset of the dual of $\mathcal{A}$. It is convex, hence by the Krein-Milman theorem it has extreme points.

Definition

A state is pure if it is an extreme point of $S(\mathcal{A})$.

the Kadison-Singer problem

Examples

- X compact topological space, $\mathcal{C}(X)$ the C^*-algebra of continuous functions $X \to \mathbb{C}$. A state on $\mathcal{C}(X)$ is a probability measure. A pure state is a Dirac measure.
- $\mathcal{B}(H)$ for a Hilbert space H. If $\xi \in H$, $\|\xi\| = 1$, then $f(a) = \langle a\xi, \xi\rangle$ is a state. These are called vector states.

the Kadison-Singer problem

Let $\mathcal{D}$ be the C^*-algebra of diagonal operators on ℓ^2. Let f be a pure state on $\mathcal{D}$.

The Kadison-Singer problem is the following:

Problem (Kadison-Singer, 1959)

Does f have a unique extension on $\mathcal{B}(\ell^2)$?

The answer is positive.

Marcus-Spielman-Srivastava (2015)

the Kadison-Singer problem

Definition

A diagonal projection is an orthogonal projection on ℓ^2 which lies in $\mathcal{D}$.

P is a diagonal projection iff there exists a subset S of $\mathbb{N}$ such that P is the orthogonal projection on the subspace of ℓ^2 spanned by $\{e_n : n \in S\}$.

Theorem

The f.a.e.:

- *The Kadison-Singer problem has a positive answer.*
- *Let $A \in \mathcal{B}(\ell^2)$ with 0 diagonal and $\epsilon > 0$. There exist $r \in \mathbb{N}$ and r pairwise orthogonal diagonal projections $P_1, P_2, \ldots, P_r$ such that $\sum_{i=1}^r P_i = 1$ and*

$$\|\sum_{i=1}^{r} P_i A P_i\| \leq \epsilon \|A\|$$

groups and operator algebras

Definition

A topological group is a group G which is a topological space such that the maps

$$(x, y) \mapsto xy$$

$$x \mapsto x^{-1}$$

are continuous.

groups and operator algebras

Examples

- G any group with the discrete topology
- $(\mathbb{R}, +)$, $(\mathbb{R}^*, \cdot)$, $(\mathbb{R}_+^*, \cdot)$
- $(\mathbb{T}, \cdot)$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$
- $GL(n, \mathbb{R}) = \{A = (a_{ij}) : n \times n \text{ matrix}, a_{ij} \in \mathbb{R},\ \det A \neq 0\}$
- $O(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : A^t A = I\}$
- the free group on n generators $\mathbb{F}_n$

groups and operator algebras

Proposition

G locally compact topological group. Then G has a left invariant measure. This measure is unique up to a scalar, is called the Haar measure and is denoted by $d\mu$.

groups and operator algebras

Definition

G topological group and H a Hilbert space . A unitary representation π of G on H is a map $G \to \mathcal{B}(H)$ such that:

1. $\pi(x)^*\pi(x) = \pi(x)\pi(x)^* = I, \quad \forall x \in G.$
2. $x \to \pi(x)$ is a homomorphism of groups from G into the group of unitary operators on H.
3. For each $v \in H$ the map $x \mapsto \pi(x)v$ is continuous.

groups and operator algebras

Examples

- The trivial representation of G on $\mathbb{C}$, $\pi(x) = 1$ for all $x \in G$.
- $L^2(G)$ the Hilbert space with inner product

$$\langle f, g \rangle = \int_G f(x)\overline{g(x)}d\mu(x).$$

The representation λ defined by:

$$\lambda(y)f(x) = f(y^{-1}x)$$

is called the left regular representation of G.

$C^*(G)$

$L^1(G) = \{f \text{ measurable} : G \to \mathbb{C}, \int_G |f(x)| d\mu(x) < +\infty\}$
Define

$$f * g(x) = \int_G f(xy^{-1}) g(y) d\mu(y)$$

$$f^*(x) = \overline{f(x^{-1})} \Delta_G(x^{-1}).$$

$C^*(G)$

Proposition

(π, H) representation of G. Define for $f \in L^1(G)$,

$$\pi(f) = \int_G f(x)\pi(x)d\mu(x).$$

Then:

1. $\pi : L^1(G) \to \mathcal{B}(H)$ *is linear.*
2. $\pi(f * g) = \pi(f)\pi(g)$
3. $\pi(f^*) = \pi(f)^*$
4. $\overline{\pi(L^1(G))H} = H$

$C^*(G)$

Define a norm on $L^1(G)$

$$\|f\| = sup_{\pi} \|\pi(f)\|,$$

where the sup is taken over the family of unitary representations of G.

Definition

The C^* algebra of G, $C^*(G)$ is the completion of $L^1(G)$ with respect to this norm.

$C^*(G)$

Examples

- $C^*(\mathbb{R}) \simeq C_0(\mathbb{R})$.
- $C^*(\mathbb{T}) \simeq C_0(\mathbb{Z})$.
- $C^*(\mathbb{Z}) \simeq C(\mathbb{T})$.

Theorem (Cuntz,1980)

$C^*(\mathbb{F}_m)$ is not isomorphic to $C^*(\mathbb{F}_n)$ for $n \neq m$.

vN(G)

G topological group, λ the left regular representation of G.
If H is a Hilbert space the weak operator topology (WOT) on $\mathcal{B}(H)$ is the topology defined by the family of seminorms $\{p_{x,y}\}_{x,y\in H}$ with $p_{x,y}(T) = \langle Tx, y\rangle$.

Definition

The von Neumann algebra vN(G) of G is the WOT closure of the linear span of $\{\lambda(x) : x \in G\}$. It is a subalgebra of $\mathcal{B}(L^2(G))$.

Examples

- $\mathrm{vN}(\mathbb{R}) \simeq L^\infty(\mathbb{R})$.
- $\mathrm{vN}(\mathbb{T}) \simeq \ell^\infty(\mathbb{Z})$.
- $\mathrm{vN}(\mathbb{Z}) \simeq L^\infty(\mathbb{T})$.

groups and operator algebras

Definition

The reduced C^*-algebra $C_r^*(G)$ of G is the norm closure of the linear span of $\{\lambda(x) : x \in G\}$. It is a subalgebra of $\mathcal{B}(L^2(G))$.

Theorem (Pimsner-Voiculescu, 1982)

$C_r^*(\mathbb{F}_m)$ is not isomorphic to $C_r^*(\mathbb{F}_n)$ for $n \neq m$.

The answer to the following question is unknown.

Question

Is $\mathrm{vN}(\mathbb{F}_n)$ isomorphic to $\mathrm{vN}(\mathbb{F}_m)$ for $n \neq m$?
In particular, is $\mathrm{vN}(\mathbb{F}_2)$ isomorphic to $\mathrm{vN}(\mathbb{F}_3)$?