Introduction to von Neumann algebras

Lecture IV

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Reminder

A C^*-algebra A is a Banach algebra with an involution such that $\|a^*a\| = \|a\|^2$ for all $a \in A$.

A von Neumann algebra M is a unital subalgebra of $B(H)$ (H: Hilbert space) which is selfadjoint and WOT-closed.

Here *unital* means $I \in M$, *selfadjoint* means $T \in M \Rightarrow T^* \in M$ and *WOT-closed* menas that if $\langle Tx, y\rangle = \lim_i \langle T_i x, y\rangle \, \forall x, y \in H$ and each $T_i \in M$ then $T \in M$.

Reminder

Theorem

Let A be a unital C-algebra and let $\pi : A \to B(H)$ be a *-homomorphism (linear, multiplicative, *-preserving). Then $\|\pi(a)\| \leq \|a\|$ for all $a \in A$. Moreover, if π is injective, then it is isometric: $\|\pi(a)\| = \|a\|$ for all $a \in A$.*

Theorem (Gelfand-Naimark)

Let A be a unital C-algebra. There exists some Hilbert space H and an injective *-homomorphism $\pi : A \to B(H)$.*

Reminder

A *-subalgebra $A \subseteq B(H)$ is called non-degenerate if

$$\overline{\mathrm{span}\{T\xi : T \in A, \xi \in H\}} = H.$$

Theorem (von Neumann's bicommutant theorem)

*Let $A \subseteq B(H)$ be a non-degenerate *-subalgebra. Then*

$$A'' = \overline{A}^{sot} = \overline{A}^{wot}.$$

Theorem (Kaplansky density theorem)

*Let $A \subseteq B(H)$ be a non-degenerate *-subalgebra. Then*

$$\mathrm{ball}(A'')_{sa} = \overline{\mathrm{ball}(A)_{sa}}^{sot} \quad \textit{and} \quad \mathrm{ball}(A'') = \overline{\mathrm{ball}(A)}^{sot}.$$

(Here $A_{sa} = \{T \in A : T = T^*\}$.)

A useful characterisation

Corollary

*A non-degenerate *-subalgebra $A \subseteq B(H)$ is a von Neumann algebra iff $\mathrm{ball}(A)$ is WOT-compact.*

Proof:

Exercise 1.

Observe that $T \in \mathrm{ball}(B(H))$ iff $|\langle Tx, y\rangle| \leq 1$ for all $x, y \in \mathrm{ball}(H)$, i.e. $\phi(T) \in \prod_{x,y \in \mathrm{ball}(H)} \overline{\mathbb{D}}$ where $\phi(T) = \{\langle Tx, y\rangle\}_{x,y \in H}$.
Conclude that $\mathrm{ball}(B(H))$ is WOT-compact (use Tihonof).

Exercise 2

If $A \subseteq B(H)$ is a von Neumann algebra, show that $\mathrm{ball}(A)$ is WOT-closed in $\mathrm{ball}(B(H))$ (use bicommutant theorem).

Exercise 3

If conversely $\mathrm{ball}(A)$ is WOT-compact, take any $T \in A''$ and show that $T \in A$, by considering $S = \frac{T}{\|T\|}$ and using Kaplansky density.

Constructions with projections

Basic Exercise

Let $p \in B(H)$ be a projection and $A \subseteq B(H)$ a von Neumann algebra. The space pH is A-invariant iff $p \in A'$.

Definition

For every $T \in B(H)$, the projection onto $\overline{ImT}$ is called the range projection of (also called the left support of T). The range projection of T is denoted $R(T)$.

Exercise

If T belongs to a von Neumann algebra A, then $R(T) \in A$.

Hint: Observe that $R(T) = \mathrm{proj}(\ker(T^*)^{\perp})$. But $\ker(T^*)$ is A'-invariant, so its projection commutes with A'. So $R(T) \in A'' = A$.

Constructions with projections

Definition

If $\{P_i\}$ is a family of projections on H, then define $\wedge P_i$ to be the projection on $\cap_i P_i(H)$ and $\vee P_i$ to be the projection on $\overline{\operatorname{span} P_i(H)}$.

Proposition

If A is von Neumann algebra and $\{P_i\} \subseteq A$ is a family of projections, then $\wedge P_i$ and $\vee P_i$ are in A.

Proof Since $P_i \in A$, its range is A' invariant. So $\operatorname{span} P_i(H)$ is A'-invariant, hence so is $\overline{\operatorname{span} P_i(H)}$. Hence the projection $\vee P_i$ onto $\overline{\operatorname{span} P_i(H)}$ is in $A'' = A$.
For $\wedge P_i$, take $\perp$.

Constructions with projections

Definition

The centre of a von Neumann algebra A is defined to be $Z(A) := A \cap A'$.
A projection $p \in A$ is said to be a central projection if it is contained in the centre of A.
A von Neumann algebra A is said to be a factor if $Z(A) = \mathbb{C}I = \{\lambda I : \lambda \in \mathbb{C}\}$.

Examples

The algebra $B(H)$ is a factor.
If G is an ICC group, then $\mathrm{vN}(G)$ is a factor.

A group G is an ICC group if for every $s \in G, s \neq e$ the conjugacy class $\{tst^{-1}; t \in G\}$ is infinite.

Ideals in von Neumann algebras

Theorem

Let A be a von Neumann algebra, and let $J \subseteq A$ be a WOT- closed (2-sided) ideal. Then $J = J^*$ and, moreover, there exists a central projection $p \in Z(A)$ such that $J = pA$.

Proof If $x \in J$, write $x = u|x|$ for the polar decomposition. Then $|x| = \sqrt{x^*x}$ is in J (functional calculus) and so $x^* = |x|u^* \in J$. So $J = J^*$.
Now let $K = \overline{J(H)}$. If $a \in J$ and $\xi \in H$, for each $b \in A$ we have $ba\xi \in K$. Hence K is A-invariant so $p := \mathrm{proj}(K) \in A'$. But also for each $c \in A'$ we have $ca\xi = ac\xi \in K$, so K is A'-invariant and thus $p \in A$. Thus $p \in Z(A)$.
Let $B = \{a|_K : a \in J\}$. Note that K is A-invariant so $B \subseteq B(K)$. Verify that B is a von Neumann algebra on K and hence contains the identity on K, namely $p|_K$.
It follows that $p \in J$ ($\exists q \in J$ s.t. $p|_K = q|_K$ or $(p - q)|_K = 0$; but also $p|_{K^\perp} = 0 = q|_{K^\perp}$ so $p = q$.) Finally, if $a \in J$ then $a = pa \in pA$, so $J \subseteq pA$, but also $pA \subseteq J$ because $p \in J$. Hence $J = pA$. □

Corollary

A is a factor iff it has no non-trivial WOT closed ideals.

Constructions with projections

Definition

Let A be a von Neumann algebra. For every $a \in A$, the central cover of a (also called the central support or central carrier) is the projection

$$c_a := \wedge\{p \in Z(A) : ap = a\}.$$

Exercise

If p is a projection in a von Neumann algebra $A \subseteq B(H)$, prove that c_p is the orthogonal projection onto the subspace

$$[ApH] := \overline{\{a\xi : \xi \in p(H)\}}.$$

Constructions with projections

If A is a von Neumann algebra on H and $p \in B(H)$ a projection, define

$$A_p = \{pa|_{pH} : a \in A\}.$$

When $p \in A$, then A_p is a *-subalgebra of $B(pH)$, called the reduced algebra.
When $p \in A'$, then A_p is a *-subalgebra of $B(pH)$, called the induced algebra.

Proposition

Let A be a von Neumann algebra on H, and let $p \in A$ be a projection. Then A_p and $(A')_p$ are von Neumann algebras on pH, and they are mutual commutants: $(A_p)' = (A')_p$.

Remark . The inclusion $A_p \subseteq (A'_p)'$ is immediate. Now take $b \in (A'_p)'$, put $c = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}$ and show that $b \in A'' = A$ to conclude that $b \in A_p$.
To show that $(A_p)' \subseteq A'_p$ is a challenge! (see the file `compress.pdf`)

Commutative von Neumann algebras

Theorem

Let H be a separable Hilbert space, $A \subseteq B(H)$ an abelian von Neumann algebra. Then there is a selfadjoint operator $a \in A$ such that $A = \{a\}''$.

Definition

*Let $A \subseteq B(H)$ be a *-algebra. A vector $\xi \in H$ is said to be cyclic for A if $A\xi$ is dense in H. It is said to be separating for A if for all $a \in A$, $a\xi = 0$ implies that $a = 0$.*

Lemma

*If $\xi \in H$ is cyclic for a *-algebra A then it is separating for A'. The converse is also true when A is non-degenerate.*

Proposition

Every commutative von Neumann algebra on a separable Hilbert space has a separating vector.

Commutative von Neumann algebras

Theorem

*Let A be a commutative von Neumann algebra on a separable Hilbert space. Then there exists a regular, compactly supported, Borel probablity measure μ on $\mathbb{R}$ such that A is *-isomorphic to $L^\infty(\mu)$.*

Theorem

*Let μ be a compactly supported and continuous regular Borel probability measure on the real line. Then $L^\infty(\mu)$ is *-isomorphic to $L^\infty([0,1])$.*

Conclusion

Every commutative von Neumann algebra on a separable Hilbert space has one of the following forms (up to *-isomorphism):

1. $L^\infty([0,1])$.
2. $\ell^\infty(\Omega)$, for a countable set Ω
3. $L^\infty([0,1]) \oplus \ell^\infty(\Omega)$, for a countable set Ω.