Interlacing polynomials: applications to restricted invertibility and multi-paving

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Abstract

Our starting point is the work of Batson, Spielman and Srivastava on the question of approximating a graph $G = (V, E, w)$ by a sparse graph H. The main technical result in this work is the following purely linear algebraic theorem: Let $d > 1$ and $v_1, \dots, v_m \in \mathbb{R}^n$ such that

$$I = \sum_{j=1}^{m} v_j v_j^T.$$

There exist non-negative reals $\{s_j\}_{1 \leqslant j \leqslant m}$, with $|\{j : s_j \neq 0\}| \leqslant dn$, such that

$$I \leq \sum_{j=1}^{m} s_j v_j v_j^T \leq \gamma_d \, I.$$

It was soon understood that this result was closely related to the restricted invertibility principle of Bourgain and Tzafriri which roughly speaking asserts that any matrix with high stable rank contains a large column submatrix with large least singular value. The same ideas led to the positive answer of the famous Kadison-Singer problem. A reduction of this problem, Anderson's paving conjecture states that for every $\varepsilon > 0$ we can find $r \in \mathbb{N}$ with the following property: For every $m \in \mathbb{N}$ and every matrix $T \in M_m(\mathbb{C})$ with $\mathrm{diag}(T) = 0$ there exist diagonal projections $Q_1, \dots, Q_r$ such that $\sum_{i=1}^{r} Q_i = I_d$ and $\|Q_i T Q_i\|_2 \leqslant \varepsilon \|T\|_2$ for all $i = 1, \dots, r$. This very strong statement was proved to be true by Marcus, Spielman and Srivastava; they extended the approach of Batson, Spielman and Srivastava, and developed the *method of interlacing families of polynomials*. In these talks we shall review the main elements of this method and discuss subsequent developments:

(i) We shall give a proof, due to Marcus, Spielman and Srivastava, of the restricted invertibility principle of Bourgain and Tzafriri, which sharpens it in two directions: it replaces the stable rank by the Schatten 4-norm and establishes tighter bounds when the number of the columns of the matrix is not much larger than the number of its rows. The proof employs an analysis of the smallest zeros of Jacobi and associated Laguerre polynomials.

(ii) We shall present a result of Ravichandran and Srivastava on the existence of non-trivial pavings for collections of matrices. An application of this multi-paving theorem is an improvement of a theorem of Johnson, Ozawa and Schechtman on commutator representations of zero trace matrices.