Connes′ Embedding Conjecture

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"We now construct an approximate imbedding of N in R. Apparently such an imbedding ought to exist for all II_1 factors because it does for the regular representation of free groups"
Alain Connes, Ann. Math. 104, (1976) p. 105.

von Neumann algebras

Definition

A von Neumann algebra is a selfadjoint unital subalgebra of $\mathcal{B}(H)$ which is WOT closed.

Examples

- $\mathcal{B}(H)$
- $H = H_1 \oplus H_2$,

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in \mathcal{B}(H_1), B \in \mathcal{B}(H_2) \right\}$$

- G discrete group, $\ell^\infty(G)$ acting on $\ell^2(G)$

von Neumann algebras

Examples

- G discrete group, $g \in G$.
  Let $\lambda(g)$ be the operator acting on $\ell^2(G)$ given by:
  $$\lambda(g)f(x) = f(g^{-1}x),$$
  $f \in \ell^2(G)$.
  vN(G) is the wot closed subalgebra of $\mathcal{B}(\ell^2(G))$ generated by the operators
  $$\{\lambda(g), \quad g \in G\}.$$
  Let $\delta_x(y) = 0$ if $x \neq y$ and $\delta_x(y) = 1$ if $x = y$.
  Then $\lambda(g)\delta_x = \delta_{gx}$.

von Neumann algebras

Definition

The center of a von Neumann algebra $\mathcal{A}$ is the set $\{z \in \mathcal{A} : za = az, \forall a \in \mathcal{A}\}$.

Definition

If the center of a von Neumann algebra $\mathcal{A}$ is equal to $\mathbb{C}I$, the von Neumann algebra is called a factor.

Examples

- $\mathcal{B}(H)$ is a factor.
- If G is a discrete group, then $\mathrm{vN}(G)$ is a factor iff every conjugacy class (except the conjugacy class of e) is infinite.
- $\mathrm{vN}(\mathbb{F}_n)$, $n \geq 2$ is a factor.

von Neumann algebras

If $\mathcal{A}$ is a factor of type II_1 there exists a linear form $\tau : \mathcal{A} \to \mathbb{C}$ such that:

- $\tau(x^*x) \geq 0$, for all $x \in \mathcal{A}$
- $\tau(1) = 1$
- If $x \in \mathcal{A}$, $\tau(x^*x) = 0 \Rightarrow x = 0$
- $\tau(xy) = \tau(yx)$, for all $x, y \in \mathcal{A}$
- the restriction of τ to the unit ball is WOT continuous
- $\tau(P(\mathcal{A})) = [0, 1]$, where $P(\mathcal{A})$ is the set of projections in $\mathcal{A}$.

The function above is called the tracial state on $\mathcal{A}$.

von Neumann algebras

Example

If G is a discrete icc group, then $\mathrm{vN}(G)$ is a factor of type II_1 and the tracial state is given by

$$\tau(x) = \langle x\delta_e, \delta_e \rangle .$$

the hyperfinite factor

Definition

$S_{\infty}^{\mathrm{fin}}$ is the group of all permutations of $\mathbb{N}$, fixing all but finitely many elements of $\mathbb{N}$.

We have

$$S_{\infty}^{\mathrm{fin}} = \cup_{n=1}^{\infty} S_n.$$

Definition

The hyperfinite factor $\mathcal{R}$ is the von Neumann algebra $\mathrm{vN}(S_{\infty}^{\mathrm{fin}})$.

the hyperfinite factor

Lemma

Let G be a discrete group and H a subgroup of G. The restriction of λ_G to H is a multiple of λ_H.

proof

Write G as disjoint union of right H-cosets, $G = \cup_{s \in S} Hs$, where S is a system of representatives of H/G in G. Then

$$\ell^2(G) = \oplus_{s \in S} \ell^2(Hs).$$

The space $\ell^2(Hs)$ is invariant under the restriction of λ_G to H and this restriction is equivalent to λ_H. $\square$

the hyperfinite factor

A factor $\mathcal{M}$ is AF if there exists an increasing sequence $(\mathcal{Q}_n)$ of finite dimensional $*$-subalgebras of $\mathcal{M}$ with the same unit, such that $\mathcal{M} = (\cup_{n=1}^{\infty} \mathcal{Q}_n)''$ (equivalently $\mathcal{M} = \overline{(\cup_{n=1}^{\infty} \mathcal{Q}_n)}^{\text{wot}}$).

Proposition

The hyperfinite factor $\mathcal{R}$ is AF.

proof The algebra $\mathcal{A}_n$ generated by $\{\lambda_G(h) : h \in S_n\}$ is finite dimensional and $\cup_{n=1}^{\infty} \mathcal{A}_n$ is dense in $\mathcal{A}$.

$\square$

the hyperfinite factor

Proposition (Murray-von Neumann)

$\mathcal{R}$ is the unique AF type II_1 factor.

the hyperfinite factor

Proposition

$\mathcal{R}$ embedds in every II_1 factor $\mathcal{M}$.

Take a projection $p \in \mathcal{M}$ such that $\tau(p) = 1/2$, where τ is the tracial state of $\mathcal{M}$. Take u a partial isometry in $\mathcal{M}$ such that
$uu^* = p, \; u^*u = 1 - p$.
The subalgebra $\mathcal{M}_2$ of $\mathcal{M}$ generated by

$$\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ u^* & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1-p \end{pmatrix}$$

is isomorphic to $M_2(\mathbb{C})$. A similar construction shows that there exists a subalgebra $\mathcal{M}_{2^n}$ of $\mathcal{M}$, isomorphic to $M_{2^n}(\mathbb{C})$.

the hyperfinite factor

One can show that

$$\overline{\cup_{n=1}^{\infty} \mathcal{M}_{2^n}}^{wot}$$

is a II_1 factor. By uniqueness of the AF type II_1 factor, this factor is isomorphic to $\mathcal{R}$. Hence $\mathcal{R}$ embedds into $\mathcal{M}$.

ultrafilters

Definition

Let X be a set. A filter $\mathcal{F}$ on X is a collection of subsets of X s.t.

- $X \in \mathcal{F}$
- $\emptyset \notin \mathcal{F}$
- $A \in \mathcal{F}, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
- $A \in \mathcal{F}, A \subseteq B \Rightarrow B \in \mathcal{F}$.

ultrafilters

Examples

- Let X be a set and $x_0 \in X$. Then $\mathcal{F} = \{A \subseteq X : x_0 \in A\}$ is a filter.
- $X = \mathbb{N}$. Set $\mathcal{F} = \{A \subseteq \mathbb{N} : \mathbb{N} - A \ \text{ is finite}\}$. Then $\mathcal{F}$ is a filter.

ultrafilters

Definition

Let X be a set. An ultrafilter $\mathcal{U}$ on X is a filter which is maximal.

Proposition

Every filter $\mathcal{F}$ is contained in an ultrafilter.

Proposition

A filter $\mathcal{F}$ is an ultrafilter iff for every $A \subseteq X$ we have $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.

ultrafilters

Example

- Let X be a set and $x_0 \in X$. Then $\mathcal{U} = \{A \subseteq X, x_0 \in A\}$ is an ultrafilter. These are called principal.

Let $\mathcal{F} = \{A \subseteq \mathbb{N} : \mathbb{N} - A \text{ is finite}\}$. There is no principal ultrafilter that contains $\mathcal{F}$. Hence there are ultrafilters that are not principal. These are called free ultrafilters.

ultrafilters

Definition

Let (a_n) be a bounded sequence in $\mathbb{R}$ and $\mathcal{U}$ an ultrafilter on $\mathbb{N}$. We say that x is a $\mathcal{U}$-limit of (a_n) if for every neighbourhood S of x, the set $\{n \in \mathbb{N} : a_n \in S\}$ is in $\mathcal{U}$.

Proposition

Let (a_n) be a bounded sequence in $\mathbb{R}$ and $\mathcal{U}$ an ultrafilter on $\mathbb{N}$. Then there exists an x which is the $\mathcal{U}$-limit of (a_n). We denote $x = \lim_{\mathcal{U}} a_n$.

ultrafilters

Proposition

Let $(a_n), (b_n)$ be bounded sequences in $\mathbb{R}$ and $\mathcal{U}$ an ultrafilter on $\mathbb{N}$. Then

- $\lim_{\mathcal{U}}(a_n + b_n) = \lim_{\mathcal{U}} a_n + \lim_{\mathcal{U}} b_n$
- $\lim_{\mathcal{U}}(\lambda a_n) = \lambda \lim_{\mathcal{U}} a_n$ *for* $\lambda \in \mathbb{R}$
- $\lim_{\mathcal{U}}(a_n b_n) = \lim_{\mathcal{U}} a_n \lim_{\mathcal{U}} b_n$.

ultrapowers

Let $\mathcal{U}$ be an ultrafilter over $\mathbb{N}$. Let X_n be a set, for $n \in \mathbb{N}$.
The set theoretic ultraproduct of the family $\{X_n\}_{n\in\mathbb{N}}$ with respect to $\mathcal{U}$ is defined as follows:
Let $\prod_{n\in\mathbb{N}} X_n$ be the product of the family $\{X_n\}_{n\in\mathbb{N}}$. Define a relation on $\prod_{n\in\mathbb{N}} X_n$:

$$x_n \sim y_n \Leftrightarrow \{n : x_n = y_n\} \in \mathcal{U}.$$

The ultraproduct of the family $\{X_n\}_{n\in\mathbb{N}}$ with respect to the ultrafilter $\mathcal{U}$ is $X/\sim$.

If $X_n = X$ for all $n \in \mathbb{N}$, we use the word ultrapower of X.
In that case, X embedds in the ultrapower of X, via the map

$$x \mapsto (x, x, x, \ldots).$$

Examples

The ultrapower of $\mathbb{R}$ with respect to a free ultrafilter on $\mathbb{N}$ is the set of hyperreal numbers. It is a field, containing $\mathbb{R}$.

remark

Let $\mathcal{U}$ be a free ultrafilter, and $\mathbb{R}^{\mathcal{U}}$ the ultrapower of $\mathbb{R}$ with respect to $\mathcal{U}$. The class of $(\frac{1}{n})_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathcal{U}}$ is strictly smaller than any positive real number (infinitesimal).

ultrapowers

Let $\mathcal{U}$ be an ultrafilter over $\mathbb{N}$. Let X_n be a Banach space for $n \in \mathbb{N}$
The ultraproduct of the family $\{X_n\}_{n\in\mathbb{N}}$ with respect to $\mathcal{U}$ is defined as follows:
Let

$$\widetilde{X} = \{x = (x_n) \in \prod_{n\in\mathbb{N}} X_n : \sup \|x_n\| < +\infty\}.$$

Let

$$J = \{x = (x_n) \in \widetilde{X} : \lim_{\mathcal{U}} \|x_n\| = 0\}.$$

The Banach space ultraproduct of the family $\{X_n\}_{n\in\mathbb{N}}$ with respect to the ultrafilter $\mathcal{U}$ is the Banach space $\widetilde{X}/J$, with norm defined by

$$\|x + J\| = \lim_{\mathcal{U}} \|x_n\|_{X_n},$$

where $x = (x_n) \in \widetilde{X}$.

ultrapowers

If $X_n = X$ for all $n \in \mathbb{N}$, we use the word Banach space ultrapower of X and we denote it $X^{\mathcal{U}}$.
In that case, X embedds in the ultrapower $X^{\mathcal{U}}$ of X, via the map

$$x \mapsto (x, x, x, \ldots).$$

Examples

- The Banach space ultrapower of $\mathbb{R}$ with respect to a free ultrafilter over $\mathbb{N}$ is isomorphic to $\mathbb{R}$.
- The Banach space ultrapower of a finite-dimensional Banach space X with respect to a free ultrafilter over $\mathbb{N}$ is isomorphic to X.

ultrapowers

Let $\mathcal{U}$ be an ultrafilter over $\mathbb{N}$. Let $\mathcal{M}_n$ be a type II_1 factor with tracial state τ_n for $n \in \mathbb{N}$.

The ultraproduct of the family $\{\mathcal{M}_n\}_{n\in\mathbb{N}}$ with respect to $\mathcal{U}$ is defined as follows:

Let

$$\widetilde{\mathcal{M}} = \{x = (x_n) \in \prod_{n\in\mathbb{N}} \mathcal{M}_n : \sup \|x_n\| < \infty\}.$$

Let

$$J = \{x = (x_n) \in \widetilde{\mathcal{M}} : \lim_{\mathcal{U}} \tau(x_n^* x_n) = 0\}.$$

The ultraproduct of the family $\mathcal{M}_n$ with respect to the ultrafilter $\mathcal{U}$ is $\widetilde{\mathcal{M}}/J$.

ultrapowers

Then $\widetilde{\mathcal{M}}/J$ is a type II_1 factor with tracial state

$$\tau(x+J) = \lim_{\mathcal{U}} \tau_n(x_n),$$

for $x = (x_n) \in \widetilde{\mathcal{M}}$.
If $\mathcal{M}_n = \mathcal{M}$ for all $n \in \mathbb{N}$, we use the word ultrapower of $\mathcal{M}$ with respect to $\mathcal{U}$ and we denote it $\mathcal{M}^{\mathcal{U}}$.
In that case, $\mathcal{M}$ embedds in the ultrapower $\mathcal{M}^{\mathcal{U}}$, via the map

$$x \mapsto (x, x, x, \ldots).$$

CEC

Definition

A factor is separable if it is faithfully representable in some $\mathcal{B}(H)$ for a separable Hilbert space H.

The Connes' Embedding Conjecture is the following:

conjecture

Every separable type II_1 factor $\mathcal{M}$ is embeddable in $\mathcal{R}^{\mathcal{U}}$, where $\mathcal{U}$ is a free ultrafilter over $\mathbb{N}$.

Embeddable means that there is an injective, trace preserving $*$-homomorphism from $\mathcal{M}$ into $\mathcal{R}^{\mathcal{U}}$.

CEC

Theorem (Ge-Hadwin, 2001)

Let $\mathcal{U}$ and $\mathcal{V}$ be two free ultrafilters over $\mathbb{N}$. Assuming Continuum Hypothesis (CH) , the factors $\mathcal{R}^{\mathcal{U}}$ and $\mathcal{R}^{\mathcal{V}}$ are isomorphic.

Theorem (Farah-Hart-Sherman, 2013)

Let $\mathcal{M}$ be a separable II_1 factor. Then, Continuum Hypothesis is equivalent to the statement that all the ultrapowers of $\mathcal{M}$ with respect to free ultrafilters over $\mathbb{N}$ are isomorphic.

CEC

Assuming CH, if the Connes' Embedding Conjecture is true, then there will be a universal type II_1 factor. Every separable II_1 factor will embedd into $\mathcal{R}^{\mathcal{U}}$, where $\mathcal{U}$ is a free ultrafilter over $\mathbb{N}$.

Ozawa (2004): If there exists a II_1 factor $\mathcal{N}$ such that every separable II_1 factor embedds into $\mathcal{N}$, then $\mathcal{N}$ is not separable.

Theorem

Let $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$. The factor $\mathcal{R}^{\mathcal{U}}$ is not separable.

equivalent statements

Let H be a separable Hilbert space. We consider the set of von Neumann algebras $\mathrm{vN}(H)$ acting on H equipped with the Effros-Marechal topology. We denote:

$\mathcal{J}_{I_{\mathrm{fin}}}$ the set of type I finite factors acting on H,

$\mathcal{J}_{I}$ the set of type I factors acting on H,

$\mathcal{J}_{\mathrm{AF}}$ the set of AF factors acting on H.

Theorem (Haagerup-Winslow, 2000)

The f.a.e.:

- *Connes´ EC is true*
- $\mathcal{J}_{\mathrm{AF}}$ *is dense in* $\mathrm{vN}(H)$
- $\mathcal{J}_{I}$ *is dense in* $\mathrm{vN}(H)$
- $\mathcal{J}_{fin}$ *is dense in* $\mathrm{vN}(H)$.

equivalent statements

If A, B are C^*-algebras we consider the following norms on the algebraic tensor product of A and B:

$$\|x\|_{\max} = \sup\{\pi(x) : \pi \ * -\text{representation of } A \otimes B\}$$

$$\|x\|_{\min} = \sup\{\pi_1 \otimes \pi_2(x) : \pi_1 \ * -\text{repr. of } A, \pi_2 \ * -\text{repr. of } B\}.$$

Let $\mathbb{F}_\infty$ be the free group with a countable number of generators.

Theorem (Kirchberg, 1993)

The f.a.e.:

- *Connes´ EC is true*
- $$C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty).$$

equivalent statements

CEC is equivalent to Tsirelson's Problem in Quantum Information Theory.