Tensor products of convex cones

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We work in a finite-dimensional real vector space V and consider a cone $\mathcal{C} \subset V$ (all cones are implicitly convex, closed, salient and generating). The dual cone $\mathcal{C}^* \subset V^*$ is defined as $\mathcal{C}^* = \{\phi \in V^* \,:\, \phi(x) \geqslant 0 \ \forall x \in \mathcal{C}\}$.

Given two such cones $\mathcal{C}_1 \subset V_1$, $\mathcal{C}_2 \subset V_2$, there are two canonical ways to define the tensor product:

1. The *minimal tensor product*

$$\mathcal{C}_1 \otimes_{\min} \mathcal{C}_2 = \operatorname{conv}\{x_1 \otimes x_2 \,:\, x_i \in \mathcal{C}_i\}.$$

2. The *maximal tensor product*

$$\mathcal{C}_1 \otimes_{\max} \mathcal{C}_2 = (\mathcal{C}_1^* \otimes_{\min} \mathcal{C}_2^*)^* \,.$$

By analogy with the terminology used in C^*-algebras, we say that $(\mathcal{C}_1, \mathcal{C}_2)$ is a nuclear pair if $\mathcal{C}_1 \otimes_{\min} \mathcal{C}_2 = \mathcal{C}_1 \otimes_{\max} \mathcal{C}_2$ (the inclusion $\subset$ always holds).

We say that a cone $\mathcal{C}$ is classical if it has a base which is a simplex. It is easy to see that a pair of cones is nuclear whenever one of them is classical. A conjecture by Barker (1976) is that the converse holds: it postulates that if a pair $(\mathcal{C}_1, \mathcal{C}_2)$ is nuclear, then either $\mathcal{C}_1$ or $\mathcal{C}_2$ is classical.

That conjecture has a strong motivation from physics. The fact that the pair (PSD, PSD) is not nuclear (PSD is the cone of positive semi-definite matrices) is related to the phenomenon of quantum entanglement. In the language of "generalized probabilistic theories", the conjecture means that entanglement exists between any two non-classical theories.

We give support to Barker's conjecture by proving its validity in the two important cases

1. For a pair of 3-dimensional cones.

2. For a pair of polyhedral cones.

We plan to present the proof of these results (based on joint work with Ludovico Lami and Carlos Palazuelos). No prior knowledge of cones will be assumed.