INTRODUCTION TO VON NEUMANN ALGEBRAS

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1. Operators and Spectrum

Let H be a complex Hilbert space with inner product $< \cdot, \cdot >$ and norm $\|\xi\|_2 = \sqrt{< \xi, \xi >}$. A linear map $T : H \to H$ is called bounded operator if

$$\|T\| = \sup\{\|T(\xi)\|_2 : \|\xi\|_2 \leq 1\} < +\infty.$$

We denote

$$B(H) = \{T : H \to H, \;\; T \textit{ is bounded operator}\}.$$

The space $B(H)$ is a Banach algebra under the composition of operators. If $T \in B(H)$ there exists unique $S \in B(H)$ such that

$$< T(\xi_1), \xi_2 >=< \xi_1, S(\xi_2) >, \quad \forall\, \xi_1,\; \xi_2 \; \in \; H.$$

We denote in this case $S = T^*$ and we call T^* adjoint of T.

Exercise: If $T \in B(H)$, then $\|T\|^2 = \|T^*T\|$.

Definition 1.1. *Let $T \in B(H)$.*

(i) If $T = T^*$ we call T selfadjoint operator.

(ii) If $TT^* = T^*T$ we call T normal operator.

(iii) If $T^*T = Id_H$ we call T isometry. (Prove that $T^*T = Id_H \Leftrightarrow \|T(\xi)\|_2 = \|\xi\|_2$, for all $\xi \in H$).

(iv) If $TT^* = T^*T = Id_H$ we call T unitary. (Thus the unitaries are the onto isometries).

Examples 1.1. (i) Let $H = l^2(\mathbb{N}) = \{\xi = (\xi_n)_n : \; \sum_{n=1}^{\infty} |\xi_n|^2\} < +\infty$ and $(a_n)_n$ be a bounded sequence. Then the operator $T : H \to H, \;\; T(\xi) = (a_1\xi_1, a_2\xi_2, ...)$ is a bounded operator and $\|T\| = \sup\{|a_n| : n \in \mathbb{N}\}$.

(ii) Let

$$S_1 : H \to H, S_1((\xi_1, \xi_2, ...)) = (0, \xi_1, \xi_2, ...), \quad S_2 : H \to H, S_1((\xi_1, \xi_2, ...)) = (\xi_2, \xi_3, ...)$$

Prove that $S_1^* = S_2, S_1$ is an isometry, S_1 is not normal.

(iii) Let

$$H = l^2(\mathbb{Z}) = \{\xi = (..., \xi_{-2}, \xi_{-1}, \xi_0, \xi_1, \xi_2, ...) : \; \sum_{n\in\mathbb{Z}}^{\infty} |\xi_n|^2 < +\infty\}.$$

The operator $S : H \to H, S((\xi_k)_k) = (\xi_{k+1})_k$ is unitary.

(iii) Let (X, μ) be a measure space, $H = L^2(X, \mu)$ and $f \in L^{\infty}(X, \mu)$. Then the operator

$$T : H \to H, \quad T(g) = fg$$

is normal and $\|T\| = \|f\|_{\infty}$.

Let H be a Hilbert space and $T \in B(H)$. We call T positive if $< T(\xi), \xi >\geq 0$ for all $\xi \in H$. We write $T \geq 0$.

Theorem 1.2. *Let $T \in B(H), T \geq 0$. Then there exists a unique positive operator $S \in B(H)$ such that $S^2 = T$. We write $S = T^{\frac{1}{2}}$.*

We shall sketch the proof later.

Definition 1.2. *If $T \in B(H)$ then we can easily see that $T^*T \geq 0$. Thus there exists $S \in B(H)$ such that $S = (T^*T)^{\frac{1}{2}}$. We denote $S = |T|$.*

Definition 1.3. *Let H be a Hilbert space and $K \subseteq H$ be a closed subspace. If $T \in B(H)$ such that the restriction $T|_K$ is an isometry and $T|_{K^\perp} = 0$ we call T partial isometry. We also call K initial space of T and $T(K)$ final space of T.*

Theorem 1.3. *(Polar decomposition) Let $T \in B(H)$. There exists a partial isometry V with initial space $\overline{|T|(H)}$ and final space $\overline{T(H)}$ such that $T = V|T|$.*

Proof. If $\xi \in H$ then

$$\|T(\xi)\|_2^2 = < T(\xi), T(\xi) > = < T^*T(\xi), \xi > = < |T|^2(\xi), \xi > = < |T|(\xi), |T|(\xi) > = \||T(\xi)|\|^2.$$

Then the map

$$V_0 : |T|(H) \to T(H), \quad |T|(\xi) \to T(\xi)$$

is a linear isometry. Therefore V_0 extends to an isometry $V_1 : \overline{|T|(H)} \to \overline{T(H)}$. We define $V : H \to H$ such that $V(\xi) = V_1(\xi)$ if $\xi \in \overline{|T|(H)}$ and $V(\xi) = 0$ if $\xi \in |T|(H)^\perp$. Then V is partial isometry and $T = V|T|$. □

Definition 1.4. *Let H be a Hilbert space and $P \in B(H)$. We call P a projection if $P^2 = P = P^*$.*

For every closed subspace $M \subseteq H$ there exists a unique projection $P \in B(H)$ such that $P(H) = M$ and vice versa.

Definition 1.5. *Let P, Q be projections. We say that P and Q are mutually orthogonal if $P(H) \perp Q(H)$. We write $P \perp Q$.*

Exercise: Prove that $P \perp Q$ iff $PQ = 0$.

Definition 1.6. *Let $\{P_i : i \in I\} \subseteq B(H)$ be projections. We denote by $\wedge_{i \in I} P_i$ the projection onto $\cap_{i \in I} P_i(H)$ and by $\vee_{i \in I} P_i$ the projection onto $\overline{span\{\cup_{i \in I} P_i(H)\}}$.*

Definition 1.7. *Let $T \in B(H)$. We call spectrum of T the subset of $\mathbb{C}$ given by*

$$\sigma(T) = \{\lambda \in \mathbb{C} : \text{ there does not exist } S \in B(H) \text{ such that } (\lambda Id_H - T)S = S(\lambda Id_H - T) = Id_H\}.$$

Theorem 1.4. *(i) The spectrum $\sigma(T)$ is a nonempty compact subset of $\mathbb{C}$.*
(ii) Let $T \in B(H)$. Then $T = T^* \Leftrightarrow \sigma(T) \subseteq \mathbb{R}$
(iii) Let $T = T^* \in B(H)$. Then $T \geq 0 \Leftrightarrow \sigma(T) \subseteq [0, +\infty)$.

Theorem 1.5. *Let $T = T^* \in B(H)$. Then $\|T\| = \sup\{|\lambda| : \lambda \in \sigma(T)\}$.*

Definition 1.8. *Let $T \in B(H)$ and $p(x) = \alpha_k x^k + ... + a_1 x + a_0 \in \mathbb{C}[x]$ be a polynomial. We denote by $p(T)$ the operator $\alpha_k T^k + ... + a_1 T + a_0 Id_H$.*

Theorem 1.6. *Let $T \in B(H)$ and $p \in \mathbb{C}[x]$. Then*

$$\sigma(p(T)) = \{p(\lambda) : \lambda \in \sigma(T)\},$$

in other words $\sigma(p(T)) = p(\sigma(T))$.

Proof. Let $\lambda \in \mathbb{C}$ and consider the polynomial $q(x) = p(x) - \lambda$. By the Fundamental Theorem of Algebra

$$q(x) = a(x - a_1)...(x - a_n).$$

Thus

$$q(T) = p(T) - \lambda Id_H = a(T - a_1 Id_H)...(T - a_n Id_H).$$

Therefore

$$\lambda \in \sigma(p(T)) \Rightarrow p(T) - \lambda Id_H \ \textit{is not invertible} \Rightarrow \exists\, k \textit{ such that } T - a_k Id_H \ \textit{is not invertible} \Rightarrow$$

$$\exists\, k \textit{ such that } a_k \in \sigma(T) \Rightarrow \lambda = p(a_k) \in p(\sigma(T)),$$

thus $\sigma(p(T)) \subseteq p(\sigma(T))$. If $\mu \in \sigma(T)$ then $T - \mu Id_H$ is not invertible. Also since

$$p(\lambda) - p(\mu) = (\lambda - \mu)\phi(\lambda) \Rightarrow p(T) - p(\mu)Id_H = (T - \mu Id_h)\phi(T)$$

the operator $p(T) - p(\mu)Id_H$ is not invertible. Thertefore $p(\mu) \in \sigma(p(T))$. Thus, $p(\sigma(T)) \subseteq \sigma(p(T))$. ☐

2. Continuous Functional Calculus, Spectral Theorem for Selfadjoint Operators

Definition 2.1. *Let $\mathcal{A}$ be a subspace of $B(H)$. We call $\mathcal{A}$ C^*-algebra if*

(i)

$$T, S \in \mathcal{A} \Rightarrow TS \in \mathcal{A},$$

(ii)

$$T \in \mathcal{A} \Rightarrow T^* \in \mathcal{A},$$

(iii)

$$\mathcal{A} = \overline{\mathcal{A}}$$

Definition 2.2. *Let $T = T^* \in B(H)$ we denote by $C^*(T)$ the unital C^*-algebra generated by T. Observe that*

$$C^*(T) = \overline{\{p(T) : p \ \textit{is polynomial}\}}.$$

If X is a compact metric space, we denote by $C(X)$ the algebra of continuous functions from X to $\mathbb{C}$. This is a Banach algebra under the norm $\|f\|_X = \sup\{|f(x)| : x \in X\}$.

Theorem 2.1. *Let $T \in B(H)$ be a selfadjoint operator. Then there exists an isometric homomorphism $\Phi : C(\sigma(T)) \to C^*(T)$ such that*

(i) $\Phi(p) = p(T)$ for all polynomials p,

(ii) $\Phi(\overline{f}) = \Phi(f)^*$ for all $f \in C(\sigma(T))$.

Proof. If p is a real polynomial then $p(T)^* = p(T)$, so

$$\|p(T)\| = \sup\{|\mu| : \ \mu \ \in \ \sigma(p(T))\}.$$

But $\sigma(p(T)) = p(\sigma(T))$. Thus,

$$\|p(T)\| = \sup\{|p(t)| : \ t \ \in \ p(\sigma(T))\} = \|p\|_{\sigma(T)}.$$

Let p be an arbitrary real polynomial. We define $q = \overline{p}p$. This is a real polynomial and

$$q(T) = \overline{p}(T)p(T) = p(T)^*p(T).$$

Thus,

$$\|p(T)\|^2 = \|p(T)^*p(T)\| = \|q(T)\| = \|q\|_{\sigma(T)} = \|\overline{p}p\|_{\sigma(T)} =$$
$$\||p|^2\|_{\sigma(T)} = \|p^2\|_{\sigma(T)}.$$

Therefore $\|p(T)\| = \|p\|_{\sigma(T)}$. We conclude that the map

$$\Phi : \mathcal{P}_{\mathbb{R}}(\sigma(T)) \to C^*(T), \quad \Phi(p) = p(T),$$

where $\mathcal{P}_{\mathbb{R}}(\sigma(T))$ is the algebra of real polynomials on $\sigma(T)$, is isometric homomorphism. Since by Stone Weierstrass Theorem $\overline{\mathcal{P}(\sigma(T))} = C(\sigma(T))$ and since

$$C^*(T) = \overline{\{p(T) : \ \ p \ \ real\ polynomial\}}$$

we conclude that Φ extends to an isometric homomorphism from $C(\sigma(T))$ onto $C^*(T)$. □

Notation: For every $f \in C(\sigma(T))$ we denote $\Phi(f) = f(T)$.

Corollary 2.2. *Let $T \in B(H)$ be a positive operator. Then there exists $S \in B(H)$ such that $S \geq 0$ and $S^2 = T$.*

Proof. We define the continuous functions $f_0(t) = t, \quad f(t) = \sqrt{t}, \ \ t \ \in \ \sigma(T)$ and we denote $S = \Phi(f) = f(T)$. Since $\sigma(S) = f(\sigma(T))$ and $f(\sigma(T)) \subseteq [0, +\infty)$ we conclude that $S \geq 0$. Since $f^2 = f_0$ we have $f^2(T) = f_0(T) \Rightarrow S^2 = T$. □

Exercise: Prove that S is the unique operator such that $S \geq 0$ and $S^2 = T$.

Remark 2.3. We know from Linear Algebra that if $T : \mathbb{C}^n \to \mathbb{C}^n$ is a selfadjoint operator, then there exists a unitary $U : \mathbb{C}^n \to \mathbb{C}^n$ such that $U^*TU = diag(\lambda_1, ..., \lambda_n)$ where $\lambda_i \in \mathbb{R}$. The set of eigenvalues of T is $\{\lambda_1, ..., \lambda_n\}$. But if $T = T^* \in B(H)$, where H is an infinite dimensional Hilbert space T does not have eigenvalues necessarily. For example the operator $T : L^2([0, 1]) \to L^2([0, 1])$ given by $T(f)(t) = tf(t)$ has not eigenvalues. Nevertheless the following theorem known as spectral theorem holds:

Theorem 2.4. *Let T be a bounded selfadjoint operator acting on a Hilbert space H. Then there exists a measure space (X, μ) a unitary $U : L^2(X, \mu) \to H$ and a function $f : X \to \mathbb{R}$ in $L^\infty(X, \mu)$ such that $U^*TU = M_f$ where $M_f(g) = fg$ for all $g \in L^2(X, \mu)$. When H is separable, X can be taken to be a locally compact Hausdorff space and μ be a regular Borel prabability measure.*

We sketch the proof:

Definition 2.3. *A vector ξ is called cyclic vector for T if $H = \overline{\{p(T)\xi : \ \ p \ \in \ \mathbb{C}[x]\}}$.*

Assume that ξ is a cyclic unit vector for T. Let $X = \sigma(T)$. By the continuous functional calculus there exists an isometric $*$-isomorphism

$$\Phi : C(X) \to C^*(T) : \Phi(f) = f(T).$$

Define the linear functional

$$\rho : C(X) \to \mathbb{C}, \ \ \rho(f) =< f(T)\xi, \xi > .$$

Then ρ is a positive linear functional on $C(X)$ and $\rho(1_X) = 1$. By Riesz's representation theorem there exists a unique regular probability measure μ such that

$$\rho(f) = \int_X f d\mu, \quad \forall \ f \ \ \in \ C(X).$$

If $g \in C(X)$ then

$$\|g\|_2^2 = \int_X |g|^2 d\mu = \rho(|g|^2) =< g(T)^* g(T)\xi, \xi >= \|g(T)\xi\|_2^2.$$

Thus the map

$$U_0 : C(X) \to H, \quad U_0(g) = g(T)\xi$$

is linear isometry in $\|\cdot\|_2$ norm. Since $C(X)$ is dense in $\|\cdot\|_2$ norm in $L^2(X,\mu), U_0$ extends to an isometry $U : L^2(X,\mu) \to H$. Since ξ is cyclic for T, U is onto, thus it is a unitary. Let $f(t) = t, t \in X$. For all $g \in C(X)$ we have

$$UM_f(g) = U_0(fg) = fg(T)\xi = f(T)g(T)\xi = Tg(T)(\xi) = TU_0(g) = TU(g).$$

Thus $UM_f = TU$. The proof is complete.

We drop the assumption of the existence of a cyclic vector. By Zorn's Lemma there exists a family $\{H_i : i \in I\}$ of mutually orthogonal closed subspaces of H such that

(i)

$$T(H_i) \subseteq H_i, \ \ \forall\, i$$

(ii) For every i there exists a cyclic vector for the operator $T|_{H_i} : H_i \to H_i$

(iii) H is equal with $\sum \oplus_i H_i$.

From the first part of the proof there exist unitaries $U_i : L^2(X_i,\mu_i) \to H_i$ such that $U_i^* T|_{H_i} U_i = M_{f_i}$ where $f_i : X \to \mathbb{R}$ are functions in $L^\infty(X_i,\mu_i)$. Define (X,μ) to be the disjoint union $\cup_i (X_i,\mu_i)$. Then $L^2(X,\mu)$ is the sum $\sum_i \oplus_i L^2(X_i,\mu_i)$. We define the operator $U = \sum_i \oplus_i U_i : L^2(X,\mu)$ and the map $f : X \to \mathbb{R}, \ \ f|_{X_i} = f_i$. Clearly $UM_f = TU$. The proof is complete.

3. Topologies on $B(H)$. Double Commutant Theorem. Kaplansky Density Theorem

Definition 3.1. *The weak operator topology (WOT) is the topology on B(H), whose the basis for every $T \in B(H)$ is the collection of sets*

$$V(T, x_1, ..., x_n, y_1, ..., y_n) = \{S \in B(H) : | < (T-S)(x_i), y_i > | < 1, 1 \le i \le n\}.$$

Observe that a net $(S_\lambda)_\lambda \subseteq B(H)$ converges to $S \in B(H)$ in the WOT iff

$$< S_\lambda(x), y > \to < S(x), y >$$

for all $x, y \in H$.

Definition 3.2. *The strong operator topology (SOT) is the topology on B(H), whose the basis for every $T \in B(H)$ is the collection of sets*

$$U(T, x_1, ..., x_n) = \{S \in B(H) : \|(T-S)(x_i)\| < 1, 1 \le i \le n\}.$$

Observe that a net $(S_\lambda)_\lambda \subseteq B(H)$ converges to $S \in B(H)$ in the SOT iff

$$\|S_\lambda(x) - y\|_2 \to 0$$

for all $x \in H$.

Remarks 3.1. (i) The SOT is strictly stronger than the WOT.
(ii) The unit ball $Ball(B(H))$ is WOT compact but not SOT compact.
(iii) If H is separable, $Ball(B(H))$ is metrizable in the WOT and in the SOT.
(iii) If C is a convex subset of $B(H)$ then $\overline{C}^{WOT} = \overline{C}^{SOT}$.

Definition 3.3. *Let $\mathcal{A}$ be a selfadjoint algebra which is WOT closed and which contains the identity operator Id_H. We call $\mathcal{A}$ von Neumann Algebra.*

Definition 3.4. *For a set $X \subseteq B(H)$, the commutant of X is the set*
$$X' = \{T \in B(H) : \quad TS = ST \ \forall \ S \ \in \ B(H)\}.$$
Also the double commutant of X is the set $X'' = (X')'$.

Remarks 3.2. (i) X' is WOT closed algebra.
(ii) $X \subseteq X''$.
(iii) $X' = X'''$.

Theorem 3.3. *Let $\mathcal{A} \subseteq B(H)$ be a $*$-subalgebra such that $\overline{span}(\mathcal{A}(H)) = H$. Then*
$$\mathcal{A}'' = \overline{\mathcal{A}}^{WOT} = \overline{\mathcal{A}}^{SOT}.$$

We can easily see that
$$\overline{\mathcal{A}}^{SOT} \subseteq \overline{\mathcal{A}}^{WOT} \subseteq \mathcal{A}''.$$
So it suffices to prove that if $T \in \mathcal{A}''$ then $T \in \overline{\mathcal{A}}^{SOT}$. Equivalently we have to prove that for all $\xi_1, ..., \xi_n \in H$ the set $U(T, \xi_1, ..., \xi_n) \cap \mathcal{A}$ is nonempty, or equivalently there exists $A \in \mathcal{A}$ such that $\|T(\xi_i) - A(\xi_i)\| < 1$ for all $i = 1, ..., n$.

Step 1: Proof for n=1.
Claim: If $P \in B(H)$ is a projection such that $\mathcal{A}P(H) \subseteq P(H)$ then $P \in \mathcal{A}'$.
Indeed if $T \in \mathcal{A}$ then for all $\xi \in H$
$$PTP(\xi) = TP(\xi) \Rightarrow P^{\perp}TP(\xi) = 0$$
Thus, $P^{\perp}TP = 0$. Since $\mathcal{A}^* \subseteq \mathcal{A}$ by the same argument $P^{\perp}T^*P = 0 \Rightarrow PTP^{\perp} = 0$. Thus $TP = PTP = PT$. Therefore $P \in \mathcal{A}'$.

Let $P = \overline{\mathcal{A}\xi_1}$. From the claim we conclude that $P \in \mathcal{A}'$. If $A \in \mathcal{A}$ then
$$AP^{\perp}\xi_1 = P^{\perp}A\xi_1 = 0.$$
Thus
$$< AP^{\perp}\xi_1, \omega >= 0, \ \forall \ \omega \ \in \ H, \ \forall \ A \ \in \ \mathcal{A} \Rightarrow < P^{\perp}\xi_1, A\omega >= 0, \ \forall \ \omega \ \in \ H, \ \forall \ A \ \in \ \mathcal{A}$$
Since $H = \overline{span}\{A(\omega) : A \in \mathcal{A}, \quad \omega \ \in \ H\}$ we have
$$P^{\perp}\xi_1 = 0 \Rightarrow \xi_1 = P\xi_1 \Rightarrow \xi_1 \in P(H).$$
Also, since $T \in \mathcal{A}'' \Rightarrow TP = PT \Rightarrow TP(H) \subseteq P(H)$. Therefore since $\xi_1 \in P(H)$ we have $T(\xi_1) \in P(H) = \overline{\mathcal{A}\xi_1}$. Therefore there exists $A \in \mathcal{A} : \|T(\xi_1) - A(\xi_1)\| < 1$.

Step 2: Proof for n arbitrary.
Define the Hilbert space $H^n = H \oplus H \oplus ... \oplus H$. Then
$$B(H^n) = M_n(B(H)) = \{(T_{ij})_{1 \leq i,j \leq n} : \quad T_{ij} \in B(H)\}.$$
If $S \in B(H)$ we denote $S^n = diag(S, ..., S)$. We denote $\mathcal{A}^n = \{S^n : S \in \mathcal{A}\}$. We can easily see that
$$(\mathcal{A}^n)' = M_n(\mathcal{A}') \Rightarrow (\mathcal{A}^n)'' = M_n(\mathcal{A}')' = (\mathcal{A}'')^n.$$

Now let $\xi = (\xi_1, ..., \xi_n)^t \in H^n$. From step 1, there exists $A^n \in \mathcal{A}^n$ such that

$$\|T^n(\xi) - A^n(\xi)\| < 1 \Rightarrow \|T(\xi_i) - A(\xi_i)\| < 1, \;\; i = 1, ..., n.$$

Definition 3.5. Let

$$C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C}, \;\; f \textit{ is continuous}\},$$

$$C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C}, \;\; f \textit{ is continuous which vanishes at } \infty\}.$$

If $T \in B(H)$ is a selfadjoint operator and $f \in C(\mathbb{R})$ then $\sigma(T) \subseteq \mathbb{R}$ and $f|_{\sigma(T)} \in C(\sigma(T))$ which implies that $f|_{\sigma(T)}(T) \in C^*(T)$. We denote $f(T) = f|_{\sigma(T)}(T)$ and we have

$$\|f(T)\| = \|f|_{\sigma(T)}\|_\infty \leq \sup\{|f(t)| : \; t \; \in \; \mathbb{R}\}.$$

Lemma 3.4. *Let $(T_\lambda)_\lambda \subseteq B(H)$ be a net of selfadjoint operators such that $SOT - \lim_\lambda T_\lambda = T$ and $f \in C_0(\mathbb{R})$. Then*

$$f(T) = SOT - \lim_\lambda f(T_\lambda).$$

Theorem 3.5. ***Kaplansky Density Theorem*** *Let $\mathcal{A} \subseteq B(H)$ be a C^*-algebra such that $\overline{span}(\mathcal{A}(H)) = H$. Then*

(i)

$$\overline{Ball(\mathcal{A})_{sa}}^{SOT} = Ball(\mathcal{A}'')_{sa}$$

(ii)

$$\overline{Ball(A)}^{SOT} = Ball(\mathcal{A}'')$$

Proof. (i) Let $T = T^* \in \mathcal{A}''$, with norm $\|T\| \leq 1$. By Double Commutant Theorem there exists a net $(T_\lambda)_\lambda \subseteq \mathcal{A}$ such that $T = WOT - \lim_\lambda T_\lambda$. Thus,

$$T = WOT - \lim_\lambda \frac{T_\lambda + T_\lambda^*}{2}.$$

Therefore we may consider that there exists a net $(S_\lambda)_\lambda \subseteq \mathcal{A}$ such that $S_\lambda^* = S_\lambda$ and $T = WOT - \lim_\lambda S_\lambda$. We proved that T is an element of the WOT closure of the set of the selfadjoint operators, $\mathcal{A}_{sa}$. But $\mathcal{A}_{sa}$ is a convex set, thus we may consider that there exists a net $(S_\lambda)_\lambda \subseteq \mathcal{A}_{sa}$ such that $T = SOT - \lim_\lambda S_\lambda$. Choose a real function $f \in C_0(\mathbb{R})$ such that $f(t) = t$ for all $-1 \leq t \leq 1$ and $\|f\|_\infty \leq 1$. Since $\sigma(T) \subseteq [-1, 1]$ we have $f(t) = t$ for all $t \in \sigma(T)$ thus $f(T) = T$. By the above Lemma

$$T = f(T) = SOT - \lim_\lambda f(S_\lambda).$$

Since $\|f|_{\sigma(S_\lambda)}\|_\infty \leq 1$ we have $\|f(S_\lambda)\| \leq 1$ for all λ. Since $f = \overline{f}$ we have

$$f(S_\lambda) \in C^*(S_\lambda)_{sa} \subseteq \mathcal{A}_{sa}.$$

The proof of (i) is complete.

(ii) Let $T = T^* \in \mathcal{A}''$ with norm $\|T\| \leq 1$. Define the selfadjoint operator $S = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}$. This operator is an element of the ball of the set of 2×2 matrices with entries in $\mathcal{A}''$: $M_2(\mathcal{A}'')$. Observe that

$$M_2(\mathcal{A}'') = M_2(\mathcal{A}^{WOT}) = \overline{M_2(\mathcal{A})}^{WOT} = M_2(\mathcal{A})''.$$

Thus $S \in Ball(M_2(\mathcal{A})'')_{sa}$. By the first part of the proof there exists a net

$$S_\lambda = \begin{pmatrix} A_\lambda & B_\lambda \\ C_\lambda & D_\lambda \end{pmatrix}$$

in the algebra $M_2(\mathcal{A})$ such that $\|S_\lambda\| \leq 1$ for all λ and $S = SOT - \lim_\lambda S_\lambda$. We conclude that $T = SOT - \lim_\lambda B_\lambda$. Since $\|B_\lambda\| \leq 1$ and $B_\lambda \in \mathcal{A}$ for all λ the proof is complete. $\square$

It remains to prove the Lemma in page 2.

Definition 3.6. *(i)Let $f \in C(\mathbb{R})$. We call f strongly continuous if for every net $S_\lambda = S_\lambda^* \subseteq B(H)$ such that $S = SOT - lim_\lambda S_\lambda \Rightarrow f(S) = SOT - \lim_\lambda f(S_\lambda)$.*
(ii) $V = \{f \in C(\mathbb{R}) : f \ \ is \ \ strongly \ \ continuous\}$
(iii) $V^b = \{f \in V; \|f\|_\infty < \infty\}$.

It suffices to prove that $C_0(\mathbb{R}) \subseteq V$. Let

$$h_s(t) = \frac{1}{1 + s^2t^2}, \ \ s \ \in \ \mathbb{R}.$$

We can prove that $h_s \in V^b$. Since the function $h(t) = t$ belongs to V and $V^bV \subseteq V$ The functions

$$k_s(t) = h(t)h_s(t) = \frac{st}{1 + s^2t^2}, \ \ s \ \in \ \mathbb{R}$$

belong to V. Observe that

$$\{k_s : s \in \mathbb{R}\} \subseteq V \cap C_0(\mathbb{R}).$$

Since the set $\{k_s : s \in \mathbb{R}\}$ separates the points of $\mathbb{R}$ the set $V \cap C_0(\mathbb{R})$ separates the points of $\mathbb{R}$. But we can easily see that $V \cap C_0(\mathbb{R})$ is an algebra. Thus by Stone-Weierstrass Theorem

$$\overline{V \cap C_0(\mathbb{R})}^{\|\cdot\|_\infty} = C_0(\mathbb{R}).$$

If (f_n) is a sequence of functions in V and $\|f_n - f\|_\infty \to 0$ then $f \in V$. Thus $C_0(\mathbb{R}) \subseteq V$. The proof is complete.