Introduction to compact quantum groups

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Motivations

Primary motivations behind introducing compact quantum groups:

- general framework of noncommutative mathematics (seeing C^*-algebras as 'noncommutative topological spaces')
- wish to generalize Pontriagin duality for locally compact abelian groups (and a common language for apparently different objects);
- natural examples of 'deformations' of (Hopf) algebras of functions on classical compact groups
- need to develop good tools to study certain operator algebras.

Commutative C*-algebras

Recall the Gelfand-Najmark duality, including morphisms: a continuous map

$$T : X \to Y$$

induces a unital *-homomorphism

$$\alpha_T : \mathrm{C}(Y) \to \mathrm{C}(X).$$

Further note that

$$\mathrm{C}(X \times X) = \mathrm{C}(X) \otimes \mathrm{C}(X)$$

Compact quantum semigroups

Definition

We call a unital C^*-algebra A the algebra of functions on a compact quantum semigroup if it admits a unital *-homomorphism (called the coproduct)

$$\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$$

which is coassociative:

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta \qquad \text{(coassociativity)}$$

Exercise

Show that if A is a commutative C^*-algebra as above, it must arise as $\mathsf{C}(S)$ for a compact semigroup S.

Classical cancellation rules

Theorem

A compact semigroup G for which the cancellation rules hold, i.e. for any $g_1, g_2, h \in G$

$$g_1 h = g_2 h \Longrightarrow g_1 = g_2,$$

$$h g_1 = h g_2 \Longrightarrow g_1 = g_2,$$

is in fact a compact group.

Quantum cancellation rules

Definition (Woronowicz, 1989)

An algebra of continuous functions on a compact quantum group is a unital C^*-algebra A with a unital $*$-algebra homomorphism $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ such that

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta \qquad \text{(coassociativity)}$$

and the quantum cancellation rules hold:

$$\overline{\mathrm{Lin}}\{\Delta(a)(b \otimes \mathbf{1}); a, b \in \mathsf{A}\} = \overline{\mathrm{Lin}}\{\Delta(a)(\mathbf{1} \otimes b); a, b \in \mathsf{A}\} = \mathsf{A} \otimes \mathsf{A}$$

The tensor products are all the time in the C^*-algebraic category. We will write $\mathsf{A} = C(\mathbb{G})$ and call $\mathbb{G}$ a compact quantum group.
Sometimes (A, Δ) is called a compact quantum group.

Quantum cancellation rules

Definition (Woronowicz, 1989)

An algebra of continuous functions on a compact quantum group is a unital C^*-algebra A with a unital $*$-algebra homomorphism $\Delta : A \to A \otimes A$ such that

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta \qquad \text{(coassociativity)}$$

and the quantum cancellation rules hold:

$$\overline{\mathrm{Lin}}\{\Delta(a)(b \otimes \mathbf{1}); a, b \in A\} = \overline{\mathrm{Lin}}\{\Delta(a)(\mathbf{1} \otimes b); a, b \in A\} = A \otimes A$$

Exercise

Check that if A is commutative, so that we have a compact semigroup S such that $A = C(S)$, the density conditions above are equivalent to cancellation rules.

Compact matrix quantum groups

Definition (Woronowicz)

An algebra of continuous functions on a compact matrix quantum group is a unital C*-algebra A together with a unitary matrix $U = (u_{ij})_{i,j=1}^n \in M_n(\mathrm{A})$ such that

- the *-algebra $\mathcal{A}$ spanned by the entries of U is dense in A;
- the formula
$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}, \quad i,j = 1, \ldots, n,$$
extends to a well-defined *-homomorphism $\Delta : \mathrm{A} \to \mathrm{A} \otimes \mathrm{A}$;
- there is a linear antimultiplicative map $S : \mathcal{A} \to \mathcal{A}$ such that
$$S \circ * \circ S \circ * = \mathrm{id}_{\mathcal{A}},$$
$$\sum_{k=1}^n S(u_{ik}) u_{kj} = \delta_{ij} 1, \ \sum_{k=1}^n u_{ik} S(u_{kj}) = \delta_{ij} 1, \quad i,j = 1, \ldots, n.$$

Compact matrix quantum groups continued

Exercise

Check that every algebra A as above yields a compact quantum group $\mathbb{G}$ (i.e. $\mathsf{A} = \mathsf{C}(\mathbb{G})$).

The unitary $U = (u_{ij})_{i,j=1}^n \in M_n(\mathsf{A})$ is then called the fundamental representation of $\mathbb{G}$. We will come back to this definition later.

Examples – classical and dual to classical

- every classical compact group G is also a quantum group; that is, $\mathrm{C}(G)$ and the map $\Delta : \mathrm{C}(G) \to \mathrm{C}(G) \otimes \mathrm{C}(G) \cong \mathrm{C}(G \times G)$

$$\Delta(f)(g,h) := f(g \cdot h), \quad f \in \mathrm{C}(G), g, h \in G,$$

satisfy Woronowicz's axioms. Moreover if $\mathbb{G}$ is a compact quantum group and $\mathrm{C}(\mathbb{G})$ is commutative, it must arise in this way.

- for Γ– discrete group both the algebras $\mathrm{C}^*(\Gamma)$ and $\mathrm{C}^*_r(\Gamma)$, with the coproducts informally given by

$$\Delta(\gamma) = \gamma \otimes \gamma, \quad \gamma \in \Gamma$$

yield compact quantum groups. They are both viewed as certain algebras of continuous functions on $\hat{\Gamma}$: the 'quantum dual' of Γ; $\mathrm{C}^*(\Gamma)$ is naturally an 'abstract' C^*-algebra, $\mathrm{C}^*_r(\Gamma) \subset B(\ell^2(\Gamma))$ a concrete one.

Deformations of classical compact Lie groups

Recall that $\mathrm{C}(SU(2))$ is a commutative unital C^*-algebra generated by the functions $\alpha, \gamma : SU(2) \to \mathbb{C}$ such that

$$\alpha^*\alpha + \gamma^*\gamma = 1.$$

Group multiplication on $SU(2)$ induces on $\mathrm{C}(SU(2))$ the coproduct

$$\Delta(\alpha) = \alpha \otimes \alpha - \gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Let $q \in [-1,1) \setminus \{0\}$. Define $\mathrm{C}(SU_q(2))$ – universal unital C^*-algebra generated by operators α, γ such that:

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma^*\gamma = 1,$$

$$\gamma^*\gamma = \gamma\gamma^*, \quad q\gamma\alpha = \alpha\gamma, \quad q\gamma^*\alpha = \alpha\gamma^*.$$

The coproduct making $SU_q(2)$ a compact quantum group is given by the formulas

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Free (liberated) compact quantum groups

Let $n \in \mathbb{N}$ and let A be the universal C*-algebra generated by the elements $\{p_{ij} : i, j = 1, \ldots, n\}$ such that

- $p_{ij} = p_{ij}^2 = p_{ij}^*$, $i, j = 1, \ldots, n$;
- $\sum_{i=1}^n p_{ij} = \sum_{i=1}^n p_{ji} = \delta_{ij} 1$, $j = 1, \ldots, n$,

Then the formula

$$\Delta(p_{ij}) = \sum_{k=1}^{n} p_{ik} \otimes p_{kj}, \quad i, j = 1, \ldots, n,$$

defines a coproduct, making A the algebra of continuous functions on a compact quantum group, usually denoted S_n^+ and called a *free permutation group*.

Exercises related to examples

Exercise

Show rigorously that all the examples above fit into the Woronowicz framework.

Exercise

Find the connection between S_n^+ and the usual permutation group S_n.

Exercise

How one could define the 'free orthogonal group' O_N^+?

Convolution of probability measures on a compact group

Let G – compact group. Given two finite measures μ, ν on G their convolution $\mu \star \nu$ is defined by

$$\int_G f(g) d_{\mu \star \nu}(g) = \int_G \int_G f(g_1 g_2) d_\mu(g_1) d_\nu(g_2), \quad f \in C(G).$$

Here finite (signed) measures – continuous functionals on $C(G)$. The convolution of probability measures remains a probability measure.

The **Haar measure** on G is the unique bi-invariant measure $\mu_h \in \mathrm{Prob}(G)$: for any $g \in G$ and a Borel set $X \subset G$

$$\mu_h(gX) = \mu_h(Xg) = \mu_h(X).$$

In other words, it is a unique measure such that

$$\nu \star \mu_h = \mu_h = \mu_h \star \nu, \quad \nu \in \mathrm{Prob}(G).$$

Convolution of probability measures on a compact quantum group

Definition

Let $\mathbb{G}$ be a compact quantum group. Given two functionals $\varphi, \psi \in C(\mathbb{G})^*$ their convolution is defined by

$$\varphi \star \psi = (\varphi \otimes \psi) \circ \Delta.$$

Convolution of states (normalised positive functionals) is a state. We view states on $C(\mathbb{G})$ as probability measures on $\mathbb{G}$ (and may write simply $\mathrm{Prob}(\mathbb{G})$).

Haar state

Definition

A state $h \in \mathrm{Prob}(\mathbb{G})$ is called a Haar state if for all $a \in \mathrm{C}(\mathbb{G})$

$$(h \otimes \mathrm{id})(\Delta(a)) = (\mathrm{id} \otimes h)(\Delta(a)) = h(a)\mathbf{1};$$

equivalently for each $\mu \in \mathrm{C}(\mathbb{G})^*$

$$h \star \mu = \mu \star h = \mu(1)h;$$

equivalently for each $\omega \in \mathrm{Prob}(\mathbb{G})$

$$h \star \omega = \omega \star h = h.$$

Haar state continued

Theorem

Every compact quantum group has a unique Haar state.

Haar state continued

Theorem

Every compact quantum group has a unique Haar state.

This uses cancellation laws! Another idea of the proof: take a faithful state $\omega \in \mathrm{Prob}(\mathbb{G})$ and show that

$$h = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \omega^{\star k}.$$

For $\mathrm{C}(G)$ the Haar state is given by the integration with respect to the Haar measure. On $\mathrm{C}^*_r(\Gamma)$ it is given by $h(\sum_{\gamma\in\Gamma} c_\gamma \lambda_\gamma) = c_e$.

Representations

A (finite-dimensional, unitary, continuous) representation of a compact group G is a continuous map $U : G \to \mathrm{U}(n)$ such that

$$U(gh) = U(g)U(h), \quad g, h \in G.$$

Looking at matrix entries we can view it as a single element $U \in M_n(\mathrm{C}(G)) \cong B(\mathbb{C}^n) \otimes \mathrm{C}(G)$.

Definition

A unitary, continuous representation of a compact quantum group $\mathbb{G}$ on a finite-dimensional Hilbert space H is a unitary $U \in B(\mathsf{H}) \otimes \mathrm{C}(\mathbb{G})$ such that

$$(\mathrm{id} \otimes \Delta)(U) = U_{12}U_{13}.$$

Equivalently, choosing an orthonormal basis in H we can write $U = [u_{ij}]_{i,j=1}^n \in M_n(C(\mathbb{G}))$ and obtain

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}, \quad i, j = 1, \ldots, n.$$

Representations continued

Definition

A unitary, continuous representation of a compact quantum group $\mathbb{G}$ on a finite-dimensional Hilbert space H is a unitary $U \in B(\mathrm{H}) \otimes \mathrm{C}(\mathbb{G})$ such that

$$(\mathrm{id} \otimes \Delta)(U) = U_{12} U_{13}.$$

Choosing an orthonormal basis in H we can write $U = [u_{ij}]_{i,j=1}^{n} \in M_n(C(\mathbb{G}))$. We will write simply $U \in \mathrm{Rep}_f(\mathbb{G})$.

Coefficients of U – linear combinations of u_{ij}.

Non-degenerate representation – invertible $U \in M_n(C(\mathbb{G}))$ + the formulas above.

Fourier transforms relative to a representation

Let $U \in \mathrm{Rep}_f(\mathbb{G})$. Define for $a \in \mathrm{C}(\mathbb{G})$ the Fourier transform of a with respect to U

$$F_a = (\mathrm{id} \otimes ha)(U^*),$$
$$\mathrm{A}(U) = \overline{\{F_a : a \in \mathrm{C}(\mathbb{G})\}}.$$

Theorem

The set $\mathrm{A}(U)$ is a non-degenerate (unital) C^*-subalgebra of $B(\mathrm{H}_U)$ and $U \in \mathrm{A}(U) \otimes \mathrm{C}(\mathbb{G})$. Moreover

$$\mathrm{A}(U) = \{(\mathrm{id} \otimes ah)(U) : a \in \mathrm{C}(\mathbb{G})\}.$$

The key formula is

$$F_a F_b^* = F_{a \star b^* h}$$

Fourier transforms relative to a representation

Recall

$$F_a = (\mathrm{id} \otimes ha)(U^*),$$
$$\mathrm{A}(U) = \overline{\{F_a : a \in \mathrm{C}(\mathbb{G})\}}.$$

Exercise

Find the interpretation of F_a if $\mathbb{G}$ is classical and U corresponds to a representation of G on H.

Invariant subspaces

Proposition

Let $U \in \mathrm{Rep}_f(\mathbb{G})$, $\mathsf{K} \subset \mathsf{H}_U$ a subspace, $P := P_{\mathsf{K}}$. The following are equivalent:

- $(P \otimes 1)U(P \otimes 1) = U(P \otimes 1)$
- $(P \otimes 1)U = U(P \otimes 1)$;
- K is invariant for $\mathrm{A}(U)$ (so that P commutes with elements of $A(U)$).

We then call K an invariant subspace. Furthermore $A(U)'$ is commutative.

We say that $U \in \mathrm{Rep}_f(\mathbb{G})$ is irreducible if H_U has no non-trivial invariant subspaces.

Theorem (Exercise)

Every $U \in \mathrm{Rep}_f(\mathbb{G})$ decomposes into a direct sum of irreducible representations.

Morphisms

Let $U, V \in \mathrm{Rep}_f(\mathbb{G})$. A morphism from U to V is an operator $T \in B(\mathsf{H}_U; \mathsf{H}_V)$ such that

$$(T \otimes 1)U = V(T \otimes 1).$$

We write $U \approx V$ if there is a morphism from U to V which is invertible.

Fact

Any non-degenerate representation V is equivalent to a unitary one (in the sense extending this above).

Exercise

Show the above statement, using the operator $y = (\mathrm{id} \otimes h)(V^* V)$.

Morphisms – Schur Lemma

Theorem (Schur Lemma)

Let $U, V \in \mathsf{Rep}_f(\mathbb{G})$ be irreducible. Then if $U \approx V$ then $\mathrm{Mor}(U, V) = \{\lambda T : \lambda \in \mathbb{C}\}$ for an invertible morphism T; and if U and V are not equivalent then $\mathrm{Mor}(U, V) = \{0\}$.

Operations on finite-dimensional representations

Operations on representations ($U, V \in \mathrm{Rep}_f(\mathbb{G})$):

- direct sum: $U \oplus V \in M_{n+m}(\mathrm{C}(\mathbb{G}))$;
- tensor product: $U \otimes V \in M_{nm}(\mathrm{C}(\mathbb{G}))$:

$$(U \otimes V)_{(i,k),(j,l)} = u_{ij} v_{kl}, \quad i,j = 1, \ldots, n_U, k, l = 1, \ldots, n_V$$

- ... there will be one more!

$\mathrm{Irr}(\mathbb{G})$ – the set of all (equivalence classes) of irreducible representations.

Definition

We write $\mathrm{Pol}(\mathbb{G})$ for the span of coefficients of all finite-dimensional unitary representations of $\mathbb{G}$. It is now easy to see it is a unital subalgebra of $\mathrm{C}(\mathbb{G})$.

Infinite-dimensional representations

Definition

If H is any Hilbert space then a representation of $\mathbb{G}$ on H is any unitary $U \in \mathrm{M}(K(\mathrm{H}) \otimes \mathrm{C}(\mathbb{G}))$ such that

$$(\mathrm{id} \otimes \Delta)(U) = U_{12} U_{13}.$$

Here M(A) denotes the multiplier algebra of A.

Theorem

Any $U \in \mathrm{Rep}(\mathbb{G})$ decomposes as a direct sum of (irreducible) finite-dimensional representations.

The right-regular representation

Let $\mathsf{L}^2(\mathbb{G})$ denote the GNS space of $\mathsf{C}(\mathbb{G})$ with respect to the Haar state h, with the GNS cyclic vector Ω_h and the representation $\pi_h : \mathsf{C}(\mathbb{G}) \to B(\mathsf{L}^2(\mathbb{G}))$. Assume further that $\mathsf{C}(\mathbb{G})$ acts on a Hilbert space H.

Theorem

There exists a unique unitary $\mathcal{U} \in B(\mathsf{L}^2(\mathbb{G}) \otimes \mathsf{H})$ such that for all $a \in \mathsf{C}(\mathbb{G})$, $\xi \in \mathsf{H}$ we have

$$\mathcal{U}(\pi_h(a)\Omega_h \otimes \xi) = (\pi_h \otimes \mathrm{id})(\Delta(a))(\Omega_h \otimes \xi).$$

Further $\mathcal{U} \in \mathsf{M}(K(\mathsf{L}^2(\mathbb{G})) \otimes \mathsf{C}(\mathbb{G}))$ is a representation of $\mathbb{G}$.

Moreover for $a \in \mathsf{C}(\mathbb{G}), \tau \in \mathsf{C}(\mathbb{G})^*$ we have

$$(\mathrm{id} \otimes \tau)(\mathcal{U})(\pi_h(a)\Omega_h) = \pi_h(\tau \star a)\Omega.$$

Usefulness of the right-regular representation

Exercise

Check that the right-regular representation as above for classical G coincides with the usual right-regular representation.

The next two results use the right-regular representation.

Theorem

The algebra $\mathrm{Pol}(\mathbb{G})$ is dense in $\mathrm{C}(\mathbb{G})$.

Theorem

Every irreducible representation of $\mathbb{G}$ is (equivalent to) a subrepresentation of the right-regular representation.

Finite-dimensional representations revisited

Theorem

The set

$$\{u_{ij}^{\alpha} : \alpha \in \mathrm{Irr}(\mathbb{G}), i, j = 1, \ldots, n_{\alpha}\}$$

is a linear basis of $\mathrm{Pol}(\mathbb{G})$.

Finite-dimensional representations revisited again

Theorem

For each $\alpha \in \mathrm{Irr}(\mathbb{G})$ there exists a (unique) $\beta \in \mathrm{Irr}(\mathbb{G})$ such that the

$$\mathrm{Lin}\{(u^{\alpha}_{ij})^* : i,j = 1,\ldots,n_\alpha\} = \mathrm{Lin}\{u^{\beta}_{ij} : i,j = 1,\ldots,n_\beta\}$$

(we have $n_\beta = n_\alpha$).

The last result implies that $\mathrm{Pol}(\mathbb{G})$ is a unital *-algebra.

Exercise

The above theorem can be now given at least two different proofs: one using the fact that non-degenerate representations are equivalent to unitary ones and using the right regular representation, and another using the density of $\mathrm{Pol}(\mathbb{G})$. Try to find them!

Theorem

The Haar state is faithful on $\mathrm{Pol}(\mathbb{G})$.

Operations on finite-dimensional representations revisited

Operations on representations ($U, V \in \mathrm{Rep}_f(\mathbb{G})$):

- direct sum: $U \oplus V \in M_{n+m}(\mathrm{C}(\mathbb{G}))$;
- tensor product: $U \otimes V \in M_{nm}(\mathrm{C}(\mathbb{G}))$:

$$(U \otimes V)_{(i,k),(j,l)} = u_{ij} v_{kl}$$

- adjoint operation:

$$\overline{U}_{ij} \text{ equals up to equivalence } U_{ij}^*.$$

Corollary

The algebra $\mathrm{Pol}(\mathbb{G})$ is a dense unital *-subalgebra of $\mathrm{C}(\mathbb{G})$.

$U \in \mathrm{Rep}_f(\mathbb{G})$ is called fundamental if its coefficients generate $\mathrm{C}(\mathbb{G})$ as a C^*-algebra.

Hopf*- algebra

Theorem

Recall that the set

$$\{u_{ij}^{\alpha} : \alpha \in \mathrm{Irr}(\mathbb{G}), i, j = 1, \ldots, n_{\alpha}\}$$

is a linear basis of $\mathrm{Pol}(\mathbb{G})$. With

$$\epsilon(u_{ij}^{\alpha}) = \delta_{ij}, \quad S(u_{ij}^{\alpha}) = (u_{ji}^{\alpha})^*$$

$\mathrm{Pol}(\mathbb{G})$ becomes a Hopf*-algebra.

Neither ϵ nor S need to extend to $C(\mathbb{G})$!

Orthogonality

Theorem

For each $\alpha \in \mathrm{Irr}(\mathbb{G})$ there exists a unique positive matrix $Q_\alpha \in GL(n_\alpha)$ such that $\mathrm{Tr}(Q_\alpha) = \mathrm{Tr}(Q_\alpha^{-1}) := d_\alpha \geq n_\alpha$ and we have for all $\alpha, \beta \in \mathrm{Irr}(\mathbb{G})$

- $h\left(u_{ij}^\alpha (u_{kl}^\beta)^*\right) = \delta_{\alpha\beta}\delta_{ik}\frac{(Q_\alpha)_{l,j}}{d_\alpha}$;
- $h\left((u_{ij}^\alpha)^* u_{kl}^\beta\right) = \delta_{\alpha\beta}\delta_{jl}\frac{(Q_\alpha^{-1})_{k,i}}{d_\alpha}$.

The matrices Q have various incarnations:

- as so-called Woronowicz characters on $\mathrm{Pol}(\mathbb{G})$
- generators of the 'scaling automorphism group' τ;
- witnesses of non-traciality of h;
- witnesses of unboundedness of S

Woronowicz characters

Theorem (Woronowicz characters)

The algebra $\mathrm{Pol}(\mathbb{G})$ admits a (unique!) family of unital multiplicative functionals $(f_z)_{z\in\mathbb{C}}$ such that

1. $\forall_{a\in \mathrm{Pol}(\mathbb{G})}\ z \mapsto f_z(a)$ is holomorphic;
2. $\forall_{z_1,z_2\in\mathbb{C}}\ f_{z_1+z_2} = f_{z_1} \star f_{z_2}$;
3. $\forall_{a\in \mathrm{Pol}(\mathbb{G}), z\in\mathbb{C}}\ f_z(S(a)) = f_{-z}(a),\ \ f_{\bar{z}}(a) = \overline{f_{-z}(a)},\ \ f_0(a) = \epsilon(a),\ \ S^2(a) = f_{-1} \star a \star f_1$;
4. the formula $a \mapsto \sigma_t(a) := f_{it} \star a \star f_{it}$, $t \in \mathbb{R}$, defines a one-parameter semigroup of automorphisms of $\mathrm{Pol}(\mathbb{G})$, which is the KMS group of the Haar state:
$$h(ab) = h(b\sigma_{-i}(a)), \quad a, b \in \mathrm{Pol}(\mathbb{G}).$$

Exercise

Find the relation between f_z and the matrices Q_α from the previous slide.

Kac property

Definition

A compact quantum group $\mathbb{G}$ is of Kac type if all $Q_\alpha = I$; equivalently, $S^2 = \mathrm{id}_{\mathrm{Pol}(\mathbb{G})}$; equivalently h is a trace; equivalently the 'quantum dimensions' d_α are equal to n_α; equivalently the Woronowicz characters trivialise.

From Pol($\mathbb{G}$) to $C(\mathbb{G})$

Definition

A Hopf *-algebra $\mathcal{A}$ is called a *CQG algebra* or a Hopf algebra *of compact type* if it is spanned by coefficients of its finite dimensional unitary corepresentations.

Theorem

Every CQG algebra arises as Pol($\mathbb{G}$) for a compact quantum group $\mathbb{G}$.

Why? And how?

From $\mathrm{Pol}(\mathbb{G})$ to $C(\mathbb{G})$ continued

We need 'good' C*-norms on $\mathrm{Pol}(\mathbb{G})$.

- universal norm:

$$\|a\|_u := \sup\{\|\pi(a)\| : \pi : \mathrm{Pol}(\mathbb{G}) \to B(\mathrm{H}), \pi \text{ unital *-homomorphism}\}$$

  Completion of $\mathrm{Pol}(\mathbb{G})$ in this norm – $\mathrm{C}_u(\mathbb{G})$ admits good Δ_u, h_u, etc..

- reduced norm:

$$\|a\|_r := \|\pi_h(a)\|,$$

  where π_h is the GNS representation of the Haar state on $\mathrm{Pol}(\mathbb{G})$.

  Completion of $\mathrm{Pol}(\mathbb{G})$ in this norm – $\mathrm{C}_r(\mathbb{G})$ admits good Δ_r, h_r, etc..

Of course $\|\cdot\|_u \geq \|\cdot\|_r$.

Definition

A compact quantum group $\mathbb{G}$ is coamenable if $\|\cdot\|_u = \|\cdot\|_r$; equivalently, h_u is faithful on $\mathrm{C}_u(\mathbb{G})$; equivalently, ϵ extends to a character on $\mathrm{C}_r(\mathbb{G})$.

And a word on a von Neumann algebraic approach

Given $\mathrm{C}(\mathbb{G})$ and its Haar state h we can construct the GNS Hilbert space $L^2(\mathbb{G})$, represent say $\mathrm{Pol}(\mathbb{G})$ on $B(L^2(\mathbb{G}))$ and consider the von Neumann algebra

$$L^\infty(\mathbb{G}) := \mathrm{Pol}(\mathbb{G})'' \subset B(L^2(\mathbb{G}))$$

Definition

We call a von Neumann algebra M with a coassociative normal unital *-homomorphism

$$\Delta : \mathrm{M} \to \mathrm{M}\overline{\otimes}\mathrm{M}$$

the algebra of essentially bounded measurable functions on a compact quantum group $\mathbb{G}$ if it admits a Haar state: a faithful normal state $h \in \mathrm{M}_*$ such that

$$(h \otimes \mathrm{id}_\mathrm{M}) \circ \Delta = (\mathrm{id}_\mathrm{M} \otimes h) \circ \Delta = h(\cdot)1.$$

One can show that each M as above arises as $L^\infty(\mathbb{G})$ for a compact quantum group $\mathbb{G}$!

So this is just the beginning...

Finally I would like to list some broad themes of study of quantum groups which are important in recent years

- extensions to locally compact quantum groups
- categorical approaches (related to intertwiners)
- study of abstract harmonic analysis: versions of Fourier multipliers
- probabilistic aspects: random walks, topological/probabilistic boundaries of the dual discrete quantum groups
- notion of quantum subgroups, quantum ergodic actions; resulting algebraic and operator algebraic ‘quantum symmetric spaces’
- study of resulting operator algebras!

Some references

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