# Operator Algebras, Operator Systems, Operator Spaces: an introduction

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# Probability, Classical & Quantum

Classical Probability: Random variables are functions $f \in \ell^\infty(\Gamma)$ on sample space $\Gamma$. Expected values $\mathbb{E}(f)$ are weighted averages $\mathbb{E}(f) = \sum f(\gamma)\mu(\gamma) = \int_\Gamma f(\gamma)d\mu(\gamma)$ of the values of the variable $f$ in the *state* described by the prob. measure $\mu$.

Quantum Probability: An event (ενδεχόμενο) is an orthogonal projection $p$ acting on a Hilbert space $H$. *[But the set of projections is not a Boolean algebra!]* Let $\omega(p) \in [0,1]$ be the probability of the event $p$ when the system is in the state described by $\omega$.

(Real) linear combinations of projections correspond to observables; they generate subspace $\mathscr{X} \subseteq \mathscr{B}(H)$; so if $x = \sum_k \lambda_k p_k \in \mathscr{X}$ we may define $\omega(x) := \sum_k \lambda_k \omega(p_k) \in \mathbb{C}$, so $\omega : \mathscr{X} \to \mathbb{C}$ linear.

Comparison: Classical Probability: Random variables form subspace $X$ of $\ell^\infty(\Gamma)$ (: Function Space), $\mu$ is a linear map on $X$. Quantum Probability: Random variables = Observables form subspace $\mathscr{X}$ of $\mathscr{B}(H)$ (: Operator Space), $\omega$ is a linear map on $\mathscr{X}$.

# Hilbert Space and Operators

Hilbert Space: $(H, \langle\cdot,\cdot\rangle)$ complete in $\|\xi\| := \langle\xi,\xi\rangle^{1/2}$.
Scalar product $\langle\xi,\eta\rangle$ linear in $\eta$, antilinear in $\xi$.

$\mathscr{B}(H,K) := \{a : H \to K$ linear, bounded$\}$ i.e.
$\|a\|_{op} := \sup\{\|a\xi\|_K : \|\xi\|_H \leq 1\} < \infty$.

Adjoint: $a : H \to K \rightsquigarrow a^* : K \to H$ given by $\langle\xi, a^*\eta\rangle_H = \langle a\xi, \eta\rangle_K$
for all $\xi \in H, \eta \in K$.
In particular, $a \in \mathscr{B}(H) \rightsquigarrow a^* \in \mathscr{B}(H)$.

# Matrices of Operators

Extra structure of operator spaces over function spaces: matrices of operators are again operators on a Hilbert space:

For $H$ Hilbert, $n \in \mathbb{N}$, let $H^n := H \oplus \cdots \oplus H$ with $\left\langle \vec{\xi}, \vec{\eta} \right\rangle := \sum_{k=1}^{n} \langle \xi_k, \eta_k \rangle$.

Every $a \in \mathscr{B}(H^n)$ gives $[a_{ij}] \in M_n(\mathscr{B}(H))$ (and vice versa).

$$M_n(\mathscr{B}(H)) \simeq \mathscr{B}(H^n).$$

So if $\mathscr{X} \subseteq \mathscr{B}(H)$, obtain norms $\|\cdot\|_n$ on $M_n(\mathscr{X})$ from $\mathscr{B}(H^n)$.

# Operator Spaces

$$\mathscr{X} \subseteq \mathscr{B}(H): \quad M_n(\mathscr{X}) \subseteq M_n(\mathscr{B}(H)) \simeq \mathscr{B}(H^n).$$

## Definition

An operator space $\mathscr{X}$ is a (closed) subspace of some $\mathscr{B}(H)$ (equipped with family of norms $\|\cdot\|_n$ on $M_n(\mathscr{X})$, $n \in \mathbb{N}$).
A map $\Phi : \mathscr{X} \to \mathscr{Y}$ between operator spaces induces for each $n \in \mathbb{N}$ a map $\Phi_n : M_n(\mathscr{X}) \to M_n(\mathscr{Y}) : [x_{ij}] \to [\Phi(x_{ij})]$.
The map $\Phi$ is said to be completely bounded if $\sup_n \|\Phi_n\|_{op} := \|\Phi\|_{cb} < \infty$.

Note: $\|\Phi_n\|_{op} = \sup\{\|[\Phi(x_{ij})]\|_n : \|[x_{ij}]\|_n \leq 1\}$.

## Examples

(i) If $\mathscr{X} \subseteq \mathscr{B}(\ell^2)$ consists of diagonal operators, any bounded linear $\Phi : \mathscr{X} \to \mathscr{X}$ is CB with $\|\Phi\|_{cb} = \|\Phi\|$.
(ii) If $\mathscr{X} \subseteq \mathscr{B}(\ell^2)$ is the union over all $n$ of all operators mapping $\mathrm{span}\{e_1, \ldots, e_n\}$ to itself, and $T([x_{ij}]) := [x_{ji}]$, then $\|T\| = 1$ but $\|T\|_{cb} = \infty$.

# The transpose map

Exercise: Prove that the transpose map $\Phi : M_n \to M_n$ has $\|\Phi\| = 1$ but $\|\Phi\|_{cb} = n$.

Hints: For $\|\Phi\|_{cb} \geq n$.

If $a = [e_{ji}] = \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{bmatrix} \in M_n(M_n)$, then $\|a\| = 1$

(a unitary) and $\frac{\Phi_n(a)}{n} = \frac{1}{n} \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{bmatrix} = \frac{1}{n} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

is the projection onto $\mathrm{span}(v)$ where $v := e_1 + e_{n+2} + e_{2n+3} + \cdots + e_{n^2} \in \ell^2(n^2)$, so $\left\| \frac{\Phi_n(a)}{n} \right\| = 1$, hence $\|\Phi\|_{cb} \geq \|\Phi_n\| \geq n$.

# The transpose map

Hints: For $\|\Phi\|_{cb} \leq n$. Let $u : \mathbb{C}^n \to \mathbb{C}^n : e_j \to e_{j+1}$ (addition mod $n$) and $E : M_n \to M_n : [a_{ij}]] \to [a_{ij}\delta_{ij}]$ the projection onto the diagonal.

Observe that if $a \in M_n$, then $a = \sum_{k=1}^{n} E(au^k)u^{-k}$ and so

$\Phi(a) = \sum_{k=1}^{n} u^k E(au^k)$.

Thus $\Phi = \sum_{k=1}^{n} U_l^k \circ E \circ U_r^k$ where $U_l(a) = ua$, $U_r(a) = au$.

But $U_r^k$, $E$ and $U_l^k$ have $\|\cdot\|_{cb} = 1$; hence this shows that $\|\Phi\|_{cb} \leq n$.

# Positivity

An operator $a \in \mathscr{B}(H)$ is positive if $\langle \xi, a\xi \rangle \geq 0$ for all $\xi \in \mathscr{H}$, equivalently if $a = a^*$ and $\sigma(a) \subseteq \mathbb{R}_+$.
Equivalently, iff $\exists b \in \mathscr{B}(H)$ s.t $a = b^* b$. (Note: when $a \geq 0$, can choose $b$ to be a limit of polys in $a$.)

## Examples

(i) A diagonal operator $\mathrm{diag}(a_n)$ is positive iff $a_n \in \mathbb{R}_+$ for all $n$.

$$(ii) \qquad A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ is positive, but}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ is not.}$$

# Operator Systems

## Definition

An operator system $\mathscr{S}$ is a linear subspace of some $\mathscr{B}(H)$ which contains the identity operator $\mathbf{1}$ and is selfadjoint, i.e. $x \in \mathscr{S} \Rightarrow x^* \in \mathscr{S}$.

Every $x \in \mathscr{S}$ is $x = x_1 + ix_2$ with $x_j \in \mathscr{S}_h$ (:=selfadjoint elements of $\mathscr{S}$).
If $x \in \mathscr{S}$ has $x = x^*$, then $\|x\| \mathbf{1} - x \in \mathscr{S}_+$ (:positive elements of $\mathscr{S}$) and $x = \|x\| \mathbf{1} - (\|x\| \mathbf{1} - x) \in \mathscr{S}_+ - \mathscr{S}_+$.
So an operator system is linearly generated by its positive cone $\mathscr{S}_+$.

## Examples

Let $\mathscr{S} \subseteq \mathscr{B}(\ell^2)$ be $\mathscr{S} = \{c_0 I + c_1 S + c_2 S^*\}$ where $S : e_n \to e_{n+1}$ is the unilateral shift on $\ell^2(\mathbb{Z}_+)$.
So $\dim \mathscr{S} = 3$. Can you represent $\mathscr{S}$ on finite-dimensional $H$?

# C* -algebras

**Definition**

An operator algebra $\mathscr{A}$ is a linear subspace of some $\mathscr{B}(H)$ which is closed under operator multiplication (synthesis).
$\mathscr{A}$ is called a (concrete) C*-algebra if it is also norm-closed and selfadjoint, i.e. $x \in \mathscr{A} \Rightarrow x^* \in \mathscr{A}$.

Objects:

unital C*-algebras $\subseteq$ Operator Systems $\subseteq$ Operator Spaces.

Morphisms:

*-morphisms $\subseteq$ Completely positive maps $\subseteq$ Completely bounded maps.

# Example: Group algebras

Let $G$ be a (countable) group (think of $\mathbb{Z}$ or $\mathbb{F}_2$). The Hilbert space $\ell^2(G)$ has o.n. basis $\{\delta_s : s \in G\}$. The group $G$ acts on $\ell^2(G)$ via

$$t \to \lambda_t \in \mathscr{B}(\ell^2(G)) \quad \text{where } \lambda_t(\delta_s) = \delta_{ts}, \; s \in G$$

(or $\lambda_t(f)(s) = f(t^{-1}s), f \in \ell^2(G)$).

• The reduced C*-algebra $C_r^*(G) := \overline{\mathrm{span}\{\lambda_s : s \in G\}}^{op}$ - closed in the norm of $\mathscr{B}(\ell^2(G))$.

Each $\lambda_s$ commutes with the right repr. $\rho$ where $\rho_t(\delta_s) = \delta_{st}$.

• The von Neumann algebra of $G$
$\mathscr{L}(G) := \{X \in \mathscr{B}(\ell^2(G)) : X\rho_t = \rho_t X \; \forall t \in G\}$.

What about a semigroup $S \subseteq G$??

# Example: $C_r^*(\mathbb{Z})$

For $G = \mathbb{Z}$: $\lambda_n(\delta_m) = \delta_{m+n}$ $(n, m \in \mathbb{Z})$

$$\lambda_1 = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \dots & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 1 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 1 & \mathbf{0} & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 1 & 0 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\sum_k a_k \lambda_k = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \dots & a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \dots \\ \dots & a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} & \dots \\ \dots & a_2 & a_1 & \mathbf{a_0} & a_{-1} & a_{-2} & \dots \\ \dots & a_3 & a_2 & a_1 & a_0 & a_{-1} & \dots \\ \dots & a_4 & a_3 & a_2 & a_1 & a_0 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \sim \quad f(z) = \sum_k a_k z^k$$

# $C_r^*(\mathbb{Z}) \simeq C(\mathbb{T})$

If $F : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ is the Fourier transform,

$$\begin{array}{ccc} \ell^2(\mathbb{Z}) & \xrightarrow{\sum_k a_k \lambda_k} & \ell^2(\mathbb{Z}) \\ \Big\downarrow F & & \Big\downarrow F \\ L^2(\mathbb{T}) & \xrightarrow[M_f]{} & L^2(\mathbb{T}) \end{array}$$

i.e. $\sum_k a_k \lambda_k \simeq M_f$, where $f(z) = \sum_k a_k z^k$ acts on $L^2(\mathbb{T})$ via $M_f g = fg,\ g \in L^2(\mathbb{T})$.
Such $f$ (: trig. polys) are norm-dense in $C(\mathbb{T})$. Hence:
$C_r^*(\mathbb{Z}) \simeq C(\mathbb{T})$.

# C* -algebras: Characterisation

**Definition**

A B*-algebra or abstract C*-algebra $\mathscr{A}$ is a complex algebra equiped with a complete submultiplicative norm:

$$\|ab\| \leq \|a\| \, \|b\|$$

and an involution[1] $a \to a^*$ satisfying the C*-condition

$$\|a^* a\| = \|a\|^2 \qquad \text{for all } \ a \in \mathscr{A}.$$

**Theorem (Gelfand, Naimark, 1943)**

*Every abstract C*-algebra $\mathscr{A}$ can be 'faithfully represented' as a concrete C*-algebra on some Hilbert space: there exists a Hilbert space $H$ and an isometric *-morphism $\pi : \mathscr{A} \to \mathscr{B}(H)$.*

There exist abstract characterisations of operator systems (Choi - Effros, 1977) operator spaces (Ruan, 1988) and operator algebras (Blecher - Ruan - Sinclair, 1990).

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[1] that is, a map on $\mathscr{A}$ such that $(a+\lambda b)^* = a^* + \bar{\lambda} b^*$, $(ab)^* = b^* a^*$, $a^{**} = a$ for all $a, b \in \mathscr{A}$ and $\lambda \in \mathbb{C}$

# Positivity and States

If $\mathscr{S} \subseteq \mathscr{B}(H)$ is an operator system, then $M_n(\mathscr{S}) \subseteq \mathscr{B}(H^n)$ is also an operator system!

**Definition**

If $\mathscr{S}, \mathscr{T}$ are operator systems, a linear map $\Phi : \mathscr{S} \to \mathscr{T}$ is positive if $\Phi(\mathscr{S}_+) \subseteq \mathscr{T}_+$. It is completely positive (CP) if $\Phi_n : M_n(\mathscr{S}) \to M_n(\mathscr{T})$ is positive.

A linear map $\phi : \mathscr{S} \to \mathbb{C}$ is a state if it is positive and $\phi(\mathbf{1}) = 1$.

A CP map is automatically CB (completely bounded).
A state is automatically completely positive, hence CB.

# Complete Positivity

• Every *-morphism $\pi : \mathscr{A} \to \mathscr{B}$ between C*-algebras is positive.
$(\pi(a^*a) = \pi(a)^*\pi(a) \geq 0 \ \forall a)$.

• Hence every *-morphism $\pi : \mathscr{A} \to \mathscr{B}$ between C*-algebras is completely positive.

• The map $\Phi : M_2(\mathbb{C}) \to \ M_2(\mathbb{C}) : a \to a^T$ (transpose) is positive, not 2-positive:

$$a = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ positive, but } \Phi_2(a) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ not positive.}$$

• For $v \in \mathscr{B}(K,H)$, the map $\Phi_v : \mathscr{B}(H) \to \mathscr{B}(K) : a \to v^*av$ is CP.

• Hence if $\pi : \mathscr{A} \to \mathscr{B}(H)$ is a *-morphism the map $\Psi : \mathscr{A} \to \mathscr{B}(K) : a \to v^*\pi(a)v$ is CP.

• Stinespring's Theorem: there are no others.

# Complete Positivity

**Theorem (Stinespring)**

*For every unital CP map* $\Phi : \mathscr{A} \to \mathscr{B}(H)$ *from a unital C*-algebra* $\mathscr{A}$ *to* $\mathscr{B}(H)$ *there exists* $(\pi, H_\phi, V)$ *where* $\pi$ *is a *-representation of* $\mathscr{A}$ *on the Hilbert space* $H_\phi$ *and* $V : H \to H_\phi$ *is an isometry so that*

$$\Phi(a) = V^* \pi(a) V \quad \textit{for all } \; a \in \mathscr{A}\,.$$

# Choi - Kraus Decomposition

## Proposition

*Let* $\mathscr{A} = M_n(\mathbb{C})$ *and* $\Phi : \mathscr{A} \to \mathscr{B}(H)$ *where* $H = \mathbb{C}^k$*.*
*If* $\Phi$ *is n-positive, then there are* $V_j : \mathbb{C}^k \to \mathbb{C}^n$ *so that*

$$\Phi(A) = \sum_{j=1}^{nk} V_j^* A V_j \quad \forall A \in \mathscr{A} \, .$$

*This is called the Kraus decomposition of* $\Phi$ *and the least possible number of nonzero* $V_j$*'s is called the Kraus rank of* $\Phi$*.*

Remark Hence $n$-positivity of $\Phi : M_n(\mathbb{C}) \to M_k(\mathbb{C})$ implies complete positivity.

# Choi - Kraus Decomposition: Απόδειξη

Ονομάζουμε $E_{rs} \in M_n(\mathbb{C})$ τον πίνακα με 1 στην θέση $(r,s)$ και 0 αλλού. Δηλαδή

$$E_{rs}(x) = e_r \langle e_s, x \rangle := |e_r\rangle \langle e_s^*|(x) \;\; (x \in \mathbb{C}^n).$$

Η $\{E_{rs} : 1 \le r, s \le n\}$ είναι βάση του γραμμ. χώρου $\mathscr{A} = M_n$.

Παρατήρηση Αν $x_1, x_2, \ldots, x_n \in \mathbb{C}^k$, έστω $V^*$ ο $k \times n$ πίνακας που έχει στήλες τα $x_1, x_2, \ldots, x_n$, οπότε

$$V^* : \mathbb{C}^n \to \mathbb{C}^k : e_r \to x_r, \; r = 1, \ldots, n.$$

Τότε

$$\begin{array}{llllllll} V^* E_{rs} V : & \mathbb{C}^k & \to & \mathbb{C}^n & \to & \mathbb{C}^n & \to & \mathbb{C}^k \\ & y & \to & Vy & \to & e_r \langle e_s, Vy \rangle & \to & V^* e_r \langle e_s, Vy \rangle \end{array}$$

Δηλαδή $(V^* E_{rs} V)(y) = V^* e_r \langle e_s, Vy \rangle = V^* e_r \langle V^* e_s, y \rangle = x_r \langle x_s, y \rangle = (|x_r\rangle \langle x_s^*|)(y)$.

# Choi - Kraus Decomposition: Απόδειξη

Αν ονομάσουμε $v$ το διάνυσμα-στήλη $v = (x_1 \oplus x_2 \oplus \cdots \oplus x_n)^\dagger \in \mathbb{C}^{nk}$, τότε ο τελεστής $|v\rangle \langle v^*| : \mathbb{C}^{nk} \to \mathbb{C}^{nk}$ που ανήκει στον $\mathscr{B}(\mathbb{C}^{nk}) = M_{nk} = M_n(M_k)$ έχει πίνακα $[|x_r\rangle \langle x_s^*|]_{r,s}$. Δηλαδή $|v\rangle \langle v^*| = |x_r\rangle \langle x_s^*| = [V^* E_{rs} V]$.

Απόδειξη της διάσπασης Kraus. Αρκεί (λόγω γραμμικότητας) να το δείξουμε για $A = E_{rs}$, $1 \leq r, s \leq n$.

Παρατηρώ ότι ο $[E_{rs}] = \begin{bmatrix} E_{11} & \dots & E_{1n} \\ \vdots & & \vdots \\ E_{n1} & \dots & E_{nn} \end{bmatrix}$ είναι θετικός στην $M_n(\mathscr{A}) = M_n(M_n)$. Επομένως αφού η $\Phi$ είναι $n$-θετική, ο $[\Phi(E_{rs})] = \begin{bmatrix} \Phi(E_{11}) & \dots & \Phi(E_{1n}) \\ \vdots & & \vdots \\ \Phi(E_{n1}) & \dots & \Phi(E_{nn}) \end{bmatrix}$ είναι θετικός στην $M_n(\mathscr{B}(H)) = M_n(M_k)$.

# Choi - Kraus Decomposition: Απόδειξη

Δηλαδή ο $B := [\Phi(E_{rs})]$ είναι θετικός στην $M_n(M_k) = M_{nk} = \mathscr{B}(\mathbb{C}^{nk})$. Από το Φασματικό Θεώρημα υπάρχει ΟΚ βάση $\{f_j : j = 1, \ldots, nk\}$ του $\mathbb{C}^{nk}$ από ιδιοδιανύσματα του $B$ με ιδιοτιμές $\lambda_j \geq 0$. Δηλαδή

$$B = \sum_{j=1}^{nk} \lambda_j \left|f_j\right\rangle \left\langle f_j^*\right| = \sum_{j=1}^{nk} \left|v_j\right\rangle \left\langle v_j^*\right| \quad \text{όπου} \quad v_j = \sqrt{\lambda_j} f_j \,.$$

Γράφοντας κάθε $v_j \in \mathbb{C}^{nk}$ ως διάνυσμα - στήλη $v_j = (x_1^j \oplus x_2^j \oplus \cdots \oplus x_n^j)^\dagger$ με $x_r^j \in \mathbb{C}^k$, έχουμε από την Παρατήρηση,

$$\left|v_j\right\rangle \left\langle v_j^*\right| = [\left|x_r\right\rangle \left\langle x_s^*\right|] = [V_j^* E_{rs} V_j]$$

και συνεπώς

$$[\Phi(E_{rs})] = B = \sum_{j=1}^{nk} [V_j^* E_{rs} V_j] \quad \text{άρα}$$

$$\Phi(E_{rs}) = \sum_{j=1}^{nk} V_j^* E_{rs} V_j \quad 1 \leq r, s \leq n$$

όπως θέλαμε. □

# Καταστάσεις

Αν $\mathscr{S}$ είναι σύστημα τελεστών, κατάσταση (state) είναι μια γραμμική μορφή $\omega : \mathscr{S} \to \mathbb{C}$ ώστε $\omega(\mathscr{S}_+) \subseteq \mathbb{R}_+$ και $\omega(\mathbf{1}) = 1$.

Αν $\mathscr{A}$ είναι "*-άλγεβρα, state είναι μια γραμμική μορφή $\omega : \mathscr{A} \to \mathbb{C}$ ώστε $\omega(\mathscr{A}_+) \subseteq \mathbb{R}_+$ και $\|\omega\| = 1$ $(= \omega(\mathbf{1})$ αν $\exists)$.

Όταν $\mathscr{S} \subseteq \mathscr{A}$, κάθε κατάσταση του $\mathscr{S}$ επεκτείνεται (Hahn - Banach / Krein) σε κατάσταση της $\mathscr{A}$.

Για $\mathscr{A} = \mathbb{C}^d$ ή γενικότερα $\mathscr{A} = C(K)$, τα states είναι ακριβώς τα μέτρα πιθανότητας.

Για $\mathscr{S} \subseteq \mathscr{B}(H)$ (subsystem!) με $\dim H = n < \infty$, τα states αντιστοιχούν ακριβώς στους πίνακες πυκνότητας (density matrices):

$$\omega_\rho(a) = \mathrm{Tr}(\rho a), \qquad \rho \in (M_n)_+, \ \mathrm{Tr}(\rho) = 1\,.$$

# Καταστάσεις

Για $\mathscr{S} \subseteq \mathscr{B}(H)$ με $\dim H = n < \infty$, τα states αντιστοιχούν ακριβώς στους πίνακες πυκνότητας (density matrices):

$$\omega_\rho(a) = \mathrm{Tr}\big(\rho^\dagger a\big), \qquad \rho \in (M_n)_+,\ \mathrm{Tr}(\rho) = 1\,.$$

Αποδ. Η απεικόνιση $\rho \to \omega_\rho : M_n \to (M_n)^*$ είναι καλά ορισμένη, γραμμική και 1-1 γιατί
$\omega_\rho = 0 \Rightarrow \mathrm{Tr}\big(\rho^\dagger \rho\big) = 0 \Rightarrow \rho^\dagger \rho = 0 \Rightarrow \rho = 0.$
Είναι επί γιατί $\dim M_n = \dim(M_n)^*$.
$\rho \geq 0 \Rightarrow \mathrm{Tr}(\rho a) = \mathrm{Tr}\big(\rho^{1/2} a \rho^{1/2}\big) \geq 0$ αν $a \geq 0 \Rightarrow \omega_\rho \geq 0.$
$\omega_\rho \geq 0 \Rightarrow \mathrm{Tr}(\rho(|\xi\rangle\langle\xi|)) \geq 0 \Rightarrow \langle \xi, \rho\xi\rangle \geq 0\ \forall \xi \Rightarrow \rho \geq 0.$
$\omega_\rho(\mathbf{1}) = 1 \iff \mathrm{Tr}(\rho) = 1.$

Παρατήρηση Όταν $\dim H = \infty$, τα $\omega_\rho$ (για κατάλληλα $\rho$) είναι τα ασθενώς-* συνεχή. Υπάρχουν και «ιδιάζοντα»: αυτά που μηδενίζουν τους τελεστές πεπερ. τάξης.

# Quantum Channels

Αν $\Phi : \mathscr{B}(H) \to \mathscr{B}(K)$ $(\dim H . \dim K < \infty)$ θέλω η δυική $\Phi^* : \mathscr{B}(K)^* \to \mathscr{B}(H)^*$ να στέλνει καταστάσεις σε καταστάσεις.

Η $\Phi^*$ ορίζεται:

$$\langle \Phi^*(\rho), T \rangle = \langle \rho, \Phi(T) \rangle \quad \forall \rho \in \mathscr{B}(K)^*, T \in \mathscr{B}(H)$$

όπου $\langle \rho, A \rangle = \mathrm{Tr}\big(\rho^\dagger A\big)$.

Ειδικότερα θέλω $\Phi^* \geq 0$ (positivity preserving $\equiv$ positive)
(ισοδύναμα $\Phi \geq 0$ - σε λίγο)
και $\mathrm{Tr}(\Phi^*(\rho)) = \mathrm{Tr}(\rho)\, \forall \rho$ (ισοδύναμα $\Phi(I) = I$.)

Επιπλέον θέλω οι ιδιότητες αυτές να διατηρούνται και στους $n \times n$ πίνακες, για κάθε $n \in \mathbb{N}$.

# Quantum Channels

Γι αυτό ορίζουμε

**Ορισμός**

*Quantum channel λέγεται μια απεικόνιση $\Phi^* : \mathscr{B}(K)^* \to \mathscr{B}(H)^*$ ($H, K$ πεπερασμένης διάστασης) αν είναι πλήρως θετική και διατηρεί το ίχνος (completely positive trace preserving, CPTP) ισοδύναμα αν η $\Phi : \mathscr{B}(H) \to \mathscr{B}(K)$ είναι πλήρως θετική και διατηρεί τη μονάδα $I$ (unital completely positive, UCP).*

**Θεώρημα (Choi)**

*Μια $\Phi : \mathscr{B}(H) \to \mathscr{B}(K)$ ($H, K$ πεπερασμένης διάστασης) είναι πλήρως θετική ($\Phi \geq_{cp} 0$) αν και μόνον αν υπάρχουν $A_1, \ldots, A_N \in \mathscr{B}(H, K)$ ώστε*

$$\Phi(X) = \sum_{i=1}^{N} A_i X A_i^\dagger \quad \text{για κάθε } X \in \mathscr{B}(H_1).$$

# Quantum Channels

Από τη σχέση $\Phi(X) = \sum_{i=1}^{N} A_i X A_i^\dagger$ εύκολα προκύπτει ότι $\Phi \geq_{cp} 0 \iff \Phi^* \geq_{cp} 0$.

Επίσης η $\Phi$ είναι UCP αν και μόνον αν $\sum_{i=1}^{N} A_i A_i^\dagger = I$ ενώ η $\Phi$ είναι CPTP (quantum channel) αν και μόνον αν $\sum_{i=1}^{N} A_i^\dagger A_i = I$.

Η συνθήκη $\sum_{i=1}^{N} A_i^\dagger A_i = I$ ισοδυναμεί με την $V^\dagger V = I$ δηλ. $V$ ισομετρία, όπου $V = [A_1, A_2, \ldots, A_N] : H^N \to K$ (εδώ $H = \mathbb{C}^n, K = \mathbb{C}^m$).