Hilbert C^*-modules

M. Anoussis, University of the Aegean

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C^*-algebras

Definition

A C*-algebra A is a Banach algebra A equipped with an involution (that is, a map $A \to A$ denoted $a \mapsto a^*$) such that

- $(a + \lambda b)^* = a^* + \bar{\lambda} b^*$
- $(ab)^* = b^* a^*$,
- $a^{**} = a$
- $\|a^* a\| = \|a\|^2$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$.

C^*-algebras

Examples

- $\mathbb{C}$
  $z^* = \bar{z}$, $\|z\| = |z|$.
- X compact, Hausdorff space
  $C(X)$ the space of continuous functions on X
  $\overline{g}(x) = \overline{g(x)}$
  $\|g\| = \sup_{x \in X} |g(x)|$.
- X locally compact, Hausdorff space
  $C_0(X)$
  $f \in C_0(X) \Leftrightarrow \forall \epsilon > 0, \exists K \subset X, K \text{ compact } : |f(x)| < \epsilon, \forall x \notin K$
  $\overline{g}(x) = \overline{g(x)}$
  $\|g\| = \sup_{x \in X} |g(x)|$.

C^*-algebras

Examples

- $B(H)$, for H Hilbert space
  $\|T\| = \sup_{x\in H, \|x\|\le 1} \|Tx\|$
  $\langle Tx, y\rangle = \langle x, T^* y\rangle$.

Theorem (Gelfand-Naimark)

Let A be a C^*-algebra. Then A is isometrically isomorphic to a closed subalgebra of $B(H)$ for some Hilbert space H.

modules

Definition

Let A be a C^*-algebra and E a vector space. E is a left A-module if there is a map $A \times E \to E$ denoted by $(a, x) \mapsto ax$ s.t. for all $a, b \in A$, $x, y \in E$, $\lambda \in \mathbb{C}$

- $a(x + y) = ax + ay$
- $(a + b)x = ax + bx$
- $(ab)x = a(bx)$
- $a(\lambda x) = (\lambda a)x = \lambda(ax)$
- $1x = x$, if A has a unit.

We also say that A acts on E.

A vector space over $\mathbb{C}$ is a $\mathbb{C}$-module.

modules

Definition

Let A be a C^*-algebra and E a vector space. E is a right A-module if there is a map $E \times A \to E$ denoted by $(x, a) \mapsto xa$ s.t. for all $a, b \in A$, $x, y \in E, \lambda \in \mathbb{C}$

- $(x + y)a = xa + ya$
- $x(a + b) = xa + xb$
- $x(ab) = (xa)b$
- $\lambda(xa) = (\lambda x)a = x(\lambda a)$
- $x1 = x$ if A has a unit.

bundles

X compact Hausdorff and H a fixed Hilbert space.
For each $x \in X$, consider a subspace H_x of H. Let

$$E = \{\xi : X \to H, \xi \ \text{continuous}, \xi(x) \in H_x, \forall x \in X\}$$

and define

$$\langle \xi, \eta \rangle (x) = (\xi(x), \eta(x)).$$

Then

$$x \mapsto (\xi(x), \eta(x))$$

is in $C(X)$
and $\langle \xi, \eta \rangle$ is an $C(X)$-valued ``inner product''.

bundles

Also if $f \in C(X)$, define

$$f\xi \in E$$

by

$$f\xi(x) = f(x)\xi(x).$$

Then E is a $C(X)$-module and moreover

$$f \langle \xi, \eta \rangle = \langle f\xi, \eta \rangle .$$

This is the prototypical example of a Hilbert $C(X)$-module.

Hilbert C^*-modules

- Kaplansky, 1953
- Paschke, 1973
- Rieffel, 1974

Hilbert C^*-modules

Definition

Let A be a C*-algebra . An inner product A-module is a complex vector space E such that
(a) E is a right A-module
(b)There is a map

$$E \times E \to A : (x, y) \to \langle x, y \rangle$$

satisfying

1. $\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, y \rangle$
2. $\langle x, y \cdot a \rangle = \langle x, y \rangle a$
3. $\langle x, y \rangle^* = \langle y, x \rangle$
4. $\langle x, x \rangle \geq 0$
5. $\langle x, x \rangle = 0 \Rightarrow x = 0 \qquad (x, y, z \in E, \; a \in A, \; \lambda \in \mathbb{C}).$

Hilbert C^*-modules

Proposition

In an inner product A-module E, for all $x, y \in E$,

$$\langle y, x\rangle \langle x, y\rangle \leq \|\langle x, x\rangle\|_A \langle y, y\rangle$$

and

$$\|\langle x, y\rangle\|_A^2 \leq \|\langle x, x\rangle\|_A \|\langle y, y\rangle\|_A .$$

Hilbert C^*-modules

Proposition

If E is a inner product A-module, we write

$$\|x\|_E = \|\langle x, x\rangle\|_A^{1/2} \qquad (x \in E).$$

This is a norm on E.

Definition

A Hilbert C*-module over A is an inner product A-module such that $(E, \|\cdot\|_E)$ is complete.

Hilbert C^*-modules

Corollary

If E is an inner product A-module, then

$$\|x \cdot a\|_E \leq \|x\|_E \, \|a\|_A \, .$$

Hilbert C^*-modules

Examples

- A Hilbert space H is a left Hilbert C^*-module over $\mathbb{C}$ with inner product
$$_{\mathbb{C}}\langle x, y\rangle = (x, y)$$
(where $(,)$ is the inner product on H which is antilinear in the second variable).
- A Hilbert space H is a right Hilbert C^*-module over $\mathbb{C}$ with inner product
$$\langle x, y\rangle_{\mathbb{C}} = (y, x)$$
(where $(,)$ is the inner product on H which is antilinear in the second variable).

Hilbert C^*-modules

Examples

- Any C*-algebra A is a Hilbert C*-module over A with $\langle a, b \rangle = a^* b$ and $a \cdot b = ab$.
- Any closed ideal J of A is an A-submodule, hence a Hilbert C^*-module over A.

Hilbert C^*-modules

Examples

$$\begin{pmatrix} A & x \\ y & \lambda \end{pmatrix}$$

$$A \quad n \times n$$

$$x \quad n \times 1$$

$$y \quad 1 \times n$$

and

$$\lambda \quad 1 \times 1.$$

Hilbert C^*-modules

Examples

$$E = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \quad n \times 1 \right\}$$

$$\left\langle \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x' \\ 0 & 0 \end{pmatrix} \right\rangle_{\mathbb{C}} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 0 & x' \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ x^* & 0 \end{pmatrix} \begin{pmatrix} 0 & x' \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & x^* x' \end{pmatrix}.$$

E is a right Hilbert C^*-module over $\mathbb{C}$.

Hilbert C^*-modules

Examples

$$E = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \quad n \times 1 \right\}$$

$$\left\langle \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x' \\ 0 & 0 \end{pmatrix} \right\rangle = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & x' \\ 0 & 0 \end{pmatrix}^*$$

$$= \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ (x')^* & 0 \end{pmatrix} = \begin{pmatrix} x(x')^* & 0 \\ 0 & 0 \end{pmatrix}.$$

E is a left Hilbert C^*-module over $M(n, \mathbb{C})$.

$$\langle Ax, x' \rangle = Ax(x')^* = A(x(x')^*) = A \langle x, x' \rangle .$$

Hilbert C^*-modules

Examples

The direct sum $\bigoplus_{k=1}^n E_k$ of finitely many Hilbert C*-modules over the same C*-algebra A is the vector space direct sum equipped with coordinate-wise inner product and module action:

$$\langle (x_k), (y_k) \rangle_E = \sum_{k=1}^n \langle x_k, x_k \rangle_{E_k} \quad \text{and} \quad (x_k) \cdot a = (x_k \cdot a).$$

Hilbert C^*-modules

Examples

The direct sum $\bigoplus E_k$ of a sequence of Hilbert C*-modules over a fixed C*-algebra A is defined to be

$$E = \bigoplus E_k =$$

$$\{x = (x_k) \in \prod_k E_k : \sum_k \langle x_k, x_k \rangle_{E_k} \text{ converges in the norm of } A\}.$$

Hilbert C^*-modules

Examples

The standard C*-module over a C*-algebra A, sometimes denoted $\mathcal{H}_A$, is the direct sum $\bigoplus E_K$, where each E_k equals the Hilbert C*-module A. Thus

$$\{x = (x_k) : \; x_k \in A : \; \sum_k x_k^* x_k \text{ converges in the norm of } \; A\}.$$

Thus, in case $A = \mathbb{C}$, the standard module is just $\ell^2(\mathbb{N})$.

Hilbert C^*-modules

If F is a submodule of E, then we may have

$$F \oplus F^{\perp} \neq E.$$

operators

Definition

Let A be a C^*-algebra and E a Hilbert C^*-module over A. A map $T : E \to E$ is called adjointable if there exists a map $T^* : E \to E$ such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all x, y in E.

remark

If follows from the definition that if T is adjointable, T^* is adjointable and $\langle T^* x, y \rangle = \langle x, Ty \rangle$. That is $(T^*)^* = T$.

operators

Proposition

Let T be an adjointable map. Then

1. *T is a linear module map.*
2. *T is bounded.*

operators

proof

- Linearity:
  If $x, y, z \in E$ and $\lambda, \mu \in \mathbb{C}$ we have:
  $$\langle T(\lambda x + \mu y), z\rangle = \langle \lambda x + \mu y, T^* z\rangle = \overline{\lambda}\,\langle x, T^* z\rangle + \overline{\mu}\,\langle y, T^* z\rangle =$$
  $$\overline{\lambda}\,\langle Tx, z\rangle + \overline{\mu}\,\langle Ty, z\rangle = \langle \lambda T(x) + \mu T(y), z\rangle$$
  and so $T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$.
- T is a module map:
  If $x, y \in E$ and $a \in A$ we have:
  $$\langle T(xa), y\rangle = \langle xa, T^* y\rangle = a^*\,\langle x, T^* y\rangle = a^*\,\langle T(x), y\rangle =$$
  $$\langle T(x)a, y\rangle$$
  and so $T(xa) = T(x)a$.

operators

- T is bounded: Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in E. Assume there exist $x, z \in E$ such that $x_n \to x$ and $Tx_n \to z$. Let $y \in E$. We have:

$$\langle T(x_n), y\rangle \to \langle z, y\rangle$$

and also

$$\langle Tx_n, y\rangle = \langle x_n, T^* y\rangle \to$$
$$\langle x, T^* y\rangle = \langle Tx, y\rangle$$

and so $T(x) = z$. Hence T is bounded by the Closed Graph Theorem.

operators

Proposition

Let T and S be adjointable operators and $\lambda \in \mathbb{C}$. Then

1. $(T + S)^* = T^* + S^*$.
2. $(\lambda T)^* = \overline{\lambda} T$.
3. *TS is adjointable and* $(TS)^* = S^* T^*$.

operators

Proposition

The algebra $\mathcal{L}(E)$ of adjointable operators is a C^*-algebra.

proof

$$\|T^*T\| \leq \|T^*\|\|T\|$$

and

$$\|T^*T\| \geq \sup_{x\in E, \|x\|\leq 1} \{\langle T^*Tx, x\rangle\} = \sup_{x\in E, \|x\|\leq 1} \{\langle Tx, Tx\rangle\} = \|T\|^2.$$

It follows that

$$\|T\| \leq \|T^*\|$$

and since $T^{**} = T$ we obtain $\|T^*\| = \|T\|$.

By the inequality above we then have:

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2 \Rightarrow \|T\|^2 = \|T^*T\|.$$

We show that $\mathcal{L}(E)$ is complete. Let $\{T_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}(E)$. Since the space of bounded linear operators on E is a Banach space, $\{T_n\}_{n\in\mathbb{N}}$ converges to a linear operator T and $\{T_n^*\}_{n\in\mathbb{N}}$ converges to a linear operator $\overline{T}$. We show that T is adjointable and $T^* = \overline{T}$. We have for $y \in E$:

$$\langle Tx, y\rangle = \lim \langle T_n x, y\rangle = \lim \langle x, T_n^* y\rangle = \langle x, \overline{T} y\rangle .$$

So, $T^* = \overline{T}$ and $\mathcal{L}(E)$ is complete. □

compact operators

Definition

Let A be a C^*-algebra and E a Hilbert C^*-module over A. Let x, y in E. Define the map $\Theta_{x,y} : E \to E$ by:

$$\Theta_{x,y}(z) = x \langle y, z \rangle .$$

compact operators

Proposition

Let A be a C^*-algebra and E a Hilbert C^*-module over A. Then for every x, y in E the map $\Theta_{x,y} : E \to E$ is adjointable and

$$\Theta_{x,y}^* = \Theta_{y,x}.$$

proof For $z, w \in E$ we have:

$$\langle \Theta_{x,y} z, w \rangle = \langle x \langle y, z \rangle , w \rangle = \langle y, z \rangle^* \langle x, w \rangle =$$

$$\langle z, y \rangle \langle x, w \rangle = \langle z, y \langle x, w \rangle \rangle = \langle z, \Theta_{y,x} w \rangle .$$

□

compact operators

Proposition

Let A be a C^*-algebra and E a Hilbert C^*-module over A. The closed linear span of the set $\{\Theta_{x,y} : x \in E, y \in E\}$ is a closed ideal in $\mathcal{L}(E)$. We call it the algebra of compact operators on E and denote it by $\mathcal{K}(E)$.

proof Let $T \in \mathcal{L}(E)$ and $x, y \in E$. We have:

$$T\Theta_{x,y} = \Theta_{Tx,y}$$

and

$$\Theta_{x,y}T = \Theta_{x,T^*y}.$$

□

Example

Let H be a Hilbert space, $A = \mathbb{C}$ and consider the Hilbert space H as a Hilbert C^*-module over A. Then the algebra of adjointable operators on the Hilbert C^*-module H over A is the algebra of bounded linear operators on H, and the algebra of compact operators on the Hilbert C^*-module H over A is the algebra of compact operators on the Hilbert space H.

$$\Theta_{x,y}z = x\langle y, z\rangle = x(z, y) = (x \otimes y)(z).$$

Example

Let A be a C^*-algebra and consider the Hilbert C^*-module A over A. Consider the map $L_a : A \to A$ defined by $L_a(x) = ax$. Then L_a is adjointable with adjoint L_{a^*} and $||L_a|| = 1$. Thus the map $a \to L_a$ is an isometric homomorphism from A onto a closed C^*-subalgebra $\mathrm{Im}L$ of $\mathcal{L}(E)$. Since $\Theta_{a,b} = L_{ab^*}$, $\mathrm{Im}L$ contains $\mathcal{K}(A)$. On the other hand, if $a \in A$ and $\{u_i\}_{i \in I}$ is a contractive approximate identity for A, we have $L_{u_i a} \to L_a$ and since $L_{u_i a}$ is in $\mathcal{K}(A)$ we see that L_a is in $\mathcal{K}(A)$. Thus $\mathrm{Im}L$ is contained in $\mathcal{K}(A)$. We conclude that $\mathcal{K}(A) = \mathrm{Im}L$ and so $\mathcal{K}(A)$ is isomorphic to A.

Example

Let A be a unital C^*-algebra and consider the Hilbert C^*-module A over A. Let T be an adjointable operator on A. Then $T(a) = T(1a) = T(1)a$ and $T = L_{T(1)}$. The map $a \to L_a$ is an isomorphism from A onto $\mathcal{L}(E)$. Hence we have $\mathcal{L}(E) = \mathcal{K}(E) \simeq A$.

Unitization

Definition

Let X be a locally compact Hausdorff space. A compactification of X is a compact Hausdorff space Y and an injective map $i : X \to Y$ such that i is a homeomorphism onto a dense, open subset of Y.

Example

If X is a locally compact Hausdorff space the one point compactification of X is a compactification. The Stone-Cěch compactification βX of X is also a compactification.

Unitization

Let X be a locally compact Hausdorff space and Y a compactification of X. If $i : X \to Y$ is the embedding of X into Y, define:
$i_* : C_0(X) \to C(Y)$ by

$$i_* f(y) = \begin{cases} 0 & \text{if } y \notin i(X) \\ f(x) & \text{if } y = i(x) \in i(X) \end{cases}$$

Then

$$i_* C_0(X) = \{f \in C(Y) : f(x) = 0, x \notin i(X)\}$$

and is an ideal of $C(Y)$.

Unitization

Let J be an ideal in $C(Y)$. There exists an open set $U \subseteq Y$ such that

$$J = \{f \in C(Y) : f(x) = 0, x \notin U\}.$$

Then $i(X)$ dense in Y implies that $i(X) \cap U \neq \emptyset$ and

$$i_* C_0(X) \cap J \neq \{0\}.$$

Unitization

Definition

Let A be a C^*-algebra and I an ideal of A. The ideal I is essential if $I \cap J \neq \{0\}$ for every ideal J of A, $J \neq \{0\}$.

Proposition

The following are equivalent for an ideal I of A.

1. *I is essential.*
2. *If $a \in A$ and $aI = \{0\}$ then $a = 0$.*

Unitization

Example

Let X be a compact Hausdorff space. Consider the C^*-algebra $C(X)$. If I is an ideal of $C(X)$ there exists an open set U such that $I = \{f \in C(X) : f(x) = 0, x \notin U\}$. The ideal I is essential if and only if U is dense in X.

Unitization

Definition

A unitization of a C^*-algebra A is a unital C^*-algebra B and an injective homomorphism $i : A \to B$ such that $i(A)$ is an essential ideal in B.

remark

If A is unital and B is a unitization of A, then $A = B$.

proof Let 1 be the unit of A and b the unit of B. If $a \in A$ we have $(b - 1)a = ba - 1a = 0$ and hence $(b - 1)A = \{0\}$. By Proposition $b = 1$ and so $A = B$. □

Unitization

Example

Let A be a C^*-algebra without unit. Set $A^1 = A \oplus \mathbb{C}$. Define $(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu)$ and $(a, \lambda)^* = (a^*, \overline{\lambda})$. Consider the embedding $L : A \to \mathcal{K}(A)$. (L_a is the operator defined by $L_a(x) = ax$ for x in A). Define $\tilde{L} : A^1 \to \mathcal{L}(A)$ by $\tilde{L}((a, \lambda)) = L_a + \lambda I$. Then, the image of A^1 by $\tilde{L}$ is closed in $\mathcal{L}(A)$ and so it is a C^*-algebra. Define the norm on A^1 by $\|(a, \lambda)\| = \|L_a + \lambda I\|$. Then, A^1 with this norm is a C^*-algebra and is a unitization of A.

Unitization

Example

Let H be a Hilbert space and $K(H)$ the algebra of compact operators on H. Then the subalgebra $K(H) + \mathbb{C}I$ of $B(H)$ is closed in $B(H)$ and is a unitization of $K(H)$.

Example

Let A be a C^*-algebra and consider the Hilbert C^*-module A over A. Consider the map $L : A \to \mathcal{K}(A)$. Then $\mathcal{L}(A)$ is a unitization of A. One has to show that $\mathcal{K}(A)$ is an essential ideal of $\mathcal{L}(A)$. Let $T \in \mathcal{L}(A)$ and assume that $T\Theta_{x,y} = 0$ for every $x, y \in A$. Then $\Theta_{Tx,y} = 0$ for every $x, y \in A$ which implies that $Tx = 0$ for every $x \in A$ and so $T = 0$. It follows that $\mathcal{K}(A)$ is an essential ideal of $\mathcal{L}(A)$.

Unitization

Example

Let X be a non compact locally compact Hausdorff space and Y a compactification of X. If $i : X \to Y$ is the embedding of X into Y, define: $i_* : C_0(X) \to C(Y)$ by

$$i_* f(y) = \begin{cases} 0 & \text{if } y \notin i(X) \\ f(x) & \text{if } y = i(x) \in i(X) \end{cases}$$

Then, $C(Y)$ and i_* is a unitization of $C_0(X)$. If Y is the one-point compactification of X, then $C_0(X)^1 = C(Y)$.

Unitization

Definition

A unitization (B, i) of a C^*-algebra A is maximal if whenever C is a C^*-algebra, $j : A \to C$ a homomorphism such that $j(A)$ is an essential ideal of C, then there exists an homomorphism $\phi : C \to B$ such that $\phi j = i$.

It is not obvious from the definition that a maximal unitization of a C^*-algebra exists.

Unitization

Theorem

Let A be a C^*-algebra. The C^*-algebra $(\mathcal{L}(A), i)$ (where $i(a) = L_a$) is a maximal unitization of A. Moreover it (B, j) is another maximal unitization, there exists an isomorphism $\phi : B \to \mathcal{L}(A)$ such that $\phi j = i$.

Definition

We will refer to $\mathcal{L}(A)$ as the multiplier algebra of A and denote it by $M(A)$.

Unitization

Corollary

Let E be a Hilbert C^*-module. Then $M(\mathcal{K}(E)) = \mathcal{L}(E)$.

Proposition

1. *Let H be a Hilbert space. Then $M(K(H)) = B(H)$.*
2. *Let T be a locally compact Hausdorff space. Then $M(C_0(T)) = C_b(T) = C(\beta T)$ where $C_b(T)$ is the space of bounded continuous functions on T and βT is the Stone Cěch compactification of T.*

Morita equivalence

Definition

Let A, B be C^*-algebras. An $A - B$ imprimitivity bimodule E is an $A - B$ bimodule s.t.

- E is a full left Hilbert C^*-module over A and a full right Hilbert C^*-module over B.
- For $x, y \in E$, $a \in A$ and $b \in B$ we have:

$$_A\langle xb, y\rangle =_A \langle x, yb^*\rangle$$

$$\langle ax, y\rangle_B = \langle x, a^* y\rangle_B$$

- For $x, y, z \in E$

$$_A\langle x, y\rangle z = x\langle y, z\rangle_B$$

Morita equivalence

Definition

The C^*-algebras A and B are Morita equivalent if there is an $A - B$ imprimitivity bimodule E.

Morita equivalence

Example

A Hilbert space H is a $K(H) - \mathbb{C}$ imprimitivity bimodule with

$$_{K(H)}\langle h, k\rangle = h \otimes k$$

where $h \otimes k(l) = h(l, k)$.

Morita equivalence

Proposition

A full Hilbert C^*-module E over B is a $\mathcal{K}(E) - B$ imprimitivity bimodule with

$$_{\mathcal{K}(E)}\langle x, y\rangle = \Theta_{x,y}.$$