# What is an Operator Algebra?

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# What is an Operator Algebra?

Short answer:

It is an algebra of bounded linear operators on a Hilbert space.

# What is an Operator Algebra?

Better short answer:
It is a normed algebra $(\mathscr{A}, \|\cdot\|)$
that can be isometrically represented as an algebra
of bounded linear operators on a Hilbert space.

So $\mathscr{A}$ is:

- a vector space,
- a ring,
- a normed space with $\|ab\| \leq \|a\| \, \|b\|$ [usually complete].

[• Sometimes closed under weaker topologies.]

Need to consider all the (completely) isometric representations of $\mathscr{A}$ as operators on Hilbert spaces.

# The algebra $\mathscr{B}(\mathscr{H})$

Let $\mathscr{H}$ be a Hilbert space. The algebra of all bounded linear operators $T : \mathscr{H} \to \mathscr{H}$ is denoted $\mathscr{B}(\mathscr{H})$. It is complete under the norm

$$\|T\| := \sup\{\|Tx\| : x \in \mathrm{b}_1(\mathscr{H})\}$$

Additionally, it has an *involution* $T \to T^*$ defined via

$$\langle T^* x, y\rangle = \langle x, Ty\rangle \quad \text{for all } x, y \in \mathscr{H}.$$

[Theorem: There exists $T^* \in \mathscr{B}(\mathscr{H})$ satisfying this equality.]
This satisfies

$$\|T^* T\| = \|T\|^2 \qquad \text{the } C^*\text{-property.}$$

# The algebras $C(K), C_0(X)$

Let $K$ be a compact Hausdorff [or metric] space.

$$C(K) := \{f : K \to \mathbb{C} : \text{continuous}\}$$

- a vector space for pointwise operations,
- a ring for to pointwise multiplication,
- a Banach space for the supremum norm $\|f\|_\infty := \sup |f(t)|$.
- has involution $f \to \bar{f}$

which determines real functions ($f = \bar{f}$), positive functions $\bar{f}f$.

Let $X$ be a locally compact Hausdorff [or metric] space.

$$C_0(X) := \{f \in C(X) : \forall \varepsilon > 0 \; \exists K \subseteq X \text{ compact s.t. } \; |f|_{K^c}| < \varepsilon\}$$

# Exercise

The algebra $C_0(X)$ can always be faithfully represented as an operator algebra (on *some* Hilbert space $\mathscr{H}$):

There exists an isometric *-morphism (a faithful *-representation) $\pi : C_0(X) \to \mathscr{B}(\mathscr{H})$:

$$\begin{aligned} \|\pi(f)\| &= \|f\|_\infty \\ \pi(\bar{f}) &= (\pi(f))^* \\ \pi(f+\lambda g) &= \pi(f) + \lambda \pi(g) \\ \pi(fg) &= \pi(f) \circ \pi(g) \qquad f, g \in C_0(X), \lambda \in \mathbb{C}. \end{aligned}$$

# Abstraction: C*-algebras

## Definition

• A Banach algebra $\mathscr{A}$ is a complex algebra equiped with a complete submultiplicative norm:

$$\|ab\| \leq \|a\| \, \|b\| \, .$$

• A C*-algebra $\mathscr{A}$ is a Banach algebra equiped with an involution[1] $a \to a^*$ and a complete submultiplicative norm satisfying the C*-condition

$$\|a^* a\| = \|a\|^2 \qquad \text{for all} \;\; a \in \mathscr{A} \, .$$

[1] that is, a map on $\mathscr{A}$ such that $(a + \lambda b)^* = a^* + \bar{\lambda} b^*$, $(ab)^* = b^* a^*$, $a^{**} = a$ for all $a, b \in \mathscr{A}$ and $\lambda \in \mathbb{C}$

# The morphisms

*A *-morphism $\phi : \mathscr{A} \to \mathscr{B}$ between C*-algebras is a linear map that preserves products and the involution.*

It can be shown that morphisms are automatically contractive, and 1-1 morphisms are isometric (algebra forces topology).

# Basic Examples of C*-algebras

- $\mathbb{C}$
- $C(K)$ : $K$ compact Hausdorff: abelian, unital.
- $C_0(X)$ : $X$ locally compact Hausdorff: abelian, nonunital (iff $X$ non-compact).
  *Commutative Gelfand-Naimark: All abelian C*-algebras can be represented as $C_0(X)$ for a unique $X$.*
- $M_n(\mathbb{C})$ : $A^* =$ conjugate transpose,
  $\|A\| = \sup\{\|Ax\|_2 : x \in \ell^2(n), \|x\|_2 = 1\}$: non-abelian, unital.
- $\mathscr{B}(\mathscr{H})$: non-abelian, unital.
  *Gelfand-Naimark: All C*-algebras can be represented as closed selfadjoint subalgebras of $\mathscr{B}(\mathscr{H})$ for 'suitable' $\mathscr{H}$.*

# Other Operator Algebras: The disc algebra

$$A(\mathbb{D}) := \{f \in C(\mathbb{T}) : \hat{f}(k) = 0 \text{ for all } k < 0\}.$$

*A closed subalgebra of the C*-algebra $C(\mathbb{T})$ but not a *-subalgebra: $A(\mathbb{D}) \cap A(\mathbb{D})^* = \mathbb{C}\mathbf{1}$: antisymmetric algebra.*

Representations on Hilbert space

• Restrict any *-representation of $C(\mathbb{T})$; for instance, multiplication operators on $L^2(\mathbb{T})$. The C*-algebra generated by this representation is abelian.

• But also, represent as operators on $\ell^2(\mathbb{Z}_+)$:
$f \to [a_{ij}]$ where $a_{ij} = \hat{f}(i-j)$.The C*-algebra generated by this representation is not abelian: It contains the unilateral shift $S$ and hence also its adjoint $S^*$, which do not commute.

# Other Operator Algebras:

- $T_n = \{(a_{ij}) \in M_n(\mathbb{C}) : a_{ij} = 0 \text{ for } i > j\}$ (upper triangular matrices).
  *A closed subalgebra of the C*-algebra $M_n(\mathbb{C})$ but not a *-subalgebra. Here $T_n \cap T_n^* = D_n$, the diagonal matrices: a maximal abelian selfadjoint algebra (masa) in $M_n$.*
- $M_{oo}(\mathbb{C})$: infinite matrices with finite support.
  *To define norm (and operations), consider its elements as operators acting on $\ell^2(\mathbb{N})$ with its usual basis. This is a selfadjoint algebra, but not complete.*
  *Its completion is $\mathscr{K}$, the set of compact operators on $\ell^2$: a non-unital, non-abelian C*-algebra.*

# Other Operator Algebras: Group algebras

Let $G$ be a (countable) group (think of $\mathbb{Z}$ or $\mathbb{F}_2$). The Hilbert space $\ell^2(G)$ has o.n. basis $\{\delta_s : s \in G\}$. The group $G$ acts on $\ell^2(G)$ via

$$t \to \lambda_t \in \mathscr{B}(\ell^2(G)) \quad \text{where } \lambda_t(\delta_s) = \delta_{ts}, \ s \in G$$

(or $\lambda_t(f)(s) = f(t^{-1}s), f \in \ell^2(G)$).

• The reduced C*-algebra $C_r^*(G) := \overline{\mathrm{span}\{\lambda_s : s \in G\}}^{op}$ - closed in the norm of $\mathscr{B}(\ell^2(G))$.

Each $\lambda_s$ commutes with the right repr. $\rho$ where $\rho_t(\delta_s) = \delta_{st}$. Hence $C_r^*(G)$ commutes with every $\rho_t$. Can consider

• The von Neumann algebra of $G$
$\mathscr{L}(G) := \{X \in \mathscr{B}(\ell^2(G)) : X\rho_t = \rho_t X \ \forall t \in G\}$.

This is larger than $C_r^*(G)$, when $|G| = \infty$ (why?)

What about a semigroup $S \subseteq G$??

# Gelfand Theory

## Theorem (Gelfand-Naimark 1)

*Every commutative C*-algebra $\mathscr{A}$ is isometrically *-isomorphic to $C_0(\sigma(\mathscr{A}))$ where $\sigma(\mathscr{A})$ is the set of nonzero morphisms $\phi : \mathscr{A} \to \mathbb{C}$ which, equipped with the topology of pointwise convergence, is a locally compact Hausdorff space. The map is the Gelfand transform:*

$$\mathscr{A} \to C_0(\sigma(\mathscr{A})) : \ a \to \hat{a} \quad \textit{where} \quad \hat{a}(\phi) = \phi(a), \, (\phi \in \sigma(\mathscr{A})).$$

*The algebra $\mathscr{A}$ is unital iff $\sigma(\mathscr{A})$ is compact.*

# Gelfand Theory

In more detail:
$\sigma(\mathscr{A})$ is the set of all *nonzero* multiplicative linear forms (*characters*) $\phi : \mathscr{A} \to \mathbb{C}$, (necessarily $\|\phi\| \leq 1$ and, when $\mathscr{A}$ is unital, $\|\phi\| = \phi(\mathbf{1}) = 1$) equipped with the w*-topology: $\phi_i \to \phi$ iff $\phi_i(a) \to \phi(a)$ for all $a \in \mathscr{A}$.

When $\mathscr{A}$ is non-abelian there may be no characters (consider $M_2(\mathbb{C})$ or $\mathscr{B}(\mathscr{H})$, for example).

When $\mathscr{A}$ is abelian there are 'many' characters: for each $a \in \mathscr{A}$ there exists $\phi \in \sigma(\mathscr{A})$ such that $\|a\| = |\phi(a)|$.

When $\mathscr{A}$ is unital $\sigma(\mathscr{A})$ is compact and $\mathscr{A}$ is isometrically *-isomorphic to $C(\sigma(\mathscr{A}))$.

# Spectrum and Positivity

Let $a$ be an element of a unital C*-algebra $\mathscr{A}$. Its spectrum is

$$\sigma(a) := \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \text{ not invertible in } \mathscr{A}\}.$$

This is a compact, nonempty subset of $\mathbb{C}$.

## Definition

An element $a$ of a C*-algebra $\mathscr{A}$ is positive $(a \geq 0)$ if

$a$ is selfadjoint $(a = a^*)$ and $\sigma(a) \subseteq \mathbb{R}_+$.

Write $\mathscr{A}_+ = \{a \in \mathscr{A} : a \geq 0\}$.

If $a, b$ are selfadjoint, we define $a \leq b$ by $b - a \in \mathscr{A}_+$.

## Examples

In $C(X)$: $f \geq 0$ iff $f(t) \in \mathbb{R}_+$ for all $t \in X$ because $\sigma(f) = f(X)$.

In $\mathscr{B}(\mathscr{H})$: $T \geq 0$ iff $\langle T\xi, \xi \rangle \geq 0$ for all $\xi \in H$.

# Positivity

## Proposition

*In a C*-algebra, every positive element has a unique positive square root.*

## Theorem

*In any C*-algebra, any element of the form $a^*a$ is positive.*

(Obvious in $\mathscr{B}(\mathscr{H})$, key result for Gelfand - Naimark Theorem)

# The GNS construction

## Definition

A state on a C*-algebra $\mathscr{A}$ is a positive linear map of norm 1, i.e. $\phi : \mathscr{A} \to \mathbb{C}$ linear such that $\phi(a^*a) \geq 0$ for all $a \in \mathscr{A}$ and $\|\phi\| = 1$. A state is called faithful if $\phi(a^*a) > 0$ whenever $a \neq 0$.

NB. When $\mathscr{A}$ is unital and $\phi$ is positive, $\|\phi\| = \phi(\mathbf{1})$.

## Examples

• On $\mathscr{B}(\mathscr{H})$, $\phi(T) = \langle T\xi, \xi \rangle$ for a unit vector $\xi \in \mathscr{H}$, or $\phi(T) = \sum_i \langle T\xi_i, \xi_i \rangle$ where the $\xi_i$ are $\perp$ and $\sum \|\xi_i\|^2 = 1$ (diagonal 'density matrix').

• On $C(K)$, $\phi(f) = \int f d\mu$ for a probability measure $\mu$ (in particular $\phi(f) = f(t_0)$ for $t_0 \in K$ - Dirac measure at $t_0$).

• For a C*-algebra $\mathscr{A}$, if $\pi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ is a *-representation and $\xi \in \mathscr{H}$ a unit vector, $\phi(a) = \langle \pi(a)\xi, \xi \rangle$.

Conversely,

# The GNS construction

Conversely,

**Theorem (Gelfand, Naimark, Segal)**

*For every state $\phi$ on a C*-algebra $\mathscr{A}$ there is a triple $(\pi_\phi, \mathscr{H}_\phi, \xi_\phi)$ where $\pi_\phi$ is a *-representation of $\mathscr{A}$ on $\mathscr{H}_\phi$ and $\xi_\phi \in \mathscr{H}_\phi$ a* cyclic [2] *unit vector such that*

$$\phi(a) = \langle \pi_\phi(a)\xi_\phi, \xi_\phi \rangle \quad \textit{for all } a \in \mathscr{A}.$$

*The GNS triple $(\pi_\phi, \mathscr{H}_\phi, \xi_\phi)$ is uniquely determined by this relation up to unitary equivalence.*

[2] i.e. $\pi_\phi(\mathscr{A})\xi_\phi$ is dense in $\mathscr{H}_\phi$.

# Consequence: The universal representation

## Theorem (Gelfand, Naimark)

*For every C*-algebra $\mathscr{A}$ there exists a *-representation $(\pi, \mathscr{H})$ which is faithful (one to one).*

Idea of proof Enough to assume $\mathscr{A}$ unital. Let $\mathscr{S}(\mathscr{A})$ be the set of all states. For each $\phi \in \mathscr{S}(\mathscr{A})$ consider $(\pi_\phi, \mathscr{H}_\phi)$ and 'add them up' to obtain $(\pi, \mathscr{H})$. Why is this faithful? Because

## Lemma

*For each nonzero $a \in \mathscr{A}$ there exists $\phi \in \mathscr{S}(\mathscr{A})$ such that $\phi(a^*a) > 0$.*

... and then

$$\left\|\pi(a)\xi_\phi\right\|^2 = \left\langle \pi(a^*a)\xi_\phi, \xi_\phi \right\rangle = \left\langle \pi_\phi(a^*a)\xi_\phi, \xi_\phi \right\rangle = \phi(a^*a) > 0$$

so $\pi(a) \neq 0$.

# Complete positivity

For $n \in \mathbb{N}$, each $A \in \mathscr{B}(\mathscr{H}^n)$ gives $n \times n$ matrix $[a_{ij}]$ with $a_{ij} \in \mathscr{B}(\mathscr{H})$ given by

$$A \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{2n} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \quad (\xi_i \in \mathscr{H})$$

The map $A \to [a_{ij}] : \mathscr{B}(H^n) \to M_n(\mathscr{B}(\mathscr{H}))$ is a *-isomorphism.
So $M_n((\mathscr{B}(\mathscr{H}))$ is a C*-algebra with the norm of $\mathscr{B}(H^n)$.

Hence if $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$ is any operator algebra, $M_n(\mathscr{A})$ becomes an operator algebra.

If $\phi : \mathscr{A} \to \mathscr{B}$ is a linear map between operator algebras, define

$$\phi_n : M_n(\mathscr{A}) \to M_n(\mathscr{B}) \text{ by } \quad \phi_n([a_{ij}]) = [\phi(a_{ij})].$$

If $\mathscr{A}, \mathscr{B}$ are C*-algebras and $\phi$ is *-linear, so is $\phi_n$.
If $\phi$ is a *-morphism, so is $\phi_n$.

# Complete positivity

$$\phi_n : M_n(\mathscr{A}) \to M_n(\mathscr{B}) \text{ by } \phi_n([a_{ij}]) = [\phi(a_{ij})].$$

## Definition

A map $\phi : \mathscr{A} \to \mathscr{B}$ between C*-algebras is positive if

$$a \geq 0 \Rightarrow \phi(a) \geq 0.$$

It does NOT follow that $\phi_n$ is positive. Example: take $\phi(a) = a^\dagger$ (transpose) on $\mathscr{A} = M_2$; clearly positive. Then

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ is +ive, but } \phi_2(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ is not +ive.}$$

## Definition

A map $\phi : \mathscr{A} \to \mathscr{B}$ between C*-algebras is completely positive if $\phi_n$ is positive for all $n \in \mathbb{N}$.

# Stinespring: Operator-valued GNS

Examples of completely positive (cp) maps:
A *-morphism $\pi$ is positive (because $\pi(a^*a) = \pi(a)^*\pi(a) \geq 0 \; \forall a$).
Hence a *-morphism is a cp map (because $\pi_n$ is a *-morphism).
A map $a \to V^*aV$ is a cp map.
(here $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$ and $V \in \mathscr{B}(\mathscr{H}, \mathscr{K})$).
Hence $a \to V^*\pi(a)V$ is a cp map. There are no others:

**Theorem (Stinespring)**

*For every unital cp map $\phi$ from a C*-algebra $\mathscr{A}$ to $B(\mathscr{H})$ there is a triple $(\pi, \mathscr{K}, V)$ where $\pi$ is a *-representation of $\mathscr{A}$ on $\mathscr{K}$ and $V : \mathscr{H} \to \mathscr{K}$ is an isometry such that*

$$\phi(a) = V^*\pi(a)V \quad \textit{for all } \; a \in \mathscr{A}.$$

We say the *-rep. $\pi$ is a dilation of the cp map $\phi$ via the embedding $V : \mathscr{H} \to \mathscr{K}$.
[There is also a uniqueness condition.]

# GNS and Stinespring

In the remainder of these notes, we will sketch the proofs of the GNS construction and of Stinespring's theorem, to emphasize the idea that Stinespring's theorem is in essence an 'operator-valued GNS construction'.

# The GNS construction

**Theorem (Gelfand, Naimark, Segal)**

*For every state $\phi$ on a C*-algebra $\mathscr{A}$ there is a triple $(\pi_\phi, \mathscr{H}_\phi, \xi_\phi)$ where $\pi_\phi$ is a *-representation of $\mathscr{A}$ on $\mathscr{H}_\phi$ and $\xi_\phi \in \mathscr{H}_\phi$ a* cyclic [3] *unit vector such that*

$$\phi(a) = \langle \pi_\phi(a)\xi_\phi, \xi_\phi \rangle \quad \textit{for all } \; a \in \mathscr{A}.$$

*The GNS triple $(\pi_\phi, \mathscr{H}_\phi, \xi_\phi)$ is uniquely determined by this relation up to unitary equivalence.*

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[3] i.e. $\pi_\phi(\mathscr{A})\xi_\phi$ is dense in $\mathscr{H}_\phi$.

# Proof of GNS and Stinespring (Sketch)

1. Consider the linear space $\mathscr{A}$.
2. Define semi-scalar product $\langle a, b \rangle_\phi := \phi(b^* a)$.
   If $\mathscr{A} = C(X)$ then $\langle a, b \rangle_\phi = \int_X a(x)\overline{b(x)} d\mu(x)$.
3. Since $\phi$ is positive, $\langle a, a \rangle_\phi = \phi(a^* a) \geq 0$.
   By Cauchy-Schwarz the set $\mathscr{N} := \{u \in \mathscr{A} \ : \langle u, u \rangle_\phi = 0\}$ is a linear space.
4. Define $\mathscr{K}_0 := \mathscr{A} / \mathscr{N}$ and complete with respect to $\|[u]\|_\phi := \sqrt{\langle [u], [u] \rangle_\phi}$ to get the Hilbert space $\mathscr{K}$ (here $[u] = u + \mathscr{N}$).

# Proof of GNS II

5 $\mathscr{A}$ acts on $\mathscr{A}$ via $\pi_0(a)(b) = ab$.

6 Now $\pi_0(a)(\mathscr{N}) \subseteq \mathscr{N}$ so $\pi_0(a)$ induces $\pi_1(a)$ on $\mathscr{K}_0$.

7 Prove that $\|\pi_1(a)([u])\|_\phi \leq \|a\| \, \|[u]\|_\phi$.
[For $\mathscr{A} = C(X)$, $\|au\|_2 \leq \|a\|_\infty \|u\|_2$.]
Hence $\pi_1(a)$ extends to a bdd operator $\pi(a)$ on $\mathscr{K}$.

Easy: $\pi : a \to \pi(a) : \mathscr{A} \to \mathscr{B}(\mathscr{K})$ is a *-representation.

8 Set $\xi_\phi = [\mathbf{1}_{\mathscr{A}}]$. Then

$$\left\langle \pi(a)\xi_\phi, \xi_\phi \right\rangle_{\mathscr{K}} = \langle \pi(a)[\mathbf{1}], [\mathbf{1}] \rangle_{\mathscr{K}}$$
$$= \langle a, \mathbf{1} \rangle_{\mathscr{K}} = \phi(\mathbf{1}^* a) = \phi(a). \qquad \square$$

# Stinespring: Operator-valued GNS

**Theorem (Stinespring)**

*For every unital cp map $\phi$ from a C*-algebra $\mathscr{A}$ to $B(\mathscr{H})$ there is a triple $(\pi, \mathscr{K}, V)$ where $\pi$ is a *-representation of $\mathscr{A}$ on $\mathscr{K}$ and $V : \mathscr{H} \to \mathscr{K}$ is an isometry such that*

$$\phi(a) = V^* \pi(a) V \quad \textit{for all } \ a \in \mathscr{A}.$$

# Proof of Stinespring (Sketch)

1. Consider the linear space
$\mathscr{A} \otimes \mathscr{H} = \mathrm{span}\{a \otimes \xi : a \in \mathscr{A}, \xi \in \mathscr{H}\}$
(see the Appendix [30]).
When $\mathscr{H} = \mathbb{C}$ then $\mathscr{A} \otimes \mathscr{H} \simeq \mathscr{A}$.
2. Define semi-inner product $\langle a \otimes \xi, b \otimes \eta \rangle_\phi := \langle \phi(b^* a)\xi, \eta \rangle_{\mathscr{H}}$
(extend linearly [30]).
When $\mathscr{H} = \mathbb{C}$ then $\langle a, b \rangle_\phi = \phi(b^* a)$.
3. Since $\phi$ is cp prove $\langle \sum_n a_n \otimes \xi_n, \sum_m a_m \otimes \xi_m \rangle_\phi \geq 0$.
By Cauchy-Schwarz the set $\mathscr{N} := \{u \in \mathscr{A} \otimes \mathscr{H} : \langle u, u \rangle_\phi = 0\}$
is a linear space.
4. Define $\mathscr{K}_0 := (\mathscr{A} \otimes \mathscr{H})/\mathscr{N}$ and complete with respect to
$\|[u]\|_\phi := \sqrt{\langle [u], [u] \rangle_\phi}$ to get the Hilbert space $\mathscr{K}$
(here $[u] = u + \mathscr{N}$).

# Proof of Stinespring II

5. $\mathscr{A}$ acts on $\mathscr{A} \otimes \mathscr{H}$ via $\pi_0(a)(b \otimes \xi) = ab \otimes \xi$
($\pi_0(a)(b) = ab$).
6. Now $\pi_0(a)(\mathscr{N}) \subseteq \mathscr{N}$ so $\pi_0(a)$ induces a map $\pi_1(a)$ on $\mathscr{K}_0$.
7. Prove that $\|\pi_1(a)([u])\|_\phi \leq \|a\| \, \|[u]\|_\phi$. Hence $\pi_1(a)$ extends to a bdd operator $\pi(a)$ on $\mathscr{K}$.
Easy: $\pi : a \to \pi(a) : \mathscr{A} \to \mathscr{B}(\mathscr{K})$ is a *-representation.
8. Define $V : \mathscr{H} \to \mathscr{K} : \xi \to \mathbf{1}_{\mathscr{A}} \otimes \xi \to [\mathbf{1}_{\mathscr{A}} \otimes \xi]$. $V$ satisfies

$$\|V\xi\|_\phi^2 = \langle [\mathbf{1} \otimes \xi], [\mathbf{1} \otimes \xi] \rangle_\phi = \langle \phi(\mathbf{1}^*\mathbf{1})\xi, \xi \rangle_{\mathscr{H}} = \|\xi\|_{\mathscr{H}}^2$$

hence is an isometry $V : \mathscr{H} \to \mathscr{K}$ and for all $\xi, \eta \in \mathscr{H}$,

$$\begin{aligned} \langle V^*\pi(a)V\xi, \eta \rangle_{\mathscr{H}} &= \langle \pi(a)V\xi, V\eta \rangle_{\mathscr{H}} = \langle \pi(a)[\mathbf{1} \otimes \xi], [\mathbf{1} \otimes \eta] \rangle_{\mathscr{K}} \\ &= \langle [a \otimes \xi], [\mathbf{1} \otimes \eta] \rangle_{\mathscr{K}} = \langle \phi(\mathbf{1}^* a)\xi, \eta \rangle_{\mathscr{H}} \end{aligned}$$

so $V^*\pi(a)V = \phi(a)$. □

# The (algebraic) tensor product

Consider the linear space $\mathscr{A}$ as a space of complex-valued functions on a set $S$ and the linear space $\mathscr{H}$ as a space of complex-valued functions on a set $T$.

If $a \in \mathscr{A}$ and $\xi \in \mathscr{H}$ define $a \otimes \xi : S \times T \to \mathbb{C}$ by $(a \otimes \xi)(s,t) := a(s)\xi(t)$.

The (algebraic) tensor product $\mathscr{A} \otimes \mathscr{H}$ is defined to be the linear span of such functions; it consists of all functions $u : S \times T \to \mathbb{C}$ of the form $u(s,t) = \sum_{i=1}^{n} a_i(s)\xi_i(t)$, where $a_i \in \mathscr{A}$ and $\xi_i \in \mathscr{H}$.

Thus the semi-inner product $\langle \cdot, \cdot \rangle_\phi$ is defined on $\mathscr{A} \otimes \mathscr{H}$ by

$$\left\langle \sum_{i=1}^{n} a_i \otimes \xi_i, \sum_{i=1}^{n} b_i \otimes \eta_i \right\rangle_\phi := \sum_{i,j=1}^{n} \left\langle \phi(b_j^* a_i)\xi_i, \eta_j \right\rangle_{\mathscr{H}}$$

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