What is an Operator Algebra?

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6th Summer School, Athens, 3-7 July 2017
What is an Operator Algebra?

Short answer:

It is an algebra of bounded linear operators on a Hilbert space.
What is an Operator Algebra?

Better short answer:
It is a normed algebra \((\mathcal{A}, \| \cdot \|)\)
that can be isometrically represented as an algebra
of bounded linear operators on a **Hilbert** space.

So \(\mathcal{A}\) is:
• a vector space,
• a ring,
• a normed space with \(\|ab\| \leq \|a\| \|b\|\) [usually complete].

[• Sometimes closed under weaker topologies.]

Need to consider all the (completely) isometric representations of \(\mathcal{A}\) as operators on Hilbert spaces.
Let $\mathcal{H}$ be a Hilbert space. The algebra of all bounded linear operators $T : \mathcal{H} \to \mathcal{H}$ is denoted $\mathcal{B}(\mathcal{H})$. It is complete under the norm

$$\| T \| := \sup\{ \| Tx \| : x \in b_1(\mathcal{H}) \}$$

Additionally, it has an *involution* $T \to T^*$ defined via

$$\langle T^* x, y \rangle = \langle x, Ty \rangle \quad \text{for all } x, y \in \mathcal{H}.$$ 

[Theorem: There exists $T^* \in \mathcal{B}(\mathcal{H})$ satisfying this equality.] This satisfies

$$\| T^* T \| = \| T \|^2$$ 

the $C^*$-property.
The algebras $C(K), C_0(X)$

Let $K$ be a compact Hausdorff [or metric] space.

$$C(K) := \{ f : K \to \mathbb{C} : \text{continuous} \}$$

- a vector space for pointwise operations,
- a ring for to pointwise multiplication,
- a Banach space for the supremum norm $\|f\|_\infty := \sup |f(t)|$.
- has involution $f \to \bar{f}$

which determines real functions ($f = \bar{f}$), positive functions $\bar{f}f$.

Let $X$ be a locally compact Hausdorff [or metric] space.

$$C_0(X) := \{ f \in C(X) : \forall \varepsilon > 0 \exists K \subseteq X \text{ compact s.t. } |f|_{K^c} < \varepsilon \}$$
The algebra $C_0(X)$ can always be **faithfully represented** as an operator algebra (on some Hilbert space $\mathcal{H}$):

There exists an isometric $*$-morphism (a **faithful $*$-representation**) $\pi : C_0(X) \rightarrow \mathcal{B}(\mathcal{H})$:

\[
\|\pi(f)\| = \|f\|_{\infty} \\
\pi(\overline{f}) = (\pi(f))^* \\
\pi(f + \lambda g) = \pi(f) + \lambda \pi(g) \\
\pi(fg) = \pi(f) \circ \pi(g) \quad f, g \in C_0(X), \lambda \in \mathbb{C}.
\]
Abstraction: C*-algebras

Definition

• A Banach algebra $\mathcal{A}$ is a complex algebra equipped with a complete submultiplicative norm:

$$\|ab\| \leq \|a\| \|b\|.$$ 

• A C*-algebra $\mathcal{A}$ is a Banach algebra equipped with an involution\(^1\) $a \rightarrow a^*$ and a complete submultiplicative norm satisfying the C*-condition

$$\|a^*a\| = \|a\|^2 \quad \text{for all} \quad a \in \mathcal{A}.$$ 

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\(^1\)that is, a map on $\mathcal{A}$ such that $(a + \lambda b)^* = a^* + \overline{\lambda} b^*$, $(ab)^* = b^* a^*$, $a^{**} = a$ for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$
A *-morphism $\phi : \mathcal{A} \to \mathcal{B}$ between C*-algebras is a linear map that preserves products and the involution.

It can be shown that morphisms are automatically contractive, and 1-1 morphisms are isometric (algebra forces topology).
Basic Examples of C*-algebras

- $\mathbb{C}$
- $C(K): K$ compact Hausdorff: abelian, unital.
- $C_0(X): X$ locally compact Hausdorff: abelian, nonunital (iff $X$ non-compact).
  
  **Commutative Gelfand-Naimark:** All abelian C*-algebras can be represented as $C_0(X)$ for a unique $X$.

- $M_n(\mathbb{C}): A^* = \text{conjugate transpose, } \|A\| = \text{sup}\{\|Ax\|_2: x \in \ell^2(n), \|x\|_2 = 1\}$: non-abelian, unital.
- $B(\mathcal{H}):$ non-abelian, unital.
  
  **Gelfand-Naimark:** All C*-algebras can be represented as closed selfadjoint subalgebras of $B(\mathcal{H})$ for ‘suitable’ $\mathcal{H}$. 
$A(\mathbb{D}) := \{ f \in C(\mathbb{T}) : \hat{f}(k) = 0 \text{ for all } k < 0 \}.$

A *-subalgebra of the C*-algebra $C(\mathbb{T})$ but not a

**Representations on Hilbert space**

- Restrict any *-representation of $C(\mathbb{T})$; for instance, multiplication operators on $L^2(\mathbb{T})$. The C*-algebra generated by this representation is abelian.

- But also, represent as operators on $\ell^2(\mathbb{Z}_+)$: $f \to [a_{ij}]$ where $a_{ij} = \hat{f}(i - j)$. The C*-algebra generated by this representation is not abelian: It contains the unilateral shift $S$ and hence also its adjoint $S^*$, which do not commute.
Other Operator Algebras:

- $T_n = \{(a_{ij}) \in M_n(\mathbb{C}) : a_{ij} = 0 \text{ for } i > j\}$ (upper triangular matrices).
  A closed subalgebra of the $C^*$-algebra $M_n(\mathbb{C})$ but not a *-subalgebra. Here $T_n \cap T_n^* = D_n$, the diagonal matrices: a maximal abelian selfadjoint algebra (masa) in $M_n$.

- $M_{\infty}(\mathbb{C})$: infinite matrices with finite support.
  To define norm (and operations), consider its elements as operators acting on $\ell^2(\mathbb{N})$ with its usual basis. This is a selfadjoint algebra, but not complete. Its completion is $\mathcal{K}$, the set of compact operators on $\ell^2$: a non-unital, non-abelian $C^*$-algebra.
Let $G$ be a (countable) group (think of $\mathbb{Z}$ or $\mathbb{F}_2$). The Hilbert space $\ell^2(G)$ has o.n. basis $\{\delta_s : s \in G\}$. The group $G$ acts on $\ell^2(G)$ via

$$t \mapsto \lambda_t \in \mathcal{B}(\ell^2(G)) \quad \text{where } \lambda_t(\delta_s) = \delta_{ts}, \ s \in G$$

(or $\lambda_t(f)(s) = f(t^{-1}s), f \in \ell^2(G)$).

- The reduced C*-algebra $C^*_r(G) := \overline{\text{span}\{\lambda_s : s \in G\}}^{op}$ - closed in the norm of $\mathcal{B}(\ell^2(G))$.

Each $\lambda_s$ commutes with the right repr. $\rho$ where $\rho_t(\delta_s) = \delta_{st}$. Hence $C^*_r(G)$ commutes with every $\rho_t$. Can consider

- The von Neumann algebra of $G$

$\mathcal{L}(G) := \{X \in \mathcal{B}(\ell^2(G)) : X \rho_t = \rho_t X \forall t \in G\}$.

This is larger than $C^*_r(G)$, when $|G| = \infty$ (why?)

What about a semigroup $S \subseteq G$??
Theorem (Gelfand-Naimark 1)

Every commutative C*-algebra $\mathcal{A}$ is isometrically *-isomorphic to $C_0(\sigma(\mathcal{A}))$ where $\sigma(\mathcal{A})$ is the set of nonzero morphisms $\phi : \mathcal{A} \to \mathbb{C}$ which, equipped with the topology of pointwise convergence, is a locally compact Hausdorff space. The map is the Gelfand transform:

$$\mathcal{A} \to C_0(\sigma(\mathcal{A})) : a \to \hat{a} \text{ where } \hat{a}(\phi) = \phi(a), \ (\phi \in \sigma(\mathcal{A})).$$

The algebra $\mathcal{A}$ is unital iff $\sigma(\mathcal{A})$ is compact.
In more detail:
\( \sigma(\mathcal{A}) \) is the set of all *nonzero* multiplicative linear forms (characters) \( \phi : \mathcal{A} \to \mathbb{C} \), (necessarily \( \|\phi\| \leq 1 \) and, when \( \mathcal{A} \) is unital, \( \|\phi\| = \phi(1) = 1 \)) equipped with the w*-topology: \( \phi_i \to \phi \) iff \( \phi_i(a) \to \phi(a) \) for all \( a \in \mathcal{A} \).

When \( \mathcal{A} \) is non-abelian there may be no characters (consider \( M_2(\mathbb{C}) \) or \( \mathcal{B}(\mathcal{H}) \), for example).

When \( \mathcal{A} \) is abelian there are ‘many’ characters: for each \( a \in \mathcal{A} \) there exists \( \phi \in \sigma(\mathcal{A}) \) such that \( \|a\| = |\phi(a)| \).

When \( \mathcal{A} \) is unital \( \sigma(\mathcal{A}) \) is compact and \( \mathcal{A} \) is isometrically *-isomorphic to \( C(\sigma(\mathcal{A})) \).
Spectrum and Positivity

Let $a$ be an element of a unital C*-algebra $\mathcal{A}$. Its spectrum is

$$\sigma(a) := \{ \lambda \in \mathbb{C} : \lambda 1 - a \text{ not invertible in } \mathcal{A} \}.$$ 

This is a compact, nonempty subset of $\mathbb{C}$.

**Definition**

An element $a$ of a C*-algebra $\mathcal{A}$ is **positive** ($a \geq 0$) if $a$ is selfadjoint ($a = a^*$) and $\sigma(a) \subseteq \mathbb{R}_+$. Write $\mathcal{A}_+ = \{ a \in \mathcal{A} : a \geq 0 \}$. If $a, b$ are selfadjoint, we define $a \leq b$ by $b - a \in \mathcal{A}_+$.

**Examples**

In $C(X)$: $f \geq 0$ iff $f(t) \in \mathbb{R}_+$ for all $t \in X$ because $\sigma(f) = f(X)$.

In $B(H)$: $T \geq 0$ iff $\langle T \xi, \xi \rangle \geq 0$ for all $\xi \in H$. 

Proposition

*In a C*-algebra, every positive element has a unique positive square root.*

Theorem

*In any C*-algebra, any element of the form \( a^*a \) is positive.*

(Obvious in \( \mathcal{B}(\mathcal{H}) \), key result for Gelfand - Naimark Theorem)
The GNS construction

**Definition**

A state on a C*-algebra $\mathcal{A}$ is a positive linear map of norm 1, i.e. $\phi : \mathcal{A} \to \mathbb{C}$ linear such that $\phi(a^*a) \geq 0$ for all $a \in \mathcal{A}$ and $\|\phi\| = 1$. A state is called faithful if $\phi(a^*a) > 0$ whenever $a \neq 0$.

NB. When $\mathcal{A}$ is unital and $\phi$ is positive, $\|\phi\| = \phi(1)$.

**Examples**

- On $\mathcal{B}(\mathcal{H})$, $\phi(T) = \langle T\xi, \xi \rangle$ for a unit vector $\xi \in \mathcal{H}$, or $\phi(T) = \sum_i \langle T\xi_i, \xi_i \rangle$ where the $\xi_i$ are $\perp$ and $\sum \|\xi_i\|^2 = 1$ (diagonal ‘density matrix’).
- On $C(K)$, $\phi(f) = \int f d\mu$ for a probability measure $\mu$ (in particular $\phi(f) = f(t_0)$ for $t_0 \in K$ - Dirac measure at $t_0$).
- For a C*-algebra $\mathcal{A}$, if $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a *-representation and $\xi \in \mathcal{H}$ a unit vector, $\phi(a) = \langle \pi(a)\xi, \xi \rangle$.

Conversely,
Conversely,

**Theorem (Gelfand, Naimark, Segal)**

For every state $\phi$ on a C*-algebra $\mathcal{A}$ there is a triple $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$ where $\pi_\phi$ is a *-representation of $\mathcal{A}$ on $\mathcal{H}_\phi$ and $\xi_\phi \in \mathcal{H}_\phi$ a cyclic \(^2\) unit vector such that

$$\phi(a) = \langle \pi_\phi(a)\xi_\phi, \xi_\phi \rangle \quad \text{for all } a \in \mathcal{A}.$$ 

The GNS triple $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$ is uniquely determined by this relation up to unitary equivalence.

\(^2\)i.e. $\pi_\phi(\mathcal{A})\xi_\phi$ is dense in $\mathcal{H}_\phi$.
Consequence: The universal representation

**Theorem (Gelfand, Naimark)**

*For every C*-algebra \( \mathcal{A} \) there exists a \(*\)-representation \((\pi, \mathcal{H})\) which is faithful (one to one).*

**Idea of proof**  Enough to assume \( \mathcal{A} \) unital. Let \( \mathcal{S}(\mathcal{A}) \) be the set of all states. For each \( \phi \in \mathcal{S}(\mathcal{A}) \) consider \((\pi_{\phi}, \mathcal{H}_{\phi})\) and ‘add them up’ to obtain \((\pi, \mathcal{H})\). Why is this faithful? Because

**Lemma**

*For each nonzero \( a \in \mathcal{A} \) there exists \( \phi \in \mathcal{S}(\mathcal{A}) \) such that \( \phi(a^*a) > 0 \).*

... and then

\[
\| \pi(a) \xi_{\phi} \|^2 = \langle \pi(a^*a) \xi_{\phi}, \xi_{\phi} \rangle = \langle \pi_{\phi}(a^*a) \xi_{\phi}, \xi_{\phi} \rangle = \phi(a^*a) > 0
\]

so \( \pi(a) \neq 0 \).
Complete positivity

For $n \in \mathbb{N}$, each $A \in \mathcal{B}(\mathcal{H}^n)$ gives $n \times n$ matrix $[a_{ij}]$ with $a_{ij} \in \mathcal{B}(\mathcal{H})$ given by

$$A \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{2n} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}$$

$(\xi_i \in \mathcal{H})$

The map $A \mapsto [a_{ij}] : \mathcal{B}(\mathcal{H}^n) \to M_n(\mathcal{B}(\mathcal{H}))$ is a *-isomorphism. So $M_n(\mathcal{B}(\mathcal{H}))$ is a C*-algebra with the norm of $\mathcal{B}(\mathcal{H}^n)$.

Hence if $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is any operator algebra, $M_n(\mathcal{A})$ becomes an operator algebra.

If $\phi : \mathcal{A} \to \mathcal{B}$ is a linear map between operator algebras, define

$$\phi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B}) \text{ by } \phi_n([a_{ij}]) = [\phi(a_{ij})].$$

If $\mathcal{A}, \mathcal{B}$ are C*-algebras and $\phi$ is *-linear, so is $\phi_n$.

If $\phi$ is a *-morphism, so is $\phi_n$. 

Complete positivity

\[ \phi_n : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B}) \text{ by } \phi_n([a_{ij}]) = [\phi(a_{ij})]. \]

**Definition**

A map \( \phi : \mathcal{A} \rightarrow \mathcal{B} \) between C*-algebras is **positive** if

\[ a \geq 0 \Rightarrow \phi(a) \geq 0. \]

It does NOT follow that \( \phi_n \) is positive. Example: take \( \phi(a) = a^\dagger \) (transpose) on \( \mathcal{A} = M_2 \); clearly positive. Then

\[
A = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}
\]

is +ive, but \( \phi_2(A) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \)

is not +ive.

**Definition**

A map \( \phi : \mathcal{A} \rightarrow \mathcal{B} \) between C*-algebras is **completely positive** if \( \phi_n \) is positive for all \( n \in \mathbb{N} \).
Examples of completely positive (cp) maps:
A $*$-morphism $\pi$ is positive (because $\pi(a^*a) = \pi(a)^*\pi(a) \geq 0 \ \forall a$). Hence a $*$-morphism is a cp map (because $\pi_n$ is a $*$-morphism).
A map $a \rightarrow V^*aV$ is a cp map. (here $A \subseteq B(H)$ and $V \in B(H,K)$).
Hence $a \rightarrow V^*\pi(a)V$ is a cp map. There are no others:

**Theorem (Stinespring)**

*For every unital cp map $\phi$ from a $C^*$-algebra $A$ to $B(H)$ there is a triple $(\pi, H, V)$ where $\pi$ is a $*$-representation of $A$ on $H$ and $V : H \rightarrow K$ is an isometry such that*

$$\phi(a) = V^*\pi(a)V \quad \text{for all } a \in A.$$

*We say the $*$-rep. $\pi$ is a dilation of the cp map $\phi$ via the embedding $V : H \rightarrow K$. [There is also a uniqueness condition.]*
In the remainder of these notes, we will sketch the proofs of the GNS construction and of Stinespring’s theorem, to emphasize the idea that Stinespring’s theorem is in essence an ‘operator-valued GNS construction’.
The GNS construction

**Theorem (Gelfand, Naimark, Segal)**

For every state $\phi$ on a C*-algebra $\mathcal{A}$ there is a triple $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$ where $\pi_\phi$ is a *-representation of $\mathcal{A}$ on $\mathcal{H}_\phi$ and $\xi_\phi \in \mathcal{H}_\phi$ a cyclic unit vector such that

$$\phi(a) = \langle \pi_\phi(a) \xi_\phi, \xi_\phi \rangle \quad \text{for all } a \in \mathcal{A}.$$ 

The GNS triple $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$ is uniquely determined by this relation up to unitary equivalence.

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3i.e. $\pi_\phi(\mathcal{A}) \xi_\phi$ is dense in $\mathcal{H}_\phi$. 
Proof of GNS and Stinespring (Sketch)

1. Consider the linear space $\mathcal{A}$.

2. Define semi-scalar product $\langle a, b \rangle_\phi := \phi(b^*a)$. If $\mathcal{A} = C(X)$ then $\langle a, b \rangle_\phi = \int_X a(x)b(x)d\mu(x)$.

3. Since $\phi$ is positive, $\langle a, a \rangle_\phi = \phi(a^*a) \geq 0$. By Cauchy-Schwarz the set $\mathcal{N} := \{ u \in \mathcal{A} : \langle u, u \rangle_\phi = 0 \}$ is a linear space.

4. Define $\mathcal{H}_0 := \mathcal{A} / \mathcal{N}$ and complete with respect to $\| [u] \|_\phi := \sqrt{\langle [u], [u] \rangle_\phi}$ to get the Hilbert space $\mathcal{H}$ (here $[u] = u + \mathcal{N}$).
5. $\mathcal{A}$ acts on $\mathcal{A}$ via $\pi_0(a)(b) = ab$.

6. Now $\pi_0(a)(\mathcal{N}) \subseteq \mathcal{N}$ so $\pi_0(a)$ induces $\pi_1(a)$ on $\mathcal{H}_0$.

7. Prove that $\|\pi_1(a)([u])\|_\phi \leq \|a\| \|[u]\|_\phi$.
   
   [For $\mathcal{A} = C(\mathcal{X})$, $\|au\|_2 \leq \|a\|_\infty \|u\|_2$.]

   Hence $\pi_1(a)$ extends to a bdd operator $\pi(a)$ on $\mathcal{H}$.

   Easy: $\pi : a \to \pi(a) : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a $*$-representation.

8. Set $\xi_\phi = [1_\mathcal{A}]$. Then

   \[
   \langle \pi(a)\xi_\phi, \xi_\phi \rangle_\mathcal{H} = \langle \pi(a)[1], [1] \rangle_\mathcal{H} = \langle a, 1 \rangle_\mathcal{H} = \phi(1^*a) = \phi(a). \]

   $\square$
Theorem (Stinespring)

For every unital cp map $\phi$ from a C*-algebra $\mathcal{A}$ to $B(\mathcal{H})$ there is a triple $(\pi, \mathcal{K}, V)$ where $\pi$ is a *-representation of $\mathcal{A}$ on $\mathcal{K}$ and $V: \mathcal{H} \to \mathcal{K}$ is an isometry such that

$$\phi(a) = V^* \pi(a) V \quad \text{for all} \quad a \in \mathcal{A}.$$
1. Consider the linear space
\[ \mathcal{A} \otimes \mathcal{H} = \text{span}\{a \otimes \xi : a \in \mathcal{A}, \xi \in \mathcal{H}\} \]
(see the Appendix [30]).
When \( \mathcal{H} = \mathbb{C} \) then \( \mathcal{A} \otimes \mathcal{H} \simeq \mathcal{A} \).

2. Define semi-inner product
\[ \langle a \otimes \xi, b \otimes \eta \rangle_\phi := \langle \phi(b^* a)\xi, \eta \rangle_\mathcal{H} \]
(extend linearly [30]).
When \( \mathcal{H} = \mathbb{C} \) then \( \langle a, b \rangle_\phi = \phi(b^* a) \).

3. Since \( \phi \) is cp prove \( \langle \sum_n a_n \otimes \xi_n, \sum_m a_m \otimes \xi_m \rangle_\phi \geq 0 \).
By Cauchy-Schwarz the set \( \mathcal{N} := \{u \in \mathcal{A} \otimes \mathcal{H} : \langle u, u \rangle_\phi = 0\} \)
is a linear space.

4. Define \( \mathcal{K}_0 := (\mathcal{A} \otimes \mathcal{H})/\mathcal{N} \) and complete with respect to
\[ \| [u] \|_\phi := \sqrt{\langle [u], [u] \rangle_\phi} \]
to get the Hilbert space \( \mathcal{K} \)
(beer \( [u] = u + \mathcal{N} \)).
5. \( A \) acts on \( A \otimes \mathcal{H} \) via \( \pi_0(a)(b \otimes \xi) = ab \otimes \xi \) \( (\pi_0(a)(b) = ab) \).

6. Now \( \pi_0(a)(\mathcal{N}) \subseteq \mathcal{N} \) so \( \pi_0(a) \) induces a map \( \pi_1(a) \) on \( \mathcal{K}_0 \).

7. Prove that \( \|\pi_1(a)([u])\|_{\phi} \leq \|a\| \|[u]\|_{\phi} \). Hence \( \pi_1(a) \) extends to a bdd operator \( \pi(a) \) on \( \mathcal{K} \).

   Easy: \( \pi : a \rightarrow \pi(a) : A \rightarrow \mathcal{B}(\mathcal{K}) \) is a \(*\)-representation.

8. Define \( V : \mathcal{H} \rightarrow \mathcal{K} : \xi \rightarrow \mathbf{1}_A \otimes \xi \rightarrow [\mathbf{1}_A \otimes \xi] \). \( V \) satisfies

\[
\|V\xi\|_{\phi}^2 = \langle [1 \otimes \xi], [1 \otimes \xi] \rangle_{\phi} = \langle \phi(1^*1)\xi, \xi \rangle_{\mathcal{H}} = \|\xi\|_{\mathcal{H}}^2
\]

hence is an isometry \( V : \mathcal{H} \rightarrow \mathcal{K} \) and for all \( \xi, \eta \in \mathcal{H} \),

\[
\langle V^*\pi(a)V\xi, \eta \rangle_{\mathcal{H}} = \langle \pi(a)V\xi, V\eta \rangle_{\mathcal{H}} = \langle \pi(a)[1 \otimes \xi], [1 \otimes \eta] \rangle_{\mathcal{H}}
= \langle [a \otimes \xi], [1 \otimes \eta] \rangle_{\mathcal{H}} = \langle \phi(1^*a)\xi, \eta \rangle_{\mathcal{H}}
\]

so \( V^*\pi(a)V = \phi(a) \). \( \square \)
The (algebraic) tensor product

Consider the linear space $\mathcal{A}$ as a space of complex-valued functions on a set $S$ and the linear space $\mathcal{H}$ as a space of complex-valued functions on a set $T$.

If $a \in \mathcal{A}$ and $\xi \in \mathcal{H}$ define $a \otimes \xi : S \times T \to \mathbb{C}$ by
\[(a \otimes \xi)(s, t) := a(s)\xi(t).\]

The (algebraic) tensor product $\mathcal{A} \otimes \mathcal{H}$ is defined to be the linear span of such functions; it consists of all functions $u : S \times T \to \mathbb{C}$ of the form $u(s, t) = \sum_{i=1}^{n} a_i(s)\xi_i(t)$, where $a_i \in \mathcal{A}$ and $\xi_i \in \mathcal{H}$.

Thus the semi-inner product $\langle \cdot, \cdot \rangle_\phi$ is defined on $\mathcal{A} \otimes \mathcal{H}$ by
\[
\left\langle \sum_{i=1}^{n} a_i \otimes \xi_i, \sum_{i=1}^{n} b_i \otimes \eta_i \right\rangle_\phi := \sum_{i,j=1}^{n} \langle \phi(b^*_j a_i)\xi_i, \eta_j \rangle_{\mathcal{H}}.
\]
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