Minimal and maximal matrix convex sets

Orr Shalit, joint with Ben Passer and Baruch Solel

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Also based on previous work with K. Davidson, A. Dor-On and B. Solel

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Compressions and dilations

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Useful picture:

$$N_i = \begin{pmatrix} X_i & * \\ * & * \end{pmatrix}$$

Classical dilation theorems I

Theorem (Halmos 1950)

If $T \in B(\mathcal{H})$ is a contraction, then

$$U := \begin{pmatrix} T & \sqrt{1 - TT^*} \\ \sqrt{1 - T^*T} & -T^* \end{pmatrix}$$

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Theorem (Sz.-Nagy 1953)

If $T \in B(\mathcal{H})$ is a contraction, then there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a unitary $U \in B(\mathcal{K})$ such that

$$T^m = P_{\mathcal{H}} U^m \big|_{\mathcal{H}} \text{ for all } m \geq 0$$

Classical dilation theorems II

Theorem (Ando 1963)

For every pair of commuting contractions $A_1, A_2 \in B(\mathcal{H})$, there exists a pair of commuting unitaries $U_1, U_2 \in B(\mathcal{K})$ such that

$$A_1^m A_2^n = P_{\mathcal{H}} U_1^m U_2^n\big|_{\mathcal{H}} \quad \text{for all} \quad m, n \geq 0$$

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Theorem (Bunce, Frazho, Popescu NC dilation theorem, 1980s)

For every row contraction $(A_1, \ldots, A_d) \in B(\mathcal{H})^d$ (i.e., $\|[A_1 \cdots A_d]\| \leq 1$), there exists a d-tuple of isometries $V_1, \ldots, V_d \in B(\mathcal{K})$ with orthogonal ranges, such that for all $n_1, \ldots, n_k \in \{1, \ldots, d\}$,

$$A_{n_1} \cdots A_{n_k} = P_{\mathcal{H}} V_{n_1} \cdots V_{n_k} \big|_{\mathcal{H}}$$

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The dimension of matrices is fixed at $n \times n$, but the number of matrices being simultaneously dilated is NOT fixed.

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This implies that each level $\mathcal{S}_m$ is convex (use $V_i = \sqrt{t_i}\, I_n$).

Matrix ranges

Example

For every $A \in B(\mathcal{H})^d$, we define its matrix range to be the free set

$$\begin{aligned}\mathcal{W}(A) &= \cup_n \{\phi(A) : \phi \in UCP(C^*(A), M_n)\} \\ &= \cup_n \{(\phi(A_1), \ldots, \phi(A_d)) : \phi \in UCP(C^*(A), M_n)\}.\end{aligned}$$

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In fact, every bounded and closed matrix convex set $\mathcal{S}$ arises as $\mathcal{W}(A)$ for some $A \in B(\mathcal{H})^d$.

Matrix convex sets and UCP interpolation

UCP interpolation problem:

Given $A \in B(\mathcal{H})^d$ and $B \in B(\mathcal{K})^d$, determine whether there exists a UCP map $OS(A) \to OS(B)$ sending A_i to B_i ($i = 1, \ldots, d$).

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Theorem (DDSS, essenitally due to Arveson)

Let $A \in B(\mathcal{H})^d$ and $B \in B(\mathcal{K})^d$. There exists a UCP map $OS(A) \to OS(B)$ sending A_i to B_i if and only if $\mathcal{W}(B) \subseteq \mathcal{W}(A)$.

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Corollary (Li and Poon (for selfadjoint matrices))

If A and B are normal, then there exists a UCP map $OS(A) \to OS(B)$ sending A_i to B_i if and only if $\sigma(B) \subseteq \operatorname{conv} \sigma(A)$.

Another sample result on matrix ranges

Definition

A tuple $A \in B(\mathcal{H})^d$ is said to be minimal if there is no reducing subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that $\mathcal{W}(A\big|_{\mathcal{H}_0}) = \mathcal{W}(A)$.

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Theorem (DDSS)

Let A, B be two minimal d-tuples of compact operators. Then: $\mathcal{W}(A) = \mathcal{W}(B)$ if and only if A and B are unitarily equivalent.

Back to the main theme...

Recall:

Definition

The set $\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$ of d-tuples is matrix convex if

1. $\mathcal{S}$ is closed under direct sums.
2. $\mathcal{S}$ is closed under the application of UCP maps.

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$\Rightarrow$ Study the minimal and maximal matrix convex sets with ground level K.

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Violating a linear inequality is detected by a state, and matrix convex sets are closed under applications of states.

Dilations via matrix convex sets

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Alternatively: for $T \in B(\mathcal{H})^d$, if the joint numerical range $\mathcal{W}_1(T)$

$$\mathcal{W}_1(T) = \{\phi(T) : \phi \in UCP(B(\mathcal{H}), \mathbb{C})\} \subseteq K$$

then T has a normal dilation N with $\sigma(N) \subseteq d \cdot K$.

Examples (instead of a definition)

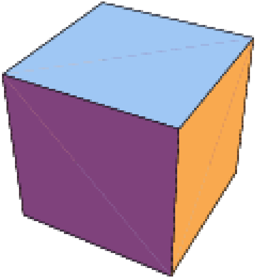
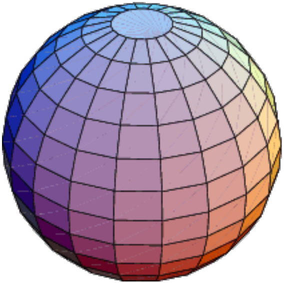
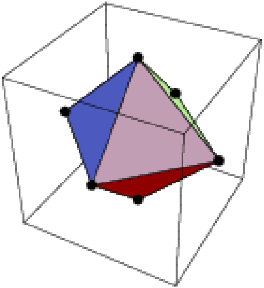
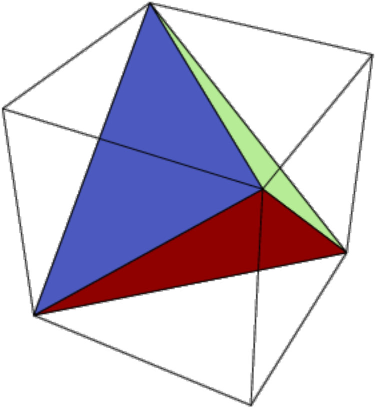




Image credit: http://mathworld.wolfram.com "Cube", "Sphere", "Octahedron"

Example - the regular tetrahedron



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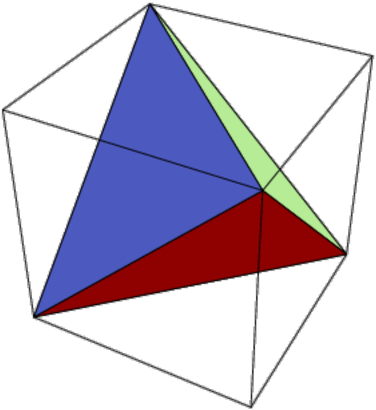

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We will be able to do much MUCH better

Image credit: http://mathworld.wolfram.com "Regular Tetrahedron"

Examples, still from DDSS

$$\overline{\mathbb{B}}_d = \text{ closed unit ball of } \ell^2 \text{ space in } \mathbb{R}^d$$

$$\Delta_d = \text{ standard } d\text{-simplex: the convex hull of } 0, e_1, \ldots, e_d \in \mathbb{R}^d$$

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$$\mathcal{W}^{\max}(\overline{\mathbb{B}}_d) \subseteq d \cdot \mathcal{W}^{\min}(\overline{\mathbb{B}}_d) \quad \mathcal{W}^{\max}(\Delta) \subseteq d \cdot \mathcal{W}^{\min}(\Delta)$$

$$\forall C, \; \mathcal{W}^{\max}(e_1 + \overline{\mathbb{B}}_d) \not\subseteq C \cdot \mathcal{W}^{\min}(e_1 + \overline{\mathbb{B}}_d)$$

$$\mathcal{W}^{\max}([-1,1]^d) \subseteq d \cdot \mathcal{W}^{\min}(D_d) \quad \mathcal{W}^{\max}(D_d) \subseteq 1 \cdot \mathcal{W}^{\min}([-1,1]^d)$$

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A key tool — anticommutation

Lemma

If $x_1, \ldots, x_d$ are pairwise anticommuting ($x_i x_j = -x_j x_i$), self-adjoint elements of a C^*-algebra, then

$$||x_1 + \ldots + x_d|| = \sqrt{||x_1^2 + \ldots + x_d^2||} \leq \sqrt{||x_1||^2 + \ldots + ||x_d||^2}$$

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There exists a tuple $(F_1, \ldots, F_d)$ of pairwise anticommuting, self-adjoint, unitary, $2^{d-1} \times 2^{d-1}$ matrices such that for any $(y_1, \ldots, y_d) \in \mathbb{R}^d$,

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$$||(F_1 - y_1 I) \otimes F_1 + \ldots + (F_d - y_d I) \otimes F_d|| \geq \sqrt{||y||^2 + (d-1)^2} + 1.$$

Cube/rectangle dilation

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For $a_1, \ldots, a_d > 0$,

$$\mathcal{W}^{max}([-1,1]^d) \subseteq \mathcal{W}^{min}([-a_1, a_1] \times \cdots \times [-a_d, a_d])$$

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Corollary (Using the above results, duality and "interpolation")

Let $\overline{\mathbb{B}}_{d,p}$ denote the closed unit ball of ℓ^p-space in $\mathbb{R}^d$. Then

$$\theta(\overline{\mathbb{B}}_{d,p}) = d^{1-|1/2-1/p|}$$

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$$Y_i := \begin{pmatrix} X_i & \sqrt{1 - X_i^2} \\ \sqrt{1 - X_i^2} & -X_i \end{pmatrix}$$

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Sharpness: the anticommuting F_1, F_2 show that can't do better than $\sqrt{2}$.

Case of a rectangle

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We seek $\mathcal{W}^{\max}([-1,1]^2) \subseteq \mathcal{W}^{\min}([-a_1,a_1] \times \cdots \times [-a_d,a_d])$ under the assumption $\sum \frac{1}{a_j^2} \leq 1$.

Let X_1 and X_2 be self-adjoint contractions. Then

$$Y_i := \begin{pmatrix} X_i & \sqrt{1-X_i^2} \\ \sqrt{1-X_i^2} & -X_i \end{pmatrix}$$

are self-adjoint and unitary. Correct the anticommutation to making commuting dilations.

$$N_1 = \begin{pmatrix} Y_1 & r \cdot \frac{1}{2}[Y_2, Y_1] \\ r \cdot \frac{1}{2}[Y_1, Y_2] & Y_1 \end{pmatrix} \qquad N_2 = \begin{pmatrix} Y_2 & \frac{1}{r} \cdot I \\ \frac{1}{r} \cdot I & -Y_2 \end{pmatrix}$$

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When does $\mathcal{W}^{\text{min}} = \mathcal{W}^{\text{max}}$?

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Theorem (Fritz, Netzer, and Thom + translation into MCS setting)

Suppose $K \subset \mathbb{R}^d$ is polyhedral. Then $\mathcal{W}^{max}(K) = \mathcal{W}^{min}(K)$ if and only if K is a simplex.

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Don't want to contradict that $\theta(\overline{\mathbb{B}}_d) = d$!!

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For what shapes (ball, diamond, cube, etc.) of the sets K and L, is simplex containment the only way a dilation hull is formed?

Example

We know that $\mathcal{W}^{\max}([-1,1]^2) \subset \sqrt{2} \cdot \mathcal{W}^{\min}([-1,1]^2)$, but there is no triangle Π with $[-1,1]^2 \subseteq \Pi \subseteq \sqrt{2} \cdot [-1,1]^2$.

Dilating a ball to a ball

Example

There exists a tuple $(F_1, \ldots, F_d)$ of pairwise anticommuting, self-adjoint, unitary, $2^{d-1} \times 2^{d-1}$ matrices such that for any $(y_1, \ldots, y_d) \in \mathbb{R}^d$,

$$||(F_1 - y_1 I) \otimes F_1 + \ldots + (F_d - y_d I) \otimes F_d|| \geq \sqrt{||y||^2 + (d-1)^2 + 1}.$$

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It is easy to add in a shift and scale of the ball $\overline{\mathbb{B}}_d^2$ on the left side, too.

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Is any circumscribing simplex of K a minimal dilation hull of K? (We can say something when K is the ball — see blackboard if time $\longrightarrow$).

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Corollary

Let $\overline{\mathbb{B}}^+_{d,p}$ denote the positive section of the ℓ^p ball in $\mathbb{R}^d$. Then $d^{1-1/p} \cdot \overline{\mathbb{B}}^+_{d,1}$ is a minimal dilation hull of $\overline{\mathbb{B}}^+_{d,p}$. Further, $\theta(\overline{\mathbb{B}}^+_{d,p}) = d^{1-1/p}$.

Thank you very much!