Shilov boundary for "holomorphic functions" on a quantum matrix ball

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(joint work with Olga Bershtein, Olof Gisselson and Daniil Proskurin)

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Let X be a compact Hausdorff space, $\mathcal{A}$ be a uniform subalgebra of $C(X)$ (i.e. $\mathcal{A}$ contains constants and separates points). The Shilov boundary of X relative $\mathcal{A}$ is the smallest closed subset $K \subset X$ such that

$$\|f\|_{C(X)} = \max_{x\in K}|f(x)| \forall f \in \mathcal{A},$$

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Shilov boundary for function algebras

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equivalently, if $J = \{f \in C(X) : f(x) = 0, x \in K\}$ then

$$j : C(X) \to C(X)/J, f \to f + J$$

is an isometry when restricted to $\mathcal{A}$.

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Definition

An ideal J in $\mathcal{B}$ is called a boundary ideal of $\mathcal{B}$ relative $\mathcal{A}$ if the canonical map $j : B \to B/J$ is a complete isometry when restricted to $\mathcal{A}$.
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Shilov boundary ideal exists and unique (Arveson '69, Hamana '79).

Example

Let $\mathbb{U} = \{z \in \mathrm{Mat}_n : zz^* < 1\}$ and $\partial\mathbb{U} = \{z \in \mathrm{Mat}_n : zz^* = 1\}$. Let $C(\overline{\mathbb{U}})$, $C(\partial\mathbb{U})$ be C^*-algebras of continuous functions on $\overline{\mathbb{U}}$ and $\partial\mathbb{U}$, and $\mathcal{A}(\mathbb{U}) \subset C(\overline{\mathbb{U}})$ be the algebra of holomorphic functions in $\mathbb{U}$. Let

$$j : C(\overline{\mathbb{U}}) \to C(\partial\mathbb{U}), f \mapsto f|_{\partial\mathbb{U}}$$

be the restriction map. By the maximum principle $j|_{\mathcal{A}(\mathbb{U})}$ is an isometry and since $C(\partial\mathbb{U})$ is commutative, it is a complete isometry. Hence $J = \ker(j)$ is a boundary ideal. Moreover, it is the Shilov boundary ideal.

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We will discuss q-analogs of the C^*-algebras $C(\overline{\mathbb{U}})$ and $C(\partial\mathbb{U})$ and a q-analog, j_q, of the homomorphism j and show that $\ker(j_q)$ is the Shilov boundary.

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and Dolbeault Complexes, ArXiV '07

The $*$-algebra $\mathrm{Pol}(\mathrm{Mat}_n)_q$

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$\mathbb{C}[\mathrm{Mat}_n]_q$, $0 < q < 1$, is defined by its generators z_i^j, $i, j = 1, \ldots, n$, and the commutation relations

$$\begin{aligned} & z_i^j z_k^l - q z_k^l z_i^j = 0, i = k \quad \& \quad j < l, \quad \text{or} \quad i < k \quad \& \quad j = l, \\ & z_i^j z_k^l - z_k^l z_i^j = 0, j < l \quad \& \quad i > k, \\ & z_i^j z_k^l - z_k^l z_i^j - (q - q^{-1}) z_i^l z_k^j = 0, j < l \quad \& \quad i < k. \end{aligned} \tag{1}$$

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$\mathbb{C}[\mathrm{Mat}_n]_q$ is a q analog of the holomorphic polynomials on Mat_n. $\mathrm{Pol}(\mathrm{Mat}_n)_q$ is a q-analog of the polynomial algebra on Mat_n. Its generators are z_i^j, $(z_i^j)^*$, $i, j = 1, \ldots, n$, and the list of relations is formed by (1) and

$$(z_k^l)^* z_i^j = q^2 \cdot \sum_{i',k'=1}^{n} \sum_{i',k'=1}^{n} R_{ki}^{k'i'} R_{lj}^{l'j'} \cdot z_{i'}^{j'} (z_{k'}^{l'})^* + (1 - q^2) \delta_{ik} \delta^{jl},$$

with δ_{ik}, δ^{jl} being the Kronecker symbols, and certain coeff. R_{ij}^{kl}

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$$\det_q \mathbf{z} = \sum_{s \in S_n} (-q)^{l(s)} z_1^{s(1)} z_2^{s(2)} \ldots z_n^{s(n)}$$

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- $\det_q \mathbf{z}$ is in the center of $\mathbb{C}[\mathrm{Mat}_n]_q$; the localization of $\mathbb{C}[\mathrm{Mat}_n]_q$ w.r.t.$(\det_q \mathbf{z})^{\mathbb{N}}$ is the algebra of regular functions on the quantum GL_n, denoted by $\mathbb{C}[\mathrm{GL}_n]_q$.

There exists a unique involution $*$ in $\mathbb{C}[\mathrm{GL}_n]_q$ such that

$$(z_a^b)^* = (-q)^{a+b-2n}(\det_q \mathbf{z})^{-1}\det_q \mathbf{z}_a^b$$

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The $*$-algebra $\mathrm{Pol}(\partial\mathbb{U})_q := (\mathbb{C}[\mathrm{GL}_n]_q, *) \simeq \mathbb{C}[U_n]_q$ is a q-analog of the polynomial algebra on the Shilov boundary of $\mathbb{U}$.

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Theorem (Vaksman)

$$j_q : \mathrm{Pol}(\mathrm{Mat}_n)_q \to \mathrm{Pol}(\partial\mathbb{U})_q, \; z_a^b \mapsto z_a^b$$

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j_q is a q-analog of the operator which restricts the polynomials to the Shilov boundary.

The co-action of $\mathbb{C}[SU_n]_q$ on $\mathrm{Pol}(\mathrm{Mat}_n)_q$

$\mathbb{C}[SU_n]_q$ is the algebra $\mathbb{C}[U_n]_q/\langle(q^{-n(n-1)/2} - \det_q \mathbf{z})\rangle$.

Letting the generators of $\mathbb{C}[SU_n]_q$ be $t_{kj} := q^{n-k} z_k^j$ we get a quantum group with co-product Δ, co-unit ϵ and antipode S

$$\Delta(t_{kj}) = \sum_m t_{km} \otimes t_{mj} \quad \epsilon(t_{kj}) = \delta_{kj} \quad S(t_{kj}) = t_{jk}^*$$

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$$\mathcal{D} : \mathrm{Pol}(\mathrm{Mat}_n)_q \to \mathrm{Pol}(\mathrm{Mat}_n)_q \otimes \mathbb{C}[SU_n]_q \otimes \mathbb{C}[SU_n]_q$$

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If $q = 1$, the co-action comes from the two different actions of SU_n on Mat_n

$$(A, X) \mapsto AX \quad (A, X) \mapsto XA^t.$$

for $X \in \mathrm{Mat}_n$ and $A \in SU_n$.

Representations of $\mathrm{Pol}(\mathrm{Mat}_n)_q$

The Fock representation: Consider a $\mathrm{Pol}(\mathrm{Mat}_n)_q$-module $\mathcal{H}$ determined by a single generator v_0 (a vacuum vector) and the relations

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Theorem (Sinelschikov, Shklyarov, Vaksman)

There exists a unique sesquilinear form $(\cdot, \cdot)$ on $\mathcal{H}$ such that

$$(v_0, v_0) = 1, (fu, v) = (u, f^* v), u, v \in \mathcal{H}, f \in \mathrm{Pol}(\mathrm{Mat}_n)_q.$$

Moreover, the form is positive definite.

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Moreover, the form is positive definite.

$\mathcal{H}$ becomes a pre-Hilbert space and for $f \in \mathrm{Pol}(\mathrm{Mat}_n)_q$

$$\pi_F(f) : v \to fv$$

is bdd and hence can be extended to a bdd operator on $\overline{\mathcal{H}}$.
$f \mapsto \pi_F(f) \in B(\overline{\mathcal{H}})$ is called the Fock representation

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For $m \leq n$ there exists a $*$-homomorphism

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in particular,

$$\rho : \mathrm{Pol}(\mathrm{Mat}_n)_q \to \mathrm{Pol}(\mathrm{Mat}_{2n})_q, \; z_k^j \mapsto z_{k+n}^{j+n}.$$

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Hence there is a $*$-homomorphism

$$\psi \circ \rho : \mathrm{Pol}(\mathrm{Mat}_n)_q \to \mathbb{C}[SU_{2n}]_q$$

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The idea is to construct a $*$-representation Π of $\mathbb{C}[SU_{2n}]_q$ s.t

$$\Pi \circ \psi \circ \rho \cong \pi_{F,n}.$$

$*$-representations of $\mathbb{C}[SU_n]_q$

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Let $\pi : \mathbb{C}[SU_2]_q \to B(\ell^2(\mathbb{Z}_+))$ be the $*$-representation given by

$$\begin{array}{ll} \pi(t_{11}) = S^* C_q, & \pi(t_{12}) = -qD_q, \\ \pi(t_{21}) = D_q, & \pi(t_{22}) = C_q S \end{array}$$

where, in the standard orthonormal basis $\{e_m\}_{m=0}^{\infty}$, we have

$$Se_m = e_{m+1}, \quad C_q e_m = \sqrt{1 - q^{2m}} e_m, \quad D_q e_m = q^m e_m.$$

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Obs! $C^*(S, C_q, D_q) = C^*(S)$, the C^*-algebra generated by the isometry S. Consider the $*$-homomorphisms
$\phi_i : \mathbb{C}[SU_n]_q \to \mathbb{C}[SU_2]_q$

$$\begin{array}{cc} \phi_i(t_{ii}) = t_{11}, & \phi_i(t_{i+1 i+1}) = t_{22}, \\ \phi_i(t_{ii+1}) = t_{12}, & \phi_i(t_{i+1 i}) = t_{21} \\ \phi_i(t_{kj}) = \delta_{kj} I & \text{otherwise.} \end{array}$$

Let $\pi_i : \mathbb{C}[SU_n]_q \to B(\ell^2(\mathbb{Z}_+))$ be the composition $\pi \circ \phi_i$. Let s_i denote the adjacent transposition $(i, i+1)$ in the symmetric group S_n.

Definition

For an element $s \in S_n$ consider a minimal decomposition of $s = s_{j_1} s_{j_2} \dots s_{j_m}$ into a product of adjacent transposition and let π_s be the $*$-representation of $\mathbb{C}[SU_n]_q$ given by $\pi_{j_1} \otimes \pi_{j_2} \otimes \cdots \otimes \pi_{j_m}$.

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Up to unitary equivalence, π_s is independent of the specific minimal decomposition. Up to action of torus, all irreducible representations of $\mathbb{C}[SU_n]_q$ arise this way (Soibelman).

Construction of Fock representation

In the symmetric group S_{2n} let

$$s = \begin{pmatrix} 1 & 2 & \dots & n & n+1 & n+2 & \dots & 2n \\ n+1 & n+2 & \dots & 2n & 1 & 2 & \dots & n \end{pmatrix}.$$

We have $s = \sigma_1 \dots \sigma_n$, where $\sigma_i = s_{n+i-1} \dots s_i$.

Theorem

$\pi_s \circ \psi \circ \rho$ is unitarily equivalent to the Fock representation π_F and $\overline{\pi_F(\mathrm{Pol}(\mathrm{Mat}_n)_q)} \subset C^*(S)^{\otimes n^2}$.

Construction of Fock representation

In the symmetric group S_{2n} let

$$s = \begin{pmatrix} 1 & 2 & \dots & n & n+1 & n+2 & \dots & 2n \\ n+1 & n+2 & \dots & 2n & 1 & 2 & \dots & n \end{pmatrix}.$$

We have $s = \sigma_1 \dots \sigma_n$, where $\sigma_i = s_{n+i-1} \dots s_i$.

Theorem

$\pi_s \circ \psi \circ \rho$ *is unitarily equivalent to the Fock representation* π_F *and* $\overline{\pi_F(\mathrm{Pol}(\mathrm{Mat}_n)_q)} \subset C^*(S)^{\otimes n^2}$.

Combining this with the faithfulness of the Fock representation gives

Corollary

$\mathrm{Pol}(\mathrm{Mat}_n)_q$ *is isomorphic to the* $*$*-sub-algebra of* $\mathbb{C}[SU_{2n}]_q$ *generated by* $\{t_{(k+n)(j+n)}\}$ *with* $1 \le k, j \le n$.

Lifting representations of $\mathrm{Pol}(\mathrm{Mat}_n)_q$

Theorem (Gisselson)

For every $*$-representation π of $\mathrm{Pol}(\mathrm{Mat}_n)_q$ there is a $*$-representation Π of $\mathbb{C}[SU_{2n}]_q$ s.t

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Generalizes

$$T \in \mathrm{Mat}_n, \|T\| \leq 1 \Rightarrow \begin{bmatrix} T^* & \sqrt{I - T^*T} \\ -\sqrt{I - TT^*} & T \end{bmatrix} \in SU_{2n}.$$

The Fock representation is faithful

Theorem (Gisselson)

The universal enveloping C^*-algebra of $\mathrm{Pol}(\mathrm{Mat}_n)_q$ exists and is isomorphic to $\overline{\pi_F(\mathrm{Pol}(\mathrm{Mat}_n)_q)}(\subset C^*(S)^{\otimes n^2})$.

Let $\mathbf{z} = ((q)^{n-k} z_k^j)_{k,j}$. Vaksman proved $||\pi_F(\mathbf{z})|| \leq 1$.

Shilov boundary

- We shall write $C(\overline{\mathbb{U}}_n)_q$ for for the universal enveloping algebra of $\mathrm{Pol}(\mathrm{Mat}_n)_q$ ($\simeq \overline{\pi_F(\mathrm{Pol}(\mathrm{Mat}_n)_q)}$) for all n.

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- The analog of holomorphic functions on $\mathbb{U}$ is the unital closed subalgebra $\mathcal{A}(\mathbb{U}_n)_q$ of $C(\overline{\mathbb{U}}_n)_q$ generated by "holomorphic coordinates" $\{z_i^j\}_{i,j=1}^n$.

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- "Algebraic" Shilov ideal $J := \ker(j_q)$ for $j_q : \mathrm{Pol}(\mathrm{Mat}_n)_q \to \mathrm{Pol}(\partial\mathbb{U}_n)_q$, $z_j^i \mapsto z_j^i$:
  - $n = 1$: $J = \langle z_1 z_1^* - 1 \rangle$
  - $n > 1$: $J = \langle \sum_{j=1}^n q^{2n-\alpha-\beta} z_j^\alpha (z_j^\beta)^* - \delta^{\alpha,\beta}, \alpha, \beta = 1, 2, \ldots, n \rangle$

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Theorem

The closure $\overline{J}_n$ in $C(\overline{\mathbb{U}}_n)_q$ is the Shilov ideal in the sense of Arveson, i.e.

$$j_q : C(\overline{\mathbb{U}}_n)_q \to C(\overline{\mathbb{U}}_n)_q/\overline{J}_n$$

is a complete isometry when restricted to $\mathcal{A}(\mathbb{U}_n)_q$ and $\overline{J}_n$ is the maximal one with this property.

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Let $\pi_{F,n}$ be the Fock representation of $\mathrm{Pol}(\mathrm{Mat}_n)_q$ and J_n its "algebraic" Shilov boundary ideal. To see that $\overline{J_n}$ is a boundary ideal it is enough to see that $\pi_{F,n}$ is a dilation of a $*$-representation ψ_n that annihilates the ideal J_n, when restricted to $\mathcal{A}(\mathbb{U}_n)_q$, i.e.

$$\pi_{F,n}(a) = P_H \psi_n(a)|_H, \forall a \in \mathcal{A}(\mathbb{U}_n)_q.$$

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In this case we would have

$$\begin{aligned} ||(\pi_{F,n}(a_{ij}))_{i,j}||_{M_k(C(\overline{\mathbb{U}}_n)_q)} &\leq ||((\psi_n(a_{ij}))||_{M_k(B(H_{\psi_n}))} \\ &\leq \|j_q^{(k)}((\pi_{F,n}(a_{ij}))\|_{M_k(C(\overline{\mathbb{U}_n}))_q/\overline{J_n})} \end{aligned}$$

for any $(a_{ij}) \in M_k(\mathcal{A}(\mathbb{U}_n)_q)$. The other direction is clear.

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We will use induction on n.

Sz-Nagy's dilation theorem and Shilov boundary for $n = 1$

Theorem (Sz-Nagy)

Let $T \in B(H)$ with $\|T\| \leq 1$. Then there exists a Hilbert space K, $K \supset H$ and a unitary operator U on K such that

$$T^n = P_H U^n|_H \forall n \geq 0.$$

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hence $p(T) = P_H p(U)|_H$.
Since $\pi : z_1 \mapsto U$ determines a $*$-representation of $\mathrm{Pol}(\mathbb{C})_q$ that annihilates $J = \langle z_1 z_1^* - 1\rangle$, we get

$$\pi_F(a) = P_H \pi(a)|_H, \; a \in \mathcal{A}(\mathbb{U})_q,$$

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- the $*$-homorphism $\Pi_\varphi : \mathrm{Pol}(\mathrm{Mat}_n)_q \to \mathrm{Pol}(\mathrm{Mat}_{n-1})_q$:

$$\Pi_\varphi : z_j^i \mapsto \begin{cases} q^{-1} z_j^i, & i,j < n \\ e^{i\varphi}, & i = j = n, \\ 0, \text{otherwise} \end{cases}$$

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Hence if $\rho \in \mathrm{Rep}(\mathrm{Pol}(\mathrm{Mat}_{n-1})_q)$, $\tau \in \mathrm{Rep}(\mathrm{Pol}(\mathrm{Mat}_n)_q)$, and π_1, $\pi_2 \in \mathrm{Rep}(\mathbb{C}[SU_n]_q)$ then $\rho \circ \Pi_\varphi$ and $(\tau \otimes \pi_1 \otimes \pi_2) \circ \mathcal{D}$ are $*$-representations of $\mathrm{Pol}(\mathrm{Mat}_n)_q$.

Assume by induction that $\pi_{F,n-1}$ is a dilation of a $*$-representation ψ s.t. $\psi(J_{n-1}) = 0$, i.e. $\pi_{F,n-1}(a) = P_H \psi(a)|_H$, $a \in \mathcal{A}(\mathbb{U}_{n-1})_q$.

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- $((\psi \circ \Pi_\varphi) \otimes \pi_\omega \otimes \pi_\omega) \circ \mathcal{D}$, $\omega \in S_n$ annihilates the ideal J_n and hence $(\pi_{F,n-1} \circ \Pi_\varphi) \otimes \pi_\omega \otimes \pi_\omega) \circ \mathcal{D}$ is a dilation of a $*$-representation that annihilates J_n;

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- combining the above steps we get that $\pi_{F,n}$ is a dilation of a $*$-representation that annihilates J_n.

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Hence $\forall a \in \mathcal{A}(\overline{\mathbb{U}_n})_q$

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We show that then
$\|\pi_{F,n}(x) + I\| = \|\pi_{F,n}(x) + \bar{J}_n\|, \forall x \in C(\overline{\mathbb{U}_n})_q \Longrightarrow I = \bar{J}_n.$

As $\mathrm{Pol}(\mathrm{Mat}_n)_q/J_n = \mathbb{C}[U_n]_q$ we have

$$(z_j^i)^* + J_n = (-q)^{i+j-2n}(\det_q \mathbf{z})^{-1} \det_q \mathbf{z}_j^i + J_n \Rightarrow$$

$$\forall x \in \mathrm{Pol}(\mathrm{Mat}_n)_q, \exists k \in \mathbb{N}; (\det_q \mathbf{z})^k x + J_n = a + J_n, a \in \mathcal{A}(\mathrm{Mat}_n)_q.$$

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Recall that $(\det_q \mathbf{z})^* \det_q \mathbf{z} + J_n = q^{-(n(n-1))} 1 + J_n$

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Thus

$$\|\pi_{F,n}(x) + I\| = q^{\frac{k(n(n-1))}{2}} \|(\det_q \mathbf{z})^k x + I\| = q^{\frac{k(n(n-1))}{2}} \|a + I\| =$$

$$q^{\frac{k(n(n-1))}{2}} \|a + \bar{J}_n\| = q^{\frac{k(n(n-1))}{2}} \|(\det_q \mathbf{z})^k x + \bar{J}_n\| = \|\pi_{F,n}(x) + \bar{J}_n\|.$$

THANK YOU!