Semicrossed Products and Reflexivity

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Part A. von Neumann algebras

Preliminaries

• $\mathscr{H}$: Hilbert space, i.e. a $\mathbb{C}$-linear space that is complete wrt an inner product $\langle\cdot,\cdot\rangle$.

Remarks

1. Every Hilbert space attains (Gram-Schmidt) an orthonormal basis $\{e_i\}_{i\in I}$ such that $\langle e_i, e_j\rangle = \delta_{i,j}$ and

$$\sum_{k=1}^{N} \lambda_{i_k} e_{i_k} \text{ is dense in } \mathscr{H}.$$

2. If $V \subset \mathscr{H}$ then

$$V^{\perp} = \{\xi \in \mathscr{H} \mid \langle \xi, v\rangle = 0 \text{ for all } v \in V\}$$

It follows that $\overline{V} = V^{\perp\perp}$. If $V = \overline{V}$ then $\mathscr{H} = V \oplus V^{\perp}$.

3. The inner product induces an identification $\mathscr{H} = \mathscr{H}^*$.

Preliminaries

- $\mathscr{B}(\mathscr{H})$: bounded linear operators $T : \mathscr{H} \to \mathscr{H}$.

Remarks

1. Since $\mathscr{H} = \mathscr{H}^*$, for every $T \in \mathscr{B}(\mathscr{H})$ there exists a $T^* \in \mathscr{B}(\mathscr{H})$ such that

$$\langle T^*\xi, \eta \rangle = \langle \xi, T\eta \rangle .$$

2. If $V \subset_{closed} \mathscr{H}$ then there exists a unique $P \in \mathscr{B}(\mathscr{H})$ such that

$$P(\xi + \eta) = \xi \text{ for all } \xi + \eta \in V \oplus V^\perp.$$

Then P is a *projection*, i.e. $P = P^2 = P^*$. In addition $1 - P$ is also a projection on $V^\perp$.

3. Moreover for such P on V we can write every $T \in \mathscr{B}(\mathscr{H})$ as

$$\begin{aligned} T &= PTP + (1-P)TP + PT(1-P) + (1-P)T(1-P) \\ &= \begin{bmatrix} PTP & PT(1-P) \\ (1-P)TP & (1-P)T(1-P) \end{bmatrix}. \end{aligned}$$

Preliminaries

- $\mathscr{B}(\mathscr{H})$: bounded linear operators $T : \mathscr{H} \to \mathscr{H}$.

Remarks

4. A (closed) $V \subset \mathscr{H}$ is called *invariant* for $T \in \mathscr{B}(\mathscr{H})$ if $TV \subset V$. Then $(1-P)TP = 0$ and so

$$T = \begin{bmatrix} PTP & PT(1-P) \\ 0 & (1-P)T(1-P) \end{bmatrix}.$$

A $V \subset \mathscr{H}$ is called *reducing* if $TV \subset V$ and $T^*V \subset V$, so that

$$T = \begin{bmatrix} PTP & 0 \\ 0 & (1-P)T(1-P) \end{bmatrix}.$$

5. A (closed) $V \subset \mathscr{H}$ is called *invariant* for $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ if $TV \subset V$ for all $T \in \mathscr{M}$. It follows that if $\mathscr{M} = \mathscr{M}^*$ then $\mathscr{M}$-invariant means $\mathscr{M}$-reducing.

$\mathscr{B}(\mathscr{H})$ admits several topologies

1. The norm topology $\|T\| = \sup\{\|T\xi\| \mid \|\xi\| \leq 1\}$.
2. The *strong-operator topology* that makes the following seminorms continuous

$$\xi \in \mathscr{H}, p_\xi(T) := \|T\xi\|.$$

A basis is given by the sets

$$N(T, \xi_1, \ldots, \xi_n, \varepsilon) := \{S \in \mathscr{B}(\mathscr{H}) \mid \|(S - T)\xi_k\| < \varepsilon, \forall k\}.$$

3. The *weak-operator topology* that makes the following seminorms continuous

$$\xi, \eta \in \mathscr{H}, \omega_{\xi,\eta}(T) := \langle T\xi, \eta \rangle.$$

A basis is given by the sets

$$N(T, \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n, \varepsilon) := \{S \in \mathscr{B}(\mathscr{H}) \mid |\langle (S - T)\xi_k, \eta_k \rangle| < \varepsilon, \forall k\}.$$

4. wot $<$ sot $<$ norm topology. Hence being wot-closed implies closed in the sot and norm topology. (Exercise: show they coincide iff $\dim \mathscr{H} < \infty$.)

Bicommutant Theorem

For $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ we write

$$\mathscr{M}' = \{S \in \mathscr{B}(\mathscr{H}) \mid ST = TS \text{ for all } T \in \mathscr{M}\}.$$

It follows that if $V \subset \mathscr{H}$ is reducing for $\mathscr{M}$ then $P_V \in \mathscr{M}'$.

Theorem (FD-version)

Let $\mathscr{H}$ with $\dim \mathscr{H} = N < \infty$ and $\mathscr{M}$ be a subalgebra of $\mathscr{B}(\mathscr{H})$ such that $1 \in \mathscr{M}$ and $\mathscr{M} = \mathscr{M}^*$. Then $\mathscr{M} = \mathscr{M}''$.

Theorem (von Neumann)

Let $\mathscr{M}$ be a subalgebra of $\mathscr{B}(\mathscr{H})$ such that $1 \in \mathscr{M}$ and $\mathscr{M} = \mathscr{M}^*$. Then $\overline{\mathscr{M}}^{\text{sot}} = \mathscr{M}''$.

Bicommutant Theorem

Theorem (FD-version)

Let $\mathscr{H}$ with $\dim \mathscr{H} = N < \infty$ and $\mathscr{M}$ be a subalgebra of $\mathscr{B}(\mathscr{H})$ such that $1 \in \mathscr{M}$ and $\mathscr{M} = \mathscr{M}^*$. Then $\mathscr{M} = \mathscr{M}''$.

Proof.

Obviously $\mathscr{M} \subset \mathscr{M}''$. Conversely fix $\{e_1, \dots, e_N\}$ an onb of $\mathscr{H}$ and define $\mathscr{K} = \oplus_{i=1}^N \mathscr{H}$. Then every $X \in \mathscr{B}(\mathscr{K})$ is written as

$$X = [X_{ij}]_{i,j=1}^N \quad \text{for } X_{ij} = P_i X P_j$$

for P_i the projection on the i-th summand of $\mathscr{K}$. We write

$$S^{(N)} = \begin{bmatrix} S & & \\ & \ddots & \\ & & S \end{bmatrix}$$

for every $S \in \mathscr{B}(\mathscr{H})$.

Bicommutant Theorem

Theorem (FD-version)

Let $\mathscr{H}$ with $\dim \mathscr{H} = N < \infty$ and $\mathscr{M}$ be a subalgebra of $\mathscr{B}(\mathscr{H})$ such that $1 \in \mathscr{M}$ and $\mathscr{M} = \mathscr{M}^*$. Then $\mathscr{M} = \mathscr{M}''$.

Proof cont'd.

We write

$$\mathscr{M}^{(N)} := \{ T^{(N)} \mid T \in \mathscr{M} \}.$$

Then

$$(\mathscr{M}^{(N)})' = \{[X_{ij}] \mid X_{ij} \in \mathscr{M}'\}.$$

and

$$(\mathscr{M}^{(N)})'' = \{ S^{(N)} \mid S \in \mathscr{M}''\}$$

(as you would do for $\mathscr{M} = \mathbb{C}$ and $\mathscr{B}(\mathscr{K}) = \mathscr{M}_N(\mathbb{C})$).

Bicommutant Theorem

Theorem (FD-version)

Let $\mathscr{H}$ with $\dim \mathscr{H} = N < \infty$ and $\mathscr{M}$ be a subalgebra of $\mathscr{B}(\mathscr{H})$ such that $1 \in \mathscr{M}$ and $\mathscr{M} = \mathscr{M}^*$. Then $\mathscr{M} = \mathscr{M}''$.

Proof cont'd.

Let $\xi = [e_1, \cdots, e_N]^t \in \mathscr{K}$ and set (the fd-space)

$$V = \mathscr{M}^{(N)}\xi \subset \mathscr{K}.$$

Then $\mathscr{M}^{(N)}V \subset V$ and so $P_V \in (\mathscr{M}^{(N)})'$. Hence if $S \in \mathscr{M}''$ then $S^{(N)}$ commutes with P_V so that $S^{(N)}V \subset V$. In particular $1 \in \mathscr{M}$ and so

$$S^{(N)}\xi \in \mathscr{M}^{(N)}\xi \quad \Rightarrow \quad \exists\, T^{(N)} \in \mathscr{M}^{(N)} \text{ such that } S^{(N)}\xi = T^{(N)}\xi.$$

Therefore $Se_k = Te_k$ implying that $S = T$. ∎

Bicommutant Theorem

Theorem (von Neumann)

Let $\mathscr{M}$ be a subalgebra of $\mathscr{B}(\mathscr{H})$ such that $1 \in \mathscr{M}$ and $\mathscr{M} = \mathscr{M}^*$. Then $\overline{\mathscr{M}}^{\text{sot}} = \mathscr{M}''$.

Proof.

As commutants are sot-closed we have that $\overline{\mathscr{M}} \subset \mathscr{M}''$. Conversely let $S \in \mathscr{M}''$ and fix a sot-n'hood

$$N(S, \xi_1, \dots, \xi_n, \varepsilon) = \{X \in \mathscr{B}(\mathscr{H}) \mid \|(X - S)\xi_k\| < \varepsilon, k = 1, \dots, n\}.$$

We want to find a $T \in \mathscr{M} \bigcap N$. Now let

$$\xi = [\xi_1, \cdots, \xi_n]^t \in \oplus_{i=1}^n \mathscr{H}$$

and consider $\mathscr{M}^{(n)} \subset \mathscr{B}(\mathscr{H})^{(n)} \subset \mathscr{B}(\mathscr{H}^{(n)})$ as in the previous theorem.

Bicommutant Theorem

Theorem (von Neumann)

Let $\mathscr{M}$ be a subalgebra of $\mathscr{B}(\mathscr{H})$ such that $1 \in \mathscr{M}$ and $\mathscr{M} = \mathscr{M}^*$. Then $\overline{\mathscr{M}}^{\text{sot}} = \mathscr{M}''$.

Proof cont'd.

Now we apply the same arguments but for the closed space

$$V = \overline{\mathscr{M}^{(n)}\xi}.$$

Since $1 \in \mathscr{M}$ then $S^{(n)}\xi \in \overline{\mathscr{M}^{(n)}\xi}$ and therefore there is a $T \in \mathscr{M}$ such that

$$\left\| (S^{(n)} - T^{(n)})\xi \right\|_{\oplus_{i=1}^{n}\mathscr{H}} < \varepsilon.$$

That is, there is a $T \in \mathscr{M}$ such that $\|(S - T)\xi_k\|_{\mathscr{H}} < \varepsilon$ for all $k = 1, \dots, n$.

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von Neumann algebras

Corollary

Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a subalgebra such that $1 \in \mathscr{M}$ and $\mathscr{M} = \mathscr{M}^*$. Then the following are equivalent:

1. $\mathscr{M} = \mathscr{M}''$.
2. $\mathscr{M}$ *is strongly closed.*
3. $\mathscr{M}$ *is weakly closed.*

Proof.

Show that (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). ∎

Definition

A subalgebra satisfying the conditions of the above corollary is called a von Neumann algebra.

von Neumann algebras

Examples

1. Any finite dimensional unital $*$-subalgebra of $\mathscr{B}(\mathscr{H})$.
2. $\mathscr{B}(\mathscr{H})$ itself.
3. Let (X, μ) be a finite measure space and consider the algebra $L^\infty(X, \mu)$ acting on $L^2(X, \mu)$ by multiplication operators. Then $L^\infty(X, \mu)' = L^\infty(X, \mu)$, i.e. it is maximal abelian and thus a von Neumann algebra.

von Neumann algebras

Definition

The *center* of a von Neumann algebra $\mathscr{M}$ is defined by $\mathscr{Z}(\mathscr{M}) := \mathscr{M} \bigcap \mathscr{M}'$.
A von Neumann algebra is called a *factor* if it has trivial center.

Definition

(i) A factor is of *type I* if it contains a minimal projection, i.e. a projection P such that there is no other projection Q with $0 < Q < P$.
(ii) A factor is of *type II* if it does not contain minimal projections but it has finite projections (i.e. there is a projection P that has an inequivalent subprojection Q).
(iii) A factor is of *type III* it is not of type I or II.

von Neumann algebras

Examples

1. $\mathscr{B}(\mathscr{H})$ is a factor.
2. Let $\mathscr{H} = \mathscr{K}_1 \otimes \mathscr{K}_2$ and $\mathscr{M} = \mathscr{B}(\mathscr{K}_1) \otimes 1$. Then $\mathscr{M}' = 1 \otimes \mathscr{B}(\mathscr{K}_2)$ so that $\mathscr{M}$ and $\mathscr{M}'$ are factors.
3. In fact any type I factor $\mathscr{M}$ on a separable Hilbert space $\mathscr{H}$ is unitarily equivalent to some $\mathscr{B}(\mathscr{K}_1) \otimes 1$ inside $\mathscr{B}(\mathscr{K}_1 \otimes \mathscr{K}_2)$.
4. (Spatial) For $\mathscr{M} \subseteq \mathscr{B}(\mathscr{H})$ we have that either the lattice of reducing spaces is $\{(0), \mathscr{H}\}$ or there is a non-trivial reducing V. Hence either $\mathscr{M}' = \mathbb{C}$ so that $\mathscr{M} = \mathscr{M}'' = \mathscr{B}(\mathscr{H})$ or there is a non-trivial projection $p \in \mathscr{M}'$.

There is a rich classification theory of von Neumann algebras and a decomposition in types. For these lectures we require a dichotomy between type I and non-type I.

Part B. Reflexivity.

Reflexive cover

Definition

Let $\mathscr{P}(\mathscr{H})$ denote the set of projections in $\mathscr{B}(\mathscr{H})$. For $\mathscr{S} \subset \mathscr{B}(\mathscr{H})$ we write

$$\operatorname{Lat} \mathscr{S} := \{P \in \mathscr{P}(\mathscr{H}) \mid (1-P)SP = 0 \text{ for all } S \in \mathscr{S}\}.$$

For $\mathscr{L} \subset \mathscr{P}(\mathscr{H})$ we write

$$\operatorname{Alg} \mathscr{L} := \{S \in \mathscr{B}(\mathscr{H}) \mid (1-P)SP = 0 \text{ for all } P \in \mathscr{L}\}.$$

It follows that $\operatorname{Lat} \mathscr{S}$ is a lattice of projections and $\operatorname{Alg} \mathscr{L}$ is a unital (strongly/weakly closed) algebra.

Reflexive cover

Remark

For any unital $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$ we have that

$$\begin{aligned}\operatorname{Alg}\operatorname{Lat}\mathscr{A} &= \{S \in \mathscr{B}(\mathscr{H}) \mid S\xi \in \overline{\mathscr{A}\xi} \text{ for all } \xi \in \mathscr{H}\} \\ &= \{S \in \mathscr{B}(\mathscr{H}) \mid \forall \xi \in \mathscr{H},\ \exists (A_n) \subset \mathscr{A} \text{ s.t. } S\xi = \lim_n A_n\xi\} \\ &= \{S \in \mathscr{B}(\mathscr{H}) \mid \forall \xi, \eta \in \mathscr{H},\ \exists (A_n) \subset \mathscr{A} \text{ s.t. } \langle S\xi, \eta\rangle = \lim_n \langle A_n\xi, \eta\rangle\}.\end{aligned}$$

Definition

A unital subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{H})$ is called *reflexive* if $\mathscr{A} = \operatorname{Alg}\operatorname{Lat}\mathscr{A}$.

Reflexive cover

Examples

1. A closer look at the BCT reveals that a von Neumann algebra is reflexive (as invariant subspaces of selfadjoint algebras are in bijection with projections in the commutant).
2. The algebra

$$\mathscr{A} := \{ \begin{bmatrix} \lambda & \mu \\ 0 & \lambda \end{bmatrix} \mid \lambda, \mu \in \mathbb{C} \}$$

is *not* reflexive. Indeed it can be checked that

$$\operatorname{Lat} \mathscr{A} = \{(0), \mathbb{C}e_1, \mathbb{C}^2\} \quad \text{and} \quad \operatorname{Alg}\operatorname{Lat} \mathscr{A} = \{ \begin{bmatrix} \lambda_1 & \mu \\ 0 & \lambda_2 \end{bmatrix} \mid \lambda_1, \lambda_2, \mu \in \mathbb{C} \}$$

3. Every $\operatorname{Alg}\operatorname{Lat} \mathscr{S}$ for $\mathscr{S} \subset \mathscr{B}(\mathscr{H})$ is reflexive.
4. If $\mathscr{A}$ is a unital subalgebra of $\mathscr{B}(\mathscr{H})$ then $\mathscr{A}^{(\infty)}$ is reflexive.

The Hardy algebra

Definition

Define $L \in \mathscr{B}(\ell^2)$ by $Le_n = e_{n+1}$. The *Hardy algebra* is defined as

$$\mathbb{H}^\infty(\mathbb{D}) := \overline{\text{span}}^{\text{wot}}\{\lambda_n L^n \mid \lambda_n \in \mathbb{C}, n \in \mathbb{N}\}.$$

Theorem (Sarason)

The Hardy algebra is reflexive.

Proof.

We will show this in greater generality but the key-points are:
1. $T \in \mathbb{H}^\infty(\mathbb{D})$ if and only if T is "lower triangular" and has constant diagonals.
2. Every $T \in \operatorname{Alg}\operatorname{Lat}\mathbb{H}^\infty(\mathbb{D})$ is "lower triangular".
3. Moreover it can be approximated à la Féjer by its diagonals, which are again in $\operatorname{Alg}\operatorname{Lat}\mathbb{H}^\infty(\mathbb{D})$.

The Hardy algebra

Proof cont'd.

4. For $T \in \operatorname{Alg}\operatorname{Lat}\mathbb{H}^\infty(\mathbb{D})$ write its m-th diagonal as

$$G_m(T) := L^m \sum_k T_{k+m,k} p_k \text{ with } T_{k+m,k} \in \mathbb{C}.$$

5. Fix $r \in (0,1)$ and notice that the vector $\xi_r = \sum_k r^k e_k$ is an eigenvector of L^*. Thus $\mathbb{C}\xi_r$ is invariant for $G_n(T)^*$ and so there is a $\lambda \in \mathbb{C}$ such that

$$\lambda \xi_r = G_m(T)^* \xi_r = (\sum_k \overline{T_{k+m,k}} p_k)(L^*)^m \xi_r = r^m \sum_k r^k \overline{T_{k+m,k}} e_k.$$

By taking inner product with every e_n we thus have

$$\lambda r^n = r^m r^n \overline{T_{n+m,n}} \quad \Rightarrow \quad T_{n+m,n} = \overline{\lambda} r^{-m} = const(n).$$

Hence all $T_{n+m,n}$ coincide, giving that the diagonals of T are constant. ∎

The free semigroup algebra

Our plan is to make this explicit for the tensor product of any factor with the regular free semigroup algebra.

We write $[d] = \{1, 2, \ldots, d\}$ allowing $d = \infty$. On the Hilbert space

$$\ell^2(\mathbb{F}_+^d) := \{\xi : \mathbb{F}_+^d \to \mathbb{C} \mid \sum_{\mu} |\xi_\mu|^2 < \infty\}.$$

define the left creation operators

$$\mathbf{l}_\mu e_w = e_{\mu w} \text{ for all } w \in \mathbb{F}_+^d.$$

The regular free semigroup algebra is defined by

$$\mathfrak{L}_d := \overline{\operatorname{span}}^{\text{wot}}\{\mathbf{l}_\mu \mid \mu \in \mathbb{F}_+^d\}.$$

Tensoring with the free semigroup algebra

Fix $\mathscr{H}$ and consider $\mathscr{H} \otimes \ell^2(\mathbb{F}_+^d)$. Then $\mathscr{B}(\mathscr{H} \otimes \ell^2(\mathbb{F}_+^d))$ admits a point-wot-continuous action induced by the unitaries

$$U_s \xi \otimes e_w = e^{i|w|s} \xi \otimes e_w \text{ for all } \xi \otimes e_w,$$

with $s \in [-\pi, \pi]$.
For $T \in \mathscr{B}(\mathscr{H} \otimes \ell^2(\mathbb{F}_+^d))$ and $m \in \mathbb{Z}_+$ *the m-th Fourier coefficient* is then given by

$$G_m(T) := \frac{1}{2\pi} \int_{-\pi}^{\pi} U_s T U_s^* e^{-ims} ds$$

where the integral is considered in the wot-topology of $\mathscr{B}(\mathscr{H} \otimes \ell^2(\mathbb{F}_+^d))$ for the Riemann sums.
An application of Fejér's Lemma implies that the Cesaro sums

$$\sigma_{n+1}(T) := \sum_{m=-n}^{n} (1 - \frac{|m|}{n+1}) G_m(T)$$

converge to T in the wot-topology.

Tensoring with the free semigroup algebra

For $T \in \mathscr{B}(\mathscr{H} \otimes \ell^2(\mathbb{F}_+^d))$ we write $T_{\mu,\nu} \in \mathscr{B}(\mathscr{H})$ for the (μ,ν)-entry given by

$$\langle T_{\mu,\nu}\xi, \eta \rangle = \langle T\xi \otimes e_\nu, \eta \otimes e_\mu \rangle \text{ for all } \xi, \eta \in \mathscr{H}.$$

We endow the free semigroup $\mathbb{F}_+^d$ with the partial order given by

$$\mu \geq \nu \text{ if there exists } z \in \mathbb{F}_+^d \text{ such that } \mu = z\nu.$$

Definition

An operator $T \in \mathscr{B}(\mathscr{H} \otimes \ell^2(\mathbb{F}_+^d))$ is a *left lower triangular operator* if $T_{\mu,\nu} = 0$ whenever $\mu \not\geq \nu$.

Tensoring with the free semigroup algebra

The next proposition shows that the Fourier co-efficients induce a graded structure on lower triangular operators. For $\mu, \nu \in \mathbb{F}_+^d$ we write

$$L_\mu := I_{\mathscr{H}} \otimes \mathbf{l}_\mu .$$

From now on we write p_w for the projection of $\ell^2(\mathbb{F}_+^d)$ to e_w.

Proposition

If T is a left lower triangular operator in $\mathscr{B}(\mathscr{H} \otimes \ell^2(\mathbb{F}_+^d))$ then

$$G_m(T) = \begin{cases} \sum_{|\mu|=m} \sum_{w \in \mathbb{F}_+^d} L_\mu (T_{\mu w, w} \otimes p_w) & \text{if } m \geq 0, \\ 0 & \text{if } m < 0. \end{cases}$$

Tensoring with the free semigroup algebra

Proof.

Fix $v, v' \in \mathbb{F}_+^d$ and $\xi, \eta \in \mathscr{H}$. Then we have that

$$\langle G_m(T)\xi \otimes e_v, \eta \otimes e_{v'} \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \langle T\xi \otimes e_v, \eta \otimes e_{v'} \rangle \, e^{i(-m-|v|+|v'|)s} ds$$
$$= \delta_{|v'|, m+|v|} \left\langle T_{v',v}\xi, \eta \right\rangle$$

for all $T \in \mathscr{B}(\mathscr{H} \otimes \ell^2(\mathbb{F}_+^d))$. Hence

$$\langle G_m(T)\xi \otimes e_v, \eta \otimes e_{v'} \rangle = 0 \text{ when } |v'| \neq m + |v|.$$

Suppose that T is in addition a left lower triangular operator.
First consider the case where $m < 0$. If $|v'| = m + |v|$ then $|v'| < |v|$ and thus $v' \not\geq v$. But then $\left\langle T_{v',v}\xi, \eta \right\rangle = 0$ since T is left lower triangular.
Hence $G_m(T) = 0$ when $m < 0$.

Tensoring with the free semigroup algebra

Proof cont'd.

Secondly for $m \geq 0$ we have that $\left\langle T_{\nu',\nu}\xi, \eta \right\rangle = 0$ whenever $\nu' \not\geq \nu$. So

$$\langle G_m(T)\xi \otimes e_\nu, \eta \otimes e_{\nu'} \rangle = \begin{cases} \left\langle T_{\nu',\nu}\xi, \eta \right\rangle & \text{if } \nu' \geq \nu \text{ and } |\nu'| - |\nu| = m, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand we compute

$$\sum_{|\mu|=m} \sum_{w \in \mathbb{F}_+^d} \left\langle L_\mu(T_{\mu w, w} \otimes p_w)\xi \otimes e_\nu, \eta \otimes e_{\nu'} \right\rangle = \sum_{|\mu|=m} \delta_{\mu\nu,\nu'} \left\langle T_{\mu\nu,\nu}\xi, \eta \right\rangle =$$

$$= \begin{cases} \left\langle T_{\nu',\nu}\xi, \eta \right\rangle & \text{if } \nu' = \mu\nu \text{ and } |\mu| = m, \\ 0 & \text{otherwise.} \end{cases}$$

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Tensoring with the free semigroup algebra

Define the subalgebra of $\mathscr{B}(\mathscr{H} \otimes \ell^2(\mathbb{F}_+^d))$

$$\mathscr{B}(\mathscr{H}) \mathbin{\overline{\otimes}} \mathscr{L}_d := \overline{\operatorname{span}}^{\mathrm{wot}}\{T \otimes \mathbf{l}_\mu \mid T \in \mathscr{B}(\mathscr{H}), \mu \in \mathbb{F}_+^d\}.$$

The following is a result of Arias-Popescu. We will see a proof à la Sarason.

Proposition (Arias-Popescu 1995)

$\mathscr{B}(\mathscr{H}) \mathbin{\overline{\otimes}} \mathscr{L}_d$ is reflexive.

Proof.

Since the gauge action of $\mathscr{B}(\mathscr{H} \otimes \ell^2(\mathbb{F}_+^d))$ restricts to a gauge action of $\mathscr{B}(\mathscr{H}) \mathbin{\overline{\otimes}} \mathscr{L}_d$, it suffices to show that every $G_m(T)$ is in $\mathscr{B}(\mathscr{H}) \mathbin{\overline{\otimes}} \mathscr{L}_d$ whenever T is in $\operatorname{Alg}\operatorname{Lat}(\mathscr{B}(\mathscr{H}) \mathbin{\overline{\otimes}} \mathscr{L}_d)$.

Tensoring with the free semigroup algebra

Proof cont'd.

For $\xi, \eta \in \mathscr{H}$ and $\nu, \mu \in \mathbb{F}_+^d$ there exist $X_n \in \mathscr{B}(\mathscr{H}) \overline{\otimes} \mathscr{L}_d$ such that

$$\begin{aligned} \langle T_{\mu,\nu}\xi, \eta \rangle &= \langle T\xi \otimes e_\nu, \eta \otimes e_\mu \rangle \\ &= \lim_n \langle X_n \xi \otimes e_\nu, \eta \otimes e_\mu \rangle = \lim_n \langle [X_n]_{\mu,\nu}\xi, \eta \rangle \end{aligned}$$

Taking $\mu \not> \nu$ gives that T is left lower triangular as every X_n is so. Therefore it suffices to show that $T_{\mu z, z} = T_{\mu,\emptyset}$ for all $z \in \mathbb{F}_+^d$. Indeed, when this holds, we can write

$$G_m(T) = \begin{cases} \sum_{|\mu|=m} L_\mu (T_{\mu,\emptyset} \otimes I) & \text{if } m \geq 0, \\ 0 & \text{if } m < 0, \end{cases}$$

and thus $G_m(T) \in \mathscr{B}(\mathscr{H}) \overline{\otimes} \mathscr{L}_d$.

Tensoring with the free semigroup algebra

Proof cont'd.

For convenience we use the notation

$$T_{(\mu)} := L_\mu^* G_m(T) = \sum_{w \in \mathbb{F}_+^d} T_{\mu w, w} \otimes p_w .$$

We treat the cases $m = 0$ and $m \geq 1$ separately. For $\lambda \in \mathbb{B}_d$ and $w = w_m \dots w_1 \in \mathbb{F}_+^d$ we write

$$w(\lambda) = \lambda_{w_m} \cdots \lambda_{w_1} .$$

We set $w(\lambda) = 1$ when $w = \emptyset$.

Tensoring with the free semigroup algebra

Proof cont'd.

• **The case** $m = 0$**.** Let $z \in \mathbb{F}_+^d$ and assume that $\{z_1, \dots, z_{|z|}\} \subseteq [d']$ for some finite d'. If $d < \infty$ then take $d' = d$. Let $\lambda \in \mathbb{B}_{d'} \subseteq \mathbb{B}_d$ such that $\lambda_i \neq 0$ for every $i \in [d']$, and consider the vector

$$g := \sum_{w \in \mathbb{F}_+^{d'}} w(\lambda) e_w .$$

As g is an eigenvector for $\mathscr{L}_d^*$ we have that

$$(L_\mu(x \otimes I))^* \xi \otimes g \in \overline{\{y\xi \otimes g \mid y \in \mathscr{B}(\mathscr{H})\}}.$$

Therefore for $\xi \in \mathscr{H}$ there exists a sequence (x_n) in $\mathscr{B}(\mathscr{H})$ such that

$$G_0(T)^* \xi \otimes g = \lim_n x_n^* \xi \otimes g. \tag{1}$$

Tensoring with the free semigroup algebra

Proof cont'd.

Hence for $\eta \in \mathscr{H}$ we take the inner product with $\eta \otimes e_w$ to get

$$\begin{aligned} w(\lambda) \langle \xi, T_{w,w} \eta \rangle &= \langle \xi, T_{w,w} \eta \rangle \langle g, e_w \rangle \\ &= \langle G_0(T)^* \xi \otimes g, \eta \otimes e_w \rangle \overset{(1)}{=} \lim_n \langle x_n^* \xi \otimes g, \eta \otimes e_w \rangle \\ &= \lim_n \langle \xi, x_n \eta \rangle \langle g, e_w \rangle \\ &= w(\lambda) \lim_n \langle \xi, x_n \eta \rangle . \end{aligned}$$

Applying for $w = \emptyset$ and $w = z$ we have that $T_{z,z} = T_{\emptyset,\emptyset}$ as $z(\lambda) \neq 0$. Since z was arbitrary we have that $G_0(T) = T_{\emptyset,\emptyset} \otimes I$.

Tensoring with the free semigroup algebra

Proof cont'd.

- **The case $m \geq 1$.** Fix $|\mu| = m$. We have to show that

$$T_{\mu z, z} = T_{\mu, \emptyset} \text{ for all } z \in \mathbb{F}_+^d.$$

Notice that μ can be written as

$$\mu = q i^{\omega} \text{ for some } i \in [d] \text{ and } \omega \geq 1.$$

By symmetry it suffices to treat just the case where $i = 1$. Hence we fix

$$\mu = q 1^{\omega} \quad \text{with} \quad \omega \geq 1 \quad \text{and} \quad q = q_{|q|} \dots q_1 \text{ with } q_1 \neq 1 \text{ or } q = \emptyset.$$

Tensoring with the free semigroup algebra

Proof cont'd.

- For $T_{\mu 1,1} = T_{\mu,\emptyset}$: Fix an $r \in (0,1)$. Let the vectors

$v := e_\emptyset + \sum_{k=1}^\infty r^k e_{1^k}$ and $\mathbf{l}_{q(t)} v = e_{q(t)} + \sum_{k=1}^\infty r^k e_{q(t)1^k}$ for $t = 1, \dots, |q|$

where

$$q(t) = q_t \dots q_1 \quad \text{for } t = 1, \dots, |q|.$$

If $q = \emptyset$ then set $\mathbf{l}_{q(t)} v = 0$. Fix $\xi \in \mathscr{H}$. As v is an eigenvector for $\mathscr{L}_d^*$:

$$X^*(\xi \otimes \mathbf{l}_q v) \in \{ x\xi \otimes v + \sum_{t=1}^{|q|} x_t \xi \otimes \mathbf{l}_{q(t)} v \mid x, x_t \in \mathscr{B}(\mathscr{H}), t = 1, \dots, |q| \}^-$$

for all $X \in \mathscr{B}(\mathscr{H}) \overline{\otimes} \mathscr{L}_d$. Hence there are (x_n) and $(x_{t,n})$ in $\mathscr{B}(\mathscr{H})$ s.t.

$$G_m(T)^* \xi \otimes \mathbf{l}_q v = \lim_n x_n^* \xi \otimes v + \sum_{t=1}^{|q|} x_{t,n}^* \xi \otimes \mathbf{l}_{q(t)} v. \tag{2}$$

Proof cont'd.

Notice that

$$\text{if } |\mu'| = m \text{ then } (\mathbf{l}_{\mu'})^* \mathbf{l}_q v = \delta_{\mu',\mu} r^\omega v.$$

Thus for all $\eta \in \mathscr{H}$ and $z \in \mathbb{F}_+^d$ we get that

$$\langle G_m(T)^* \xi \otimes \mathbf{l}_q v, \eta \otimes e_z \rangle = r^\omega \langle \xi, T_{q1^\omega z,z} \eta \rangle \langle v, e_z \rangle .$$

On the other hand every $\mathbf{l}_{q(t)} v$ is supported on $q(t)1^k$ with $|q(t)1^k| \geq t \geq 1$ and so

$$\left\langle \mathbf{l}_{q(t)} v, e_\emptyset \right\rangle = 0 \text{ for all } t.$$

By taking the inner product with $\eta \otimes e_\emptyset$ in equation (2) we get

$$r^\omega \langle \xi, T_{q1^\omega,\emptyset} \eta \rangle = \lim_n \langle \xi, x_n \eta \rangle .$$

Tensoring with the free semigroup algebra

Proof cont'd.

Now the only vector of length 1 in the support of $\mathbf{I}_{q(t)}v$ is achieved when $t = 1$ and $k = 0$, in which case it is $q(1) \neq 1$ by assumption. Hence taking inner product with $\eta \otimes e_1$ in (2) gives

$$r^{\omega+1} \langle \xi, T_{q1^\omega 1,1} \eta \rangle = \lim_n r \langle \xi, x_n \eta \rangle .$$

Therefore

$$\langle \xi, T_{q1^\omega 1,1} \eta \rangle = \lim_n r^{-\omega} \langle \xi, x_n \eta \rangle = \langle \xi, T_{q1^\omega,\emptyset} \eta \rangle$$

which implies that $T_{\mu 1,1} = T_{\mu,\emptyset}$ when $q \neq \emptyset$.

Tensoring with the free semigroup algebra

Proof cont'd.

- For $T_{\mu 2,2} = T_{\mu,\emptyset}$: Now let the vectors

$$w = e_\emptyset + \sum_{k=1}^{\infty} r^k e_{2^k} \quad \text{and} \quad \mathbf{l}_{\mu(s)} w = e_{\mu(s)} + \sum_{k=1}^{\infty} r^k e_{\mu(s)2^k} \text{ for } s = 1, \dots, m.$$

As above, for $\xi \in \mathscr{H}$ there are sequences (y_n) and $(y_{s,n})$ in $\mathscr{B}(\mathscr{H})$ such that

$$G_m(T)^* \xi \otimes \mathbf{l}_\mu w = \lim_n y_n^* \xi \otimes w + \sum_{s=1}^{m} y_{s,n}^* \xi \otimes \mathbf{l}_{\mu(s)} w \tag{3}$$

since w is an eigenvector of $\mathscr{L}_d^*$. Now for $\eta \in \mathscr{H}$ and $z \in \mathbb{F}_+^d$ we get

$$\left\langle G_m(T)^* \xi \otimes \mathbf{l}_\mu w, \eta \otimes e_z \right\rangle = \left\langle \xi, T_{\mu z, z} \eta \right\rangle \left\langle w, e_z \right\rangle.$$

Tensoring with the free semigroup algebra

Proof cont'd.

For $z = \emptyset$ we have that $\left\langle \mathbf{l}_{\mu(s)} w, e_\emptyset \right\rangle = 0$ for all $s \in [m]$ and so (3) gives

$$\left\langle \xi, T_{\mu,\emptyset} \eta \right\rangle = \lim_n \langle \xi, y_n \eta \rangle .$$

For $z = 2$ we have that $\left\langle \mathbf{l}_{\mu(1)} w, e_2 \right\rangle = \langle \mathbf{l}_1 w, e_2 \rangle = 0$. Moreover we have that $\left\langle \mathbf{l}_{\mu(s)} w, e_2 \right\rangle = 0$ when $s \geq 2$. Therefore equation (3) gives

$$r \langle \xi, T_{q 1^\omega 2, 2} e_2 \rangle = \lim_n r \langle \xi, y_n \eta \rangle .$$

Hence $\left\langle \xi, T_{\mu 2,2} e_2 \right\rangle = \left\langle \xi, T_{\mu,\emptyset} \eta \right\rangle$, and thus $T_{\mu 2,2} = T_{\mu,\emptyset}$.
Applying for $i \in \{3, \dots, d\}$ yields $T_{\mu i,i} = T_{\mu,\emptyset}$ for all $i \in [d]$.

Tensoring with the free semigroup algebra

Proof cont'd.

- Inductive hypothesis: Assume that $T_{q1^\omega z,z} = T_{q1^\omega,\emptyset}$ when $|z| \leq N$. We have to show that the same is true for words of length $N+1$. Consider the word iz with $|z| = N$. Then we apply the same arguments for the vectors

$$v_z := e_z + \sum_{k=1}^{\infty} r^k e_{i^k z}.$$

Then we get $T_{\mu iz,iz} = T_{\mu z,z}$ which is $T_{\mu,\emptyset}$ from the inductive hypothesis.

■

Part C. Semicrossed Products.

Endomorphisms

Let $\mathscr{M}$ be a von Neumann algebra and define $\mathrm{End}(\mathscr{M})$ the wot- wot-continuous unital endomorphisms of $\mathscr{M}$.

Remarks

1. As $\mathscr{M}$ is a C*-algebra then every $\alpha \in \mathrm{End}(\mathscr{M})$ is a contractive.
2. If $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ is a factor then every α is injective (and isometric). Indeed if $\mathscr{I} \lhd \mathscr{M}$ is a wot-closed ideal then $\mathscr{I}$ acts non-degenerately on $[\mathscr{I}\mathscr{H}]$ and thus it is a von Neumann algebra there. The projection P on $[\mathscr{I}\mathscr{H}]$ is in $\mathscr{I} \subset \mathscr{M}$ and $\mathscr{I} = \mathscr{M}P$. Moreover $[\mathscr{I}\mathscr{H}]$ is $\mathscr{M}$-invariant and thus P is in $\mathscr{M}'$. Hence $P = 1_{\mathscr{H}}$ and so $\mathscr{I} = \mathscr{M}$.
3. If $\mathscr{M} \neq \mathscr{B}(\mathscr{H})$ then there is a non-trivial projection $P \in \mathscr{M}'$ on a subspace V. Then for $\alpha \in \mathrm{End}(\mathscr{M})$ we have that the maps

$$x \mapsto \alpha(x)|_V \quad \text{and} \quad x \mapsto \alpha(x)|_{V^\perp}$$

are $*$-representations. If $\mathscr{M}$ is not of type I then they are injective.

Endomorphisms

Remarks

4. (Arveson) If $\mathscr{M} = \mathscr{B}(\mathscr{H})$ then there are $\{s_j\}_{j\in[n]}$ such that

$$\alpha(x) = \sum_{j\in[n]} s_j x s_j^* \quad \text{and} \quad s_i^* s_j = \delta_{i,j}, \sum_{j\in[n]} s_j s_j^* = 1.$$

Indeed $\alpha\colon \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ is a $*$-representation of $\mathscr{B}(\mathscr{H})$ and so it decomposes into the sum of irreducible representations, i.e.

$$\alpha = \oplus_j \alpha_j \text{ with } \alpha_j\colon \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H}_j) \text{ such that } \oplus_j \mathscr{H}_j = \mathscr{H}.$$

However $\alpha_j \simeq \mathrm{id}$ and so there is a unitary $s_j\colon \mathscr{H} \to \mathscr{H}_j$ such that

$$\alpha_j(x) = s_j x s_j^*.$$

Seeing $s_j\colon \mathscr{H} \to \mathscr{H}$ as an isometry completes the proof.

Semicrossed Products

From now on we fix d $*$-endomorphisms $\{\alpha_1, \dots, \alpha_d\}$ on $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$. For a word $w = w_k \dots w_1 \in \mathbb{F}_+^d$ we write

$$\overline{w} = w_1 w_2 \dots w_k \text{ for the reversed word.}$$

On $\mathscr{H} \otimes \ell^2(\mathbb{F}_+^d)$ we define the diagonal $*$-representation

$$\pi \colon \mathscr{M} \to \mathscr{B}(\mathscr{H} \otimes \ell^2(\mathbb{F}_+^d)) \text{ such that } \pi(x)\xi \otimes e_w = \alpha_{\overline{w}}(x)\xi \otimes e_w.$$

By considering the reversed words now we can check that

$$\pi(x) L_i \xi \otimes e_w = \alpha_{\overline{iw}}(x)\xi \otimes e_{iw} = \alpha_{\overline{w}}\alpha_i(x)\xi \otimes e_{iw} = L_i \pi(\alpha_i(x))\xi \otimes e_w.$$

These are called *covariant relations*.

Definition

For $\alpha_1, \dots, \alpha_d \in \operatorname{End}(\mathscr{M})$ we write

$$\mathscr{M} \overline{\times}_\alpha \mathscr{L}_d := \overline{\operatorname{span}}^{\text{wot}} \{ L_\mu \pi(x) \mid \mu \in \mathbb{F}_+^d, x \in \mathscr{M} \}$$

for the *semicrossed product* algebra.

Semicrossed Products

Theorem

Let $\mathscr{M}$ be a factor on a separable Hilbert space $\mathscr{H}$. Then $\mathscr{M} \overline{\times}_\alpha \mathscr{L}_d$ is reflexive.

Proof.

Case 1. (Bickerton-Kakariadis 2017) Let $\mathscr{M} = \mathscr{B}(\mathscr{H})$ and suppose that

$$\alpha_i \simeq \{s_{i,j}\}_{j \in [n_i]} \text{ such that } \alpha_i(x) = \sum_{j \in [n_i]} s_{i,j} x s_{i,j}^*.$$

Let the *multiplicity* of the system be given by

$$N = n_1 + \cdots + n_d.$$

We aim to show that $\mathscr{B}(\mathscr{H}) \overline{\times}_\alpha \mathscr{L}_d$ is unitarily equivalent to $\mathscr{B}(\mathscr{H}) \overline{\otimes} \mathscr{L}_N$.

Semicrossed Products

Proof cont'd.

To this end let

$$U : \mathscr{H} \otimes \ell^2(\mathbb{F}_+^N) \to \mathscr{H} \otimes \ell^2(\mathbb{F}_+^d)$$

by $U\xi \otimes e_\emptyset = \xi \otimes e_\emptyset$ and

$$U\xi \otimes e_{(\mu_k, j_k)\dots(\mu_1, j_1)} = s_{\mu_1, j_1} \cdots s_{\mu_k, j_k} \xi \otimes e_{\mu_k \dots \mu_1}.$$

For words of length k we define the spaces

$$\mathscr{K}_k := \overline{\operatorname{span}}\{\xi \otimes e_{(\mu_k, j_k)\dots(\mu_1, j_1)} \mid \xi \in \mathscr{H}, (\mu_i, j_i) \in [d] \times [n_{\mu_i}]\}.$$

The ranges of $\mathscr{K}_k$ under U are orthogonal and thus U is a unitary with $U^* \xi \otimes e_\emptyset = \xi \otimes e_\emptyset$ and

$$U^* \xi \otimes e_{\mu_k \dots \mu_1} = \sum_{j_1 \in [n_{\mu_1}]} \cdots \sum_{j_k \in [n_{\mu_k}]} s_{\mu_k, j_k}^* \cdots s_{\mu_1, j_1}^* \xi \otimes e_{(\mu_k, j_k)\dots(\mu_1, j_1)}.$$

Semicrossed Products

Proof cont'd.

For $x \in \mathscr{B}(\mathscr{H})$ we obtain

$$\begin{aligned} \pi(x) U \xi \otimes e_{(\mu_k, j_k) \dots (\mu_1, j_1)} &= \alpha_{\mu_1} \cdots \alpha_{\mu_k}(x) s_{\mu_1, j_1} \cdots s_{\mu_k, j_k} \xi \otimes e_{\mu_k \dots \mu_1} \\ &= s_{\mu_1, j_1} \cdots s_{\mu_k, j_k} x \xi \otimes e_{\mu_k \dots \mu_1} \\ &= U(x \otimes 1) \xi \otimes e_{(\mu_k, j_k) \dots (\mu_1, j_1)} \end{aligned}$$

where we used that

$$\alpha_{\mu_i}(x) s_{\mu_i, j_i} = \sum_j s_{\mu_i, j} x s^*_{\mu_i, j} s_{\mu_i, j_i} = s_{\mu_i, j_i} x.$$

This shows that

$$U^* \pi(x) U = x \otimes 1.$$

Semicrossed Products

Proof cont'd.

On the other hand we have that

$$L_i U \xi \otimes e_{(\mu_k, j_k)\dots(\mu_1, j_1)} = s_{\mu_1, j_1} \cdots s_{\mu_k, j_k} \xi \otimes e_{i \mu_k \dots \mu_1}$$

whereas

$$\begin{aligned} U \sum_{j_i \in [n_i]} L_{i,j_i}(s_{i,j_i}^* \otimes 1) \xi \otimes e_{(\mu_k, j_k)\dots(\mu_1, j_1)} &= \sum_{j_i \in [n_i]} s_{\mu_1, j_1} \dots s_{\mu_k, j_k} s_{i,j_i} s_{i,j_i}^* \xi \otimes e_{i \mu_k \dots \mu_1} \\ &= s_{\mu_1, j_1} \dots s_{\mu_k, j_k} \xi \otimes e_{i \mu_k \dots \mu_1}. \end{aligned}$$

Hence we obtain that

$$U^* L_i U = \sum_{j_i \in [n_i]} L_{i,j_i}(s_{i,j_i}^* \otimes 1) \text{ for all } i \in [d].$$

Semicrossed Products

Proof cont'd.

Therefore the generators of $\mathscr{B}(\mathscr{H})\overline{\times}_\alpha\mathscr{L}_d$ are mapped into $\mathscr{B}(\mathscr{H})\overline{\otimes}\mathbb{F}_+^N$. We need to show that the elements $x\otimes 1$ and U^*L_iU also generate the elements

$$L_{i,j_i} \text{ for all } (i,j_i)\in([d],[n_i]).$$

Since every s_{i,j_i} is in $\mathscr{B}(\mathscr{H})$ we have that

$$U^*L_iU(s_{i,j_i}\otimes 1)=\sum_{j_i\in[n_i]} L_{i,j_i}(s_{i,j_i}^*\otimes 1)(s_{i,j_i}\otimes 1)=L_{i,j_i}$$

and the proof is complete.

Proof cont'd.

Case 2. (Helmer 2015) Suppose that $\mathscr{M} \neq \mathscr{B}(\mathscr{H})$. Fix an operator $T \in \operatorname{Alg}\operatorname{Lat}(\mathscr{M} \overline{\times}_\alpha \mathscr{L}_d)$. We will show that $G_m(T) \in \mathscr{M} \overline{\times}_\alpha \mathscr{L}_d$ for all $m \in \mathbb{Z}$. Recall that

$$\operatorname{Alg}\operatorname{Lat}(\mathscr{A}) = \operatorname{Ref}(\mathscr{A}) := \{ T \in \mathscr{B}(\mathscr{K}) \mid T\xi \in \overline{\mathscr{A}\xi} \text{ for all } \xi \in \mathscr{K} \}$$

for every unital algebra $\mathscr{A} \subset \mathscr{B}(\mathscr{K})$.
As before T has to be a lower triangular operator, i.e. $G_m(T) = 0$ if $m < 0$. If $m \geq 0$ then $T_{\mu,\emptyset} \in \mathscr{M}$ by the same arguments and it suffices to show that

$$T_{\mu\nu,\nu} = \alpha_{\overline{\nu}}(T_{\mu,\emptyset}) \text{ for all } \nu \in \mathbb{F}_+^d.$$

Semicrossed Products

Proof cont'd.

By assumption let V a non-trivial subspace such that $P_V \in \mathscr{M}'$. We henceforth fix the word $v \in \mathbb{F}_+^d$ and we define the subspaces of $\mathscr{K}$

$$\mathscr{K}_0 := \overline{\operatorname{span}}\{\xi \otimes e_w \mid \xi \in V, w \in \mathbb{F}_+^d\}$$

and

$$\mathscr{K}_v := \overline{\operatorname{span}}\{\eta \otimes e_{wv} \mid \eta \in V^{\perp}, w \in \mathbb{F}_+^d\}.$$

Both $\mathscr{K}_0$ and $\mathscr{K}_v$ are invariant subspaces of $\mathscr{M} \overline{\times}_\alpha \mathscr{L}_d$.
Let P is the projection on $\mathscr{K}_0 \oplus K_v$; then we have

$$G_m(T)P \in \operatorname{Ref}((\mathscr{M} \overline{\times}_\alpha \mathscr{L}_d)P).$$

We will use the unitary

$$U\colon P(\mathscr{K}) \to \mathscr{K} : \xi \otimes e_w + \eta \otimes e_{wv} \mapsto (\xi + \eta) \otimes e_w.$$

Proof cont'd.

A straightforward computation shows that

$$U\pi(x)PU^* = \sum_{w\in\mathbb{F}_+^d} (\alpha_w(x)|_V + \alpha_{w\overline{v}}(x)|_{V^\perp}) \otimes p_w$$

and that $UL_i pU^* = L_i$.
In particular P is reducing for L_i and we get

$$UG_m(T)PU^* = \sum_{|\mu|=m} \sum_{w\in\mathbb{F}_+^d} L_\mu(T_{\mu w,w}|_V + T_{\mu wv,wv}|_{V^\perp}) \otimes p_w.$$

Proof cont'd.

By taking compressions we thus have that the $(\overline{\mu},\emptyset)$-entry of the operator $UG_m(T)pU^*$ is in the reflexive cover of the $(\overline{\mu},\emptyset)$-block of the algebra $\operatorname{Ref}(U(\mathscr{M}\overline{\times}_\alpha \mathscr{L}_d)PU^*)$. However the latter coincides with

$$\left\{ \begin{bmatrix} \alpha_{\overline{v}}(x)|_V & \\ & \alpha_{\overline{v}}(x)|_{V^\perp} \end{bmatrix} \mid x \in \mathscr{M} \right\}$$

Hence there is a $x \in \mathscr{M}$ such that

$$T_{\mu,\emptyset}|_V + T_{\mu v,v}|_{V^\perp} = x|_V + \alpha_{\overline{v}}(x)|_{V^\perp}.$$

Since the restrictions to V and $V^\perp$ are injective we derive that

$$T_{\mu,\emptyset} = x \quad \text{and} \quad T_{\mu v,v} = \alpha_{\overline{v}}(x) = \alpha_{\overline{v}}(T_{\mu,\emptyset}),$$

which completes the proof. ∎