From Spectral Synthesis to Harmonic Operators

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Classical spectral synthesis

Example: $G = \widehat{\mathbb{R}}$.

$$A(\widehat{\mathbb{R}}) := \{\hat{f} : f \in L^1(\mathbb{R})\} \subseteq C_0(\widehat{\mathbb{R}})$$

This is a (selfadjoint) algebra under pointwise operations and is complete in the norm $\|\hat{f}\|_A = \|f\|_1$.

$$\begin{array}{ccc} L^1(\mathbb{R}) & \xrightarrow{\text{dual}} & L^\infty(\mathbb{R}) \\ \mathcal{F}\downarrow & & \downarrow \\ A(\widehat{\mathbb{R}}) & \xrightarrow{\text{dual}} & A(\widehat{\mathbb{R}})^* \end{array}$$

(Notice that since $L^\infty(\mathbb{R})$ is w*-generated by exponentials $e_x(t) = e^{ixt},\ x \in \mathbb{R}$, the space $A(\widehat{\mathbb{R}})^*$ is w*-generated by evaluations (characters) δ_x given by $\left\langle \delta_x, \hat{f} \right\rangle = \hat{f}(x)$.)

Now consider any locally compact abelian group G in place of $\widehat{\mathbb{R}}$.

Spectral synthesis

Given a closed set $E \subseteq G$, we say that a $\tau \in A(G)^*$ is *supported in E* if $\langle \tau, g \rangle = 0$ for all $g \in A(G)$ whose closed support $\operatorname{supp} g := \overline{\{x \in G : g(x) \neq 0\}}$ is (compact and) disjoint from E.

We say that E is *a set of synthesis (S-set)* if every $\tau \in A(G)^*$ which is supported in E is 'synthesisable' from characters in E:

$$E \text{ S-set}: \quad \operatorname{supp} \tau \subseteq E \;\Rightarrow\; \tau \in \overline{\operatorname{span}\{\delta_x : x \in E\}}^{w*}.$$

Equivalently (Hahn-Banach), E is a set of synthesis if, for $\tau \in A(G)^*$ and $g \in A(G)$,

$$\operatorname{supp} \tau \subseteq E \subseteq \operatorname{null} g \;\Rightarrow\; \langle \tau, g \rangle = 0$$

(here $\operatorname{null} g := \{x \in G : g(x) = 0\}$).

It was discovered by L. Schwartz (1948) that the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ does not satisfy synthesis.

Spectral synthesis

Formulation in terms of $A(G)$: Given a closed set $E \subseteq G$, we consider

$$I(E) = \{g \in A(G) : g|_E = 0\} \lhd A(G).$$

$$J(E) = \overline{\{g \in A_c(G) : \operatorname{supp} g \cap E = \emptyset\}}.$$

(here, $g \in A_c(G)$ means $\operatorname{supp} g$ is compact). Then:

$$E \text{ S-set} \iff J(E) = I(E).$$

Support of an operator

An operator $S \in \mathcal{B}(\ell^2(\mathbb{Z}))$ vanishes on a rectangle $A \times B \subseteq \mathbb{Z} \times \mathbb{Z}$ if the matrix entries $s_{i,j} = \big(Se_j, e_i\big)$ of S are 0 for all $(j, i) \in A \times B$.
This is equivalent to $P(B)SP(A) = 0$, where $P(A)$ is the projection onto the space spanned by the basis elements $\{e_j : j \in A\}$.

S is supported in a set $\Omega \subseteq \mathbb{Z} \times \mathbb{Z}$ iff it vanishes in Ω^c, i.e.

$$A \times B \subseteq \Omega^c \quad \Rightarrow \quad P(B)SP(A) = 0.$$

Laurent operators and invariance

An $S \in \mathcal{B}(\ell^2(\mathbb{Z}))$ is called a Laurent operator if it is constant along diagonals, i.e. $s_{i,j} = s_{i+k,j+k}$ for all k.
(Invariant under the action of $\mathbb{Z}$)

Then $s_{i,j} = f(j - i)$ for some single-variable f.

Laurent operators have invariant supports:

$$\exists E \subseteq \mathbb{Z} \ : \ \operatorname{supp} S = \{(i,j) \in \mathbb{Z} : j - i \in E\} := E^*.$$

To generalise this for “continuous” coordinate system:

Supports of operators and masa bimodules

Say $T : L^2(X,\mu) \to L^2(Y,\nu)$ vanishes in a Borel rectangle $A \times B$ if $P(B)TP(A) = 0$. This happens iff T vanishes on $(A \cup M) \times (B \cup N)$ where $\mu(M) = 0 = \nu(N)$.

Say T is supported in a set $\Omega \subseteq X \times Y$ if

$$(A \times B) \cap \Omega \simeq_\omega \emptyset \quad \Rightarrow \quad P(B)SP(A) = 0.$$

This means $\Omega \cap (A \times B) \subseteq M \times Y \cup X \times N$ or $(\Omega \cap (A \times B))^c \supseteq M^c \times N^c$ where $\mu(M) = 0 = \nu(N)$ (marg. null).
draw

Fix $\Omega \subseteq X \times Y$. If T is supported in Ω then $M_f T M_g$ is supported in Ω for all $f, g \in L^\infty$.

The set $\mathcal{M} = \mathcal{M}_{max}(\Omega)$ of all T which are supported in Ω is a w*-closed masa bimodule: $\mathcal{D}_x \mathcal{M} \mathcal{D}_y \subseteq \mathcal{M}$.

Supports of masa bimodules

Given a w*-closed masa bimodule $\mathcal{M}$, 'the support' ought to be the complement of the union of all Borel rectangles on which every $T \in \mathcal{M}$ vanishes. Measurability?

There is a countable family $\mathcal{E}$ of Borel rectangles whose union ω-contains (i.e. up to a marg. null set) every Borel $A \times B$ s.t. $P(B)\mathcal{M}P(A) = \{0\}$.

The complement of this union (such a set is called ω-closed) is called the ω-support $\operatorname{supp}\mathcal{M}$ of $\mathcal{M}$.

Predual formulation

(Shulman-Turowska, 2004)
The *predual* $T(X, Y)$ of $\mathcal{B}(L^2(X), L^2(Y))$ can be identified with the space of all functions of the form

$$h(x, y) = \sum_i f_i(x) g_i(y)$$

where $f_i, g_i \in L^2$ and $\sum_i \|f_i\|_2 \|g_i\|_2 < \infty$ (identify functions differing on a marginally null set). i.e. agreeing on $M^c \times N^c$ with M, N null. (draw)

$$\text{Duality} \quad \langle T, h \rangle = \sum_i (Tf_i, \bar{g}_i) .$$

Failure of operator synthesis: Arveson's example

If $\Omega \subseteq X \times Y$ is ω-closed, and $\mathcal{M}$ *is a w*-closed masa bimodule with* $\operatorname{supp}\mathcal{M} = \Omega$, does it follow that *every T which is supported in Ω must lie in $\mathcal{M}$*?

Arveson (1974): No! Take $\Omega = \{(s, t) \in \mathbb{R}^3 \times \mathbb{R}^3 : t - s \in \mathbb{S}^2\}$ where $\mathbb{S}^2 \subseteq \mathbb{R}^3$.

Operator synthesis

An ω-closed $\Omega \subseteq X \times Y$ is callled a set of operator synthesis (OS-set) if, for $T \in \mathcal{B}(L^2(X), L^2(Y))$ and $h \in T(X \times Y)$,

$$\mathrm{supp}\, T \subseteq \Omega \subseteq \mathrm{null}\, h \;\Rightarrow\; \langle T, h \rangle = 0.$$

Equivalently, if $\mathcal{M}_1, \mathcal{M}_2$ are w*-closed masa bimodules with $\mathrm{supp}\, \mathcal{M}_i = \Omega$, then $\mathcal{M}_1 = \mathcal{M}_2$.

Equivalently, the only w*-closed masa bimodule $\mathcal{M}$ are with $\mathrm{supp}\, \mathcal{M} = \Omega$ is $\mathcal{M}_{max}(\Omega)$ (reflexivity).

Synthesis and $\Sigma\upsilon\nu\theta\epsilon\sigma\iota\varsigma$

Theorem

Let G be locally compact second countable. Assume A(G) has this approximation property: $u \in \overline{A(G)u} \quad \forall u \in A(G)$.
Let $E \subseteq G$ *be closed.*

$$E \text{ is an S-set} \iff E^* = \{(s,t) \in G \times G : ts^{-1} \in E\} \text{ is an OS-set}$$

Due to : Froelich (1988) for abelian G,
Spronk-Turowska (2002) for compact G,
Ludwig-Turowska (2006) for general G but with local synthesis.

• Are there any groups s.t. $A(G)$ fails this approximation property?

NB Various sets of operator multiplicity are also studied (Shulman-Todorov-Turowska).

• What is $A(G)$?

The Fourier algebra $A(G)$ for non abelian groups

Represent G on $L^2(G)$ by $(\lambda_s f)(t) = f(s^{-1}t)$, $f \in L^2(G)$.

Definition (Eymard, 1964)

The Fourier algebra $A(G)$ is the set of all functions $u : G \to \mathbb{C}$ of the form $u(s) = (\lambda_s f, g)$ with $f, g \in L^2(G)$.

- This is a linear space, in fact an algebra of functions on G, complete in the norm is given by $\|u\|_A = \inf \|f\|_2 \|g\|_2$.
- Its dual is (isom. & w*-homeo.) to the von Neumann algebra of G:

$$\mathrm{VN}(G) = \text{w*-span}\{\lambda_s : s \in G\}.$$

Duality: $\langle \lambda_s, u \rangle_a := u(s)$.

Our approach (w. Anoussis & Todorov)

For $E \subseteq G$ closed, recall the ideals of $A(G)$

$$I(E) = \{g \in A(G) : g|_E = 0\}$$
$$J(E) = \overline{\{g \in A_c(G) : \operatorname{supp} g \cap E = \emptyset\}}^{\|\cdot\|_A}.$$

They are largest (resp. smallest) ideals J with $\operatorname{null}(J) = E$.
For a closed ideal $J \lhd A(G)$, consider $J^\perp \subseteq \mathrm{VN}(G) \subseteq \mathcal{B}(L^2(G))$ and 'saturate it' to get

$$\mathrm{Bim}(J^\perp) := \text{w*-span}\{M_f T M_g : T \in J^\perp, f, g \in L^\infty(G)\}.$$

Theorem

Let $E \subseteq G$ closed. If $\mathcal{M} \subseteq \mathcal{B}(L^2(G))$ w-closed masa bimodule with $\operatorname{supp} \mathcal{M} = E^*$, then*

$$\mathrm{Bim}(I(E)^\perp) \subseteq \mathcal{M} \subseteq \mathrm{Bim}(J(E)^\perp).$$

Corollary E S-set $\Rightarrow$ E^* OS-set: Immediate.
E^* OS-set $\Rightarrow$ E S-set : when G has above approx. property, have $\mathrm{Bim}(J^\perp) \cap \mathrm{VN}(G) = J^\perp$.

Harmonic functions, Harmonic operators

Let $\mu \in M(G)$.
• Say a function $\phi : G \to \mathbb{C}$ is a μ-harmonic function if

$$\int_G \phi(st)d\mu(t) = \phi(s)\,.$$

• Quantisation: Say an operator $T \in \mathcal{B}(L^2(G))$ is a μ-harmonic operator if

$$\int_G \rho_t T \rho_t^* d\mu(t) = T\,.$$

($t \to \rho_t$: right regular rep. of G.)

NB. A $\phi \in L^\infty(G)$ is μ-harmonic iff M_ϕ is μ-harmonic.

Harmonic functions, Harmonic operators

If $\Sigma \subseteq M(G)$, define

$$\mathcal{H}(\Sigma) := \{\phi \in L^\infty(G) : \mu\text{-harmonic for all } \mu \in \Sigma\}$$

$$\tilde{\mathcal{H}}(\Sigma) := \{T \in \mathcal{B}(L^2(G)) : \mu\text{-harmonic for all } \mu \in \Sigma\}.$$

Theorem

Suppose G is abelian, discrete or compact. If $\Sigma \subseteq M(G)$,

$$\tilde{\mathcal{H}}(\Sigma) = \mathrm{Bim}_{vn}(\mathcal{H}(\Sigma)).$$

Here $\mathrm{Bim}_{vn}(\mathcal{H}(\Sigma))$ is the weak-* closed linear span of all products $AM_\phi B$ with $A, B \in \mathrm{VN}(G)$ and $\phi \in \mathcal{H}(\Sigma)$.

Left Ideals and vn(G) bimodules

Let $J \subseteq L^1(G)$ be a left ideal. Consider

$$\ker \Theta(J) := \{T \in \mathcal{B}(L^2(G)) : \int_G \rho_t T \rho_t^* f(t) dt = 0 \text{ for all } f \in J\}$$

It is easy to see that if $A, B \in \mathrm{VN}(G)$ and $\int \phi(t) f(t) dt = 0$ for all $f \in J$ then $AM_\phi B \in \ker \Theta(J)$; Thus,

$$\mathrm{Bim}_{vn}(J^\perp) \subseteq \ker \Theta(J)$$

where $\mathrm{Bim}_{vn}(J^\perp)$ is the weak-* closed linear span of all products $AM_\phi B$ as above.

Theorem

If G is abelian, discrete or compact, then equality holds for every left ideal $J \subseteq L^1(G)$.