Dynamical systems and (semi)-crossed products

Athens, July 2016

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(Classical) Dynamical System (X, ϕ)

X : compact Hausdorff space (compact metric space)

$\phi : X \to X$ homeomorphism (more generally: continuous)

Study behaviour of $\phi^n := \phi \circ \phi \circ \cdots \circ \phi$, $n \in \mathbb{N}$, so study action of *group* $\mathbb{Z}$ or *semigroup* $\mathbb{Z}_+$ on X.
Can also study actions of more general (semi-) groups G on X.

Consider the C*-algebra $\mathscr{C} = C(X) = \{f : X \to \mathbb{C} : \text{continuous}\}$.
Transfer action from X to $\mathscr{C}$ by defining:

$$\alpha : \mathscr{C} \to \mathscr{C} : f \to f \circ \phi \,.$$

Advantage: α preserves algebraic as well as topological structure: preserves sum, product, involution as well as norm.

C*-Dynamical System $(\mathscr{C}, \alpha)$

$\mathscr{C}$: (unital) C*-algebra (possibly non-abelian)

$\alpha : \mathscr{C} \to \mathscr{C}$ *-automorphism (more generally: *-endomorphism)

Aim: To construct algebra encoding the dynamical behaviour of $(\mathscr{C}, \alpha)$: the algebra should ‘contain’ both $\mathscr{C}$ and $\mathbb{Z}$ (or $\mathbb{Z}_+$) so that the action is reflected in the algebraic/topological structure.

Construction

For each $x \in X$, the forward orbit is $\mathscr{O}_+(x) := \{x, \phi(x), \phi^{(2)}(x), \dots\}$.

Let $f \in C(X)$ act on $\mathscr{O}_+(x)$ to get $(f(x), f(\phi(x)), f(\phi^{(2)}(x)), \dots) \in \ell^\infty(\mathbb{Z}_+)$.

This gives repr. of $\mathscr{C} = C(X)$ on $\ell^2(\mathbb{Z}_+)$ by diagonal operators:

Let $\xi = (\xi_0, \xi_1, \dots) \in \ell^2(\mathbb{Z}_+)$. Define

$$\pi_x(f)\xi = (f(x)\xi_0, (f \circ \phi)(x)\xi_1, (f \circ \phi^{(2)})(x)\xi_2, \dots).$$

Let S be the forward shift

$$S\xi = (0, \xi_0, \xi_1, \xi_2, \dots).$$

The generators acting on $H_x := \ell^2(\mathbb{Z}_+)$

$$\pi_x(f) = \begin{pmatrix} f(x) & 0 & 0 & \dots & \dots \\ 0 & f(\phi(x)) & 0 & \dots & \dots \\ 0 & 0 & f(\phi^{(2)}(x)) & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & 0 & 0 & \dots & \dots \\ 1 & 0 & 0 & \dots & \dots \\ 0 & 1 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

The semicrossed product

The operator algebra $C(X) \times_\phi \mathbb{Z}^+$ is the norm closed algebra generated by the operators
$\tilde{\pi}(f) := \bigoplus_{x \in X} \pi_x(f)$, $f \in C(X)$ and
$\tilde{S} := \bigoplus_{x \in X} S$,
acting on the space $\bigoplus_{x \in X} H_x$ (where each $H_x := \ell^2(\mathbb{Z}_+)$).

It is the closure of the space of 'analytic trigonometric polynomials'

$$\sum_{k=0}^{n} \tilde{S}^k \tilde{\pi}(f_k)$$

in the norm of $\mathscr{B}(\bigoplus_{x \in X} H_x)$.

The generators satisfy the covariance relation

$$\tilde{\pi}(f)\tilde{S} = \tilde{S}\tilde{\pi}(\alpha(f)) \qquad (f \in \mathscr{C}). \tag{C}$$

Another representation

Let X be a (locally) compact Hausdorff space, ϕ a homeomorphism of X, μ a ϕ-invariant Borel measure on X (thus $\mu(\phi^{-1}(S)) = \mu(S)$ for all $S \subseteq X$ Borel).
Let $\mathscr{C} = C_0(X)$ and $\alpha(f) = f \circ \phi$.
Represent $\mathscr{C}$ on $H = L^2(X, \mu)$ as multiplication operators:

$$\rho(f)\xi = f\xi \qquad (f \in \mathscr{C},\ \xi \in H).$$

Represent $\mathbb{Z}$ on H *by composition:* [1]

$$U\xi = \xi \circ \phi^{-1}$$

The covariance relation: $\rho(f)U = U\rho(\alpha(f))$.

If $\mu(S) > 0$ for every nonempty open set $S \subseteq X$ then the closed algebra generated by $\{\rho(f) : f \in \mathscr{C}\}$ and U in $\mathscr{B}(L^2(X, \mu)$ is (completely isometrically) isomorphic to $C(X) \times_\phi \mathbb{Z}^+$.

[1] The fact that U is unitary follows from the fact that μ is ϕ-invariant.

The classification problem for $C_0(X) \times_\phi \mathbb{Z}^+$

When are two semicrossed products $C_0(X) \times_\phi \mathbb{Z}^+$ and $C_0(Y) \times_\psi \mathbb{Z}^+$ isomorphic as algebras?

A sufficient condition: Assume that ϕ and ψ are topologically conjugate, i.e., there exists a homeomorphism

$$\gamma : X \to Y$$

so that

$$\begin{array}{ccc} X & \overset{\gamma}{\rightarrowtail\!\!\!\twoheadrightarrow} & Y \\ \downarrow \phi & & \downarrow \psi \\ X & \overset{\gamma}{\rightarrowtail\!\!\!\twoheadrightarrow} & Y \end{array}$$

Then the semicrossed products $C_0(X) \times_\phi \mathbb{Z}^+$ and $C_0(Y) \times_\psi \mathbb{Z}^+$ are isomorphic as algebras.

The classification problem for $C_0(X) \times_\phi \mathbb{Z}^+$

Necessity, under the following assumptions:

- Arveson and Josephson (1969): X, Y locally compact; ϕ, ψ homeomorphisms; periodic points of ϕ (and ψ) form null set for some invariant ergodic separable non-atomic measure of full support.
- Peters (1985): X, Y compact, ϕ, ψ continuous, no periodic points.
- Hadwin and Hoover (1988): X, Y compact, the set

  $$\{x \in X \mid \phi(x) \neq x, \phi^{(2)}(x) = \phi(x)\}$$

  has empty interior.
- Power (1992): X, Y locally compact, ϕ, ψ homeomorphisms.

The classification problem for $C_0(X) \times_\phi \mathbb{Z}^+$

Davidson - Katsoulis, 2008 Let X, Y be locally compact Hausdorff spaces and let ϕ, ψ be proper continuous maps on X and Y respectively. Then the dynamical systems (X, ϕ) and (Y, ψ) are conjugate if and only if the semicrossed products $C_0(X) \times_\phi \mathbb{Z}^+$ and $C_0(Y) \times_\psi \mathbb{Z}^+$ are isomorphic as algebras.

(ϕ proper: $K \subseteq X$ compact $\Rightarrow \phi^{-1}(K) \subseteq X$ compact)

Semisimplicity and recurrence

A point $x \in X$ is recurrent for the dynamical system (X, ϕ) if for every neighbourhood V of x, there is $n \geq 1$ so that $\phi^n(x) \in V$. Equivalently, for X metric space, if there is a sequence $n_k \to \infty$ so that $\phi^{n_k}(x) \to x$. Let X_r denote the recurrent points of (X, ϕ).

If X is compact, recurrent points exist ($\sim$ Poincaré).

Aim To show: X_r is dense in X iff $\mathscr{A} := C_0(X) \times_\phi \mathbb{Z}^+$ is semisimple (i.e. $\mathrm{Rad}\,\mathscr{A} = \{0\}$).

The radical

Let $\mathscr{A}$ be a unital Banach algebra. The Jacobson Radical of $\mathscr{A}$ is defined by

$$\operatorname{Rad}\mathscr{A} = \{q \in \mathscr{A} : \sigma(aq) = 0 \text{ for all } a \in \mathscr{A}\}.$$

$$\operatorname{Rad}\mathscr{A} = \{q \in \mathscr{A} : aq \text{ is quasinilpotent for all } a \in \mathscr{A}\}$$

where $x \in \mathscr{A}$ is called quasinilpotent if $\lim \|x^n\|^{1/n} = 0$.

Example In M_n the radical is $\{0\}$.

In T_n (= upper triangular matrices), the radical is large: all strictly upper triangular matrices (so $T_n/\operatorname{Rad}(T_n)$ is commutative).

If $\mathscr{A}$ is a C*-algebra, then $\operatorname{Rad}\mathscr{A} = \{0\}$.

‘Fourier’ coefficients

Let $\mathscr{A} = \mathscr{C} \times_\alpha \mathbb{Z}^+$ and $\mathscr{A}_0 = \{\sum_{k=0}^{n} \tilde{S}^k \tilde{\pi}(f_k) : f_k \in \mathscr{C}, n \in \mathbb{Z}_+\}$: analytic trig. polys. (Write (π, S) for $(\tilde{\pi}, \tilde{S})$.)

Every $a \in \mathscr{A}$ has a ‘formal Fourier series’

$$a \sim \sum S^n E_n(a)$$

where each $E_n(a)$ is in $\pi(\mathscr{C})$, found as follows:

On $\mathscr{A}_0$, define, for $t \in \mathbb{R}$, $\theta_t(\sum S^n \pi(f_n)) = \sum (e^{it} S)^n \pi(f_n)$.

Observe that $\frac{1}{2\pi} \int_{-\pi}^{\pi} \theta_t(\sum S^n \pi(f_n)) e^{-ikt} dt = S^k \pi(f_k)$ and θ_t extends (!) to $\mathscr{A}$, so for each $a \in \mathscr{A}$ may recover $S^k E_k(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta_t(a) e^{-ikt} dt$.

‘Fourier’ coefficients and the radical

Moreover, ‘Féjer’s Theorem’ holds: if $\sigma_n(a)$ is the average of the first n terms of the ‘Fourier’ series, then $\|\sigma_n(a) - a\| \to 0$. (Proof: as in classical case!)

Proposition *Let $\mathscr{A} = \mathscr{C} \times_\alpha \mathbb{Z}^+$. An element $a \in \mathscr{A}$ is in the radical of $\mathscr{A}$ iff $\tilde{S}^n E_n(a) \in \mathrm{Rad}\,\mathscr{A}$ for all $n \geq 0$. In particular all elements of the radical satisfy $E_0(a) = 0$.*

Strategy for locating $\mathrm{Rad}\,\mathscr{A}$:

(a) Recurrent points give monomials outside the radical.

(b) Wandering points give monomials in the radical.

(b) Wandering sets give elements of the radical

An open set $V \subseteq X$ is ϕ-wandering if $V, \phi^{-1}(V), \phi^{-2}(V), \dots$ are pairwise disjoint.
Suppose f lives in a wandering V. Then $S^n\pi(f) \in \mathrm{Rad}\mathscr{A}$ if $n \neq 0$. Indeed, $\forall\ S^k\pi(g)$,

$$(S^n\pi(f))(S^k\pi(g))(S^n\pi(f)) = S^{2n+k}\pi((f \circ \phi^{k+n})(g \circ \phi^n)f) = 0$$

hence $(S^n\pi(f)a)^2 = 0$ for all $a \in \mathscr{A}$, hence $S^n\pi(f) \in \mathrm{Rad}\mathscr{A}$.

(P. Muhly, proof: M. Anoussis)

(a) Recurrent points give elements outside the radical

Claim: If $S^n\pi(f) \in \operatorname{Rad}\mathscr{A}$ then $f(x) = 0$ for every recurrent $x \in X$.

Idea: If $f(x) \neq 0$, multiply $S^n\pi(f)$ by suitable (convergent) series $\sum_k S^{m_k}\frac{\pi(g)}{2^k}$ to obtain an element

$$a = \sum_k S^{n_k}\frac{\pi(hf)}{2^k}$$

with nonzero spectral radius. To choose the n_k appropriately: Estimate $\|a^m\|$ from below by norms of products of the form

$$p_1 = S^{n_1}\pi(hf),\ p_2 = p_1\left(S^{n_2}\frac{\pi(hf)}{2}\right)p_1, \ldots.$$

The exponents that appear in simplifying this have the structure $n_1, n_1 + n_2, 2n_1 + n_2, \ldots$ and one needs a combinatorial / topological lemma showing that $(hf) \circ \phi^s$ is large enough for all such exponents s.

Theorem (Donsig, K., Manoussos)
Assume that X is a metric space. Then $\mathscr{A} := C_0(X) \times_\alpha \mathbb{Z}^+$ is semisimple iff the ϕ-recurrent points are dense in X.

Proof. If $\overline{X_r} = X$ then by (a) there are no monomials in $\mathrm{Rad}\,\mathscr{A}$ (their coefficients would vanish on X_r) ; hence (Féjer) $\mathrm{Rad}\,\mathscr{A} = \{0\}$.

If $\mathrm{Rad}\,\mathscr{A} = \{0\}$, there can be no wandering open sets by (b); but in l.c. metric spaces, this implies that the recurrent points must be dense. $\square$

The radical

Theorem (Donsig, K., Manoussos)

$\mathrm{Rad}\,\mathscr{A}$ *is the ideal generated by all monomials* $S^n\pi(f)$ *where* $n > 0$ *and* f *vanishes on the recurrent points of* X*.*

So $a \sim \sum S^n E_n(a)$ is in $\mathrm{Rad}\,\mathscr{A}$ iff
$E_0(a) = 0$ and $E_n(a) = \pi(f_n)$ where $f_n(x) = 0$ for all recurrent $x \in X$.