Operator Algebras and Ergodic Theory

Fourth Summer School, Athens, July 2015

Aristides Katavolos

History

1904-06 Hilbert et al. (Göttingen): Integral equations, spectral theory etc.

1925-26 Quantum Mechanics: *Matrix Mechanics* (Heisenberg with Born & Jordan) vs *Wave Mechanics* (Schrödinger)

Equivalent for finitely many particles (Stone - von Neumann theorem, 1931).

Not equivalent for infinitely many particles (quantum stat. mech., quantum field theory). $\rightsquigarrow$ Inequivalent representations of the observables and their algebraic relations.

P.A.M. Dirac, *The Principles of Quantum Mechanics* (1930)

J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (1932)

Matrix Mechanics $\rightsquigarrow$ Operator Algebras

Classical mechanics: Observables are functions f: they correspond to one-dimensional arrays $[\hat{f}(n)]$.

Quantum mechanics: Observables now correspond to two-dimensional arrays $[X_{n,m}]$.

Simple harmonic oscillator $\rightsquigarrow$

$$[X_{n,m}] = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & \sqrt{2} & 0 & \dots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \dots \\ 0 & 0 & \sqrt{3} & 0 & \dots \end{bmatrix}$$

Two crucial points
(1) Infinite matrices

(2) Unbounded matrices

(Heisenberg's Canonical Commutation Relations (CCR) cannot be represented by finite matrices or infinite bounded matrices.)

Operator Algebras

$\rightsquigarrow$ Represent the observables and their algebraic relations by Hilbert space operators.

$\rightsquigarrow$ Operator Algebras

Time evolution gives a representation of the group $\mathbb{R}$ (reversible systems) or the semigroup $\mathbb{R}_+$ (irreversible systems) as acting on the algebra of observables (Heisenberg picture) or on the states of the system (Schrödinger picture).

Symmetries of the system also form a group G acting on the algebra or the states.

Preliminaries

H: Hilbert space
$\langle x, y\rangle$: scalar product,
H is complete in the norm $\|\xi\| := \sqrt{\langle \xi, \xi\rangle}$.
$\mathscr{B}(H) := \{T : H \to H$ linear, continuous$\}$ with norm
$\|T\| := \sup\{\|T\xi\| : \|\xi\| = 1\}$
Involution $T \to T^*$ where $\langle T^*\xi, \eta\rangle = \langle \xi, T\eta\rangle$, all $\xi, \eta \in H$.
A set $\mathscr{S} \subseteq \mathscr{B}(H)$ is selfadjoint if $\mathscr{S} = \mathscr{S}^*$
The commutant of $\mathscr{S}$ is

$$\mathscr{S}' := \{T \in \mathscr{B}(H) : TS = ST \ \forall S \in \mathscr{S}\}$$

Always an algebra containing $I := I_H$

The object

A von Neumann algebra $\mathscr{M}$ (or $\mathscr{M} \curvearrowright H$) is a selfadjoint subset of $\mathscr{B}(H)$ with $\mathscr{M} = \mathscr{M}''$.

Examples

- The algebra $\mathscr{B}(H)$ for some Hilbert space H.
- The algebra M_n of all $n \times n$ complex matrices.
- The algebra D_n of $n \times n$ *diagonal* matrices.
- The algebra $\mathscr{B}(H) \oplus \mathscr{B}(H) = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A, B \in \mathscr{B}(H) \right\}$.

The multiplication algebra

(X, μ): a *countably separated* measure (probability) space.

$L^2(X, \mu)$: the space of (equiv. classes, mod. equality μ-a.e.) of measurable functions $f : X \to \mathbb{C}$ with $\|f\|_2 := \left(\int |f|^2 d\mu\right)^{1/2} < \infty$.

$L^\infty(X, \mu)$: the space of (equiv. classes, mod. equality μ-a.e.) of essentially bounded measurable functions $f : X \to \mathbb{C}$ with $\|f\|_\infty := \mathrm{esssup}\,|f|$.

The multiplication algebra

For $f \in L^\infty(X,\mu)$ put $M_f : g \to fg : L^2(X,\mu) \to L^2(X,\mu)$.
Then $M_f \in \mathscr{B}(L^2(X,\mu))$ (since $\|fg\|_2 \le \|f\|_\infty \|g\|_2$) and in fact $\|M_f\| = \|f\|_\infty$.

Definition

$\mathscr{M}_\mu = \{M_f : f \in L^\infty(X,\mu)\} \subseteq \mathscr{B}(L^2(X,\mu))$

Then

$$\mathscr{M}_\mu' = \mathscr{M}_\mu.$$

Indeed if $T \in \mathscr{M}_\mu'$ setting $f := T(\mathbf{1})$, for all $h \in L^\infty(X,\mu)$ we have $T(h) = TM_h(\mathbf{1}) = M_h T(\mathbf{1}) = hf$ and $\|fh\|_2 \le \|T\| \, \|h\|_2$ so $f \in L^\infty(X,\mu)$ and hence $T = M_f$.

Conclusion

$\mathscr{M}_\mu \curvearrowright L^2(X,\mu)$ is a von Neumann algebra.

The von Neumann algebra of a group

Let G be a countable (discrete) group. (Think of $\mathbb{Z}$ or $\mathbb{F}_2$.)

$$H = \ell^2(G) = \{f : G \to \mathbb{C} : \sum_{t \in G} |f(t)|^2 < \infty\}.$$

Then $\ell^2(G)$ has ON basis $\{\delta_t : t \in G\}$.
For $s \in G$ define a map

$$\lambda_s : \delta_t \to \delta_{st}$$

and extend linearly. This is an ℓ^2 isometry, so extends to $\lambda_s : \ell^2(G) \to \ell^2(G)$. But it is onto because $\lambda_s \lambda_t = \lambda_{st}$ so $\lambda_s \lambda_{s^{-1}} = I$, hence unitary.

$$\text{For } \ f \in \ell^2(G), \quad (\lambda_s f)(t) = f(s^{-1}t).$$

The von Neumann algebra of a group (contd.)

Definition

The von Neumann algebra generated by the set of unitaries

$$\{\lambda_t : t \in G\}$$

is called the von Neumann algebra $\mathrm{VN}(G) = \mathscr{L}(G)$ *of the group.*

So $T \in \mathscr{L}(G)$ iff $TX = XT$ for all X satisfying $X\lambda_t = \lambda_t X$ for all $t \in G$.

The bicommutant Theorem

... connects algebraic with topological property.

For $x, y \in H$ consider the lin. functional

$$\omega_{xy} : \mathscr{B}(H) \to \mathbb{C} : T \to \omega_{xy}(T) := \langle Tx, y \rangle \, .$$

The weak operator topology (WOT) on $\mathscr{B}(H)$: weakest making all functionals $\{\omega_{xy} : x, y \in H\}$ continuous.

Note any $\mathscr{S}'$ is WOT-closed: If $T_i S = S T_i$ and $\langle T_i x, y \rangle \to \langle Tx, y \rangle$ for all x, y, then

$$\begin{aligned} \langle (ST - TS)x, y \rangle &= \langle Tx, S^* y \rangle - \langle TSx, y \rangle \\ &= \lim(\langle T_i x, S^* y \rangle - \langle T_i Sx, y \rangle) \\ &= \lim \langle (ST_i - T_i S)x, y \rangle = 0 \end{aligned}$$

Theorem

If $\mathscr{M} \subseteq \mathscr{B}(H)$ is a selfadjoint subalgebra with unit, TFAE
(i) $\mathscr{M} = \mathscr{M}''$
(ii) $\mathscr{M}$ is WOT closed.

The bicommutant Theorem: Proof

(i)$\Rightarrow$ (ii) is clear.
To show (ii)$\Rightarrow$ (i): take $A \in \mathscr{M}''$ and show it is in the WOT closure of $\mathscr{M}$. If not, there would exist a WOT-continuous functional $\omega : \mathscr{B}(H) \to \mathbb{C}$ annihilating $\mathscr{M}$ but not A (Hahn-Banach).

All WOT-continuous functionals are of the form $\omega = \sum_{k=1}^{n} \omega_{x_k,y_k}$.

Identify $\mathscr{B}(H^n)$ with $n \times n$ matrices over $\mathscr{B}(H)$. Represent $\mathscr{M}$ as $\mathscr{N} := \pi(\mathscr{M}) \subseteq \mathscr{B}(H^n)$, where

$$\pi(T) = \begin{bmatrix} T & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & T \end{bmatrix}.$$

Verify that the commutant $\mathscr{N}'$ of $\mathscr{N}$ is the algebra of $n \times n$ matrices over $\mathscr{M}'$. Thus $\pi(A) \in \mathscr{N}''$.

The bicommutant Theorem: Proof contd.

Now for all $T \in \mathscr{M}$,

$$0 = \omega(T) = \sum_{k=1}^{n} \langle Tx_k, y_k \rangle = \langle \pi(T)\vec{x}, \vec{y} \rangle \,.$$

Hence $\vec{y} \perp \mathscr{N}\vec{x}$. Now $\overline{\mathscr{N}\vec{x}}$ is $\mathscr{N}$-invariant, so the projection P_x on it is in $\mathscr{N}'$. But $\pi(A)P_x = P_x\pi(A)$ since $\pi(A) \in \mathscr{N}''$. So $\overline{\mathscr{N}\vec{x}}$ is $\pi(A)$-invariant and hence (since $\vec{x} \in \mathscr{N}\vec{x}$) $\pi(A)\vec{x} \in \overline{\mathscr{N}\vec{x}}$. Therefore $\langle \pi(A)\vec{x}, \vec{y} \rangle = 0$, i.e. $\omega(A) = 0$, contrary to assumption.

Tracial von Neumann algebras

A state ϕ on a von Neumann algebra $\mathscr{M}$ is a linear map $\phi : \mathscr{M} \to \mathbb{C}$ such that $\phi(A^*A) \geq 0$ for all $A \in \mathscr{M}$ and $\phi(I) = 1$. It is called faithful if $\phi(A^*A) > 0$ when $A \neq 0$ and normal if it is WOT-continuous on bounded subsets of $\mathscr{M}$.

A tracial state is a state τ such that $\tau(AB) = \tau(BA)$ for all $A, B \in \mathscr{M}$.

Example

On $\mathscr{M}_\mu$ define $\tau(M_f) = \int_X f d\mu$ for all $f \in L^\infty(X, \mu)$.

Example

On $\mathscr{L}(G)$ define $\tau(A) = \langle A\delta_e, \delta_e \rangle$ for all $A \in \mathscr{L}(G)$.

Definition

A tracial von Neumann algebra $(\mathscr{M}, \tau)$ *is a von Neumann algebra equipped with a faithful normal tracial state.*

The trace on $\mathscr{L}(G)$

$$\tau(A) = \langle A\delta_e, \delta_e\rangle \quad \text{for all } A \in \mathscr{L}(G).$$

It is a WOT-continuous state, it is faithful because

$$\tau(A^*A) = 0 \iff A\delta_e = 0 \iff A\delta_t = A\rho_t\delta_e = \rho_t A\delta_e = 0 \iff A = 0$$

(here $\rho_t : \delta_s \to \delta_{st}$ so it commutes with $\mathscr{L}(G)$) and it is a trace:
Enough (bicommutant theorem) to check $\tau(AB) = \tau(BA)$ when $A = \lambda_s, B = \lambda_t$.
This is obvious:
$\langle \lambda_s\lambda_t\delta_e, \delta_e\rangle = \langle \delta_{st}, \delta_e\rangle = \delta_{st,e} = \delta_{ts,e} = \langle \lambda_t\lambda_s\delta_e, \delta_e\rangle$.

Tracial von Neumann algebras (contd.)

Example: $\mathscr{L}(\mathbb{Z}) \simeq \mathscr{M}_m$

Let $F : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ be the Fourier transform. Then $\mathscr{L}(\mathbb{Z}) = F\mathscr{M}_m F^{-1}$, where $\mathscr{M}_m$ is the multiplication algebra of $L^\infty(\mathbb{T})$. Thus $\tau(FM_f F^{-1}) = \int_{\mathbb{T}} fdm$ for all $f \in L^\infty(\mathbb{T})$.

Indeed if $\{f_k : k \in \mathbb{Z}\}$ is the o.n. basis of $L^2(\mathbb{T})$ with $f_k(e^{it}) = e^{ikt}$, then $Ff_k = \delta_k$ so $F^{-1}\lambda_n F = M_{f_n}$ for $n \in \mathbb{Z}$.

Standard form

Let $(\mathscr{M}, \tau)$ be a tracial von Neumann algebra. On $\mathscr{M}$, define

$$\langle a, b \rangle_\tau := \tau(b^* a) \quad \text{and} \quad \|a\|_2 := (\tau(a^* a))^{1/2}.$$

The completion of $(\mathscr{M}, \|\cdot\|_2)$ is called $L^2(\mathscr{M}, \tau)$ or H_τ.
Thus $\mathscr{M}$ embeds densely in $L^2(\mathscr{M}, \tau)$; write $\hat{a}$ when $a \in \mathscr{M}$ is viewed as a vector in $L^2(\mathscr{M}, \tau)$.

[When $\mathscr{M}$ is abelian, it is isomorphic to some $L^\infty(X, \mu)$ and $L^2(\mathscr{M}, \tau)$ is naturally isomorphic to $L^2(X, \mu)$.]

Represent $\mathscr{M}$ on H_τ: For $a \in \mathscr{M}$, the map $\hat{b} \to \widehat{ab}$ is $\|\cdot\|_2$-bounded on $\widehat{\mathscr{M}}$, so extends to $\pi(a) \in \mathscr{B}(H_\tau)$.

Proof Since $b^*(a^* a) b \leq \|a^* a\| \, b^* b$ as operators,we have $\tau(b^*(a^* a) b) \leq \|a^* a\| \, \tau(b^* b)$ so

$$\left\|\widehat{ab}\right\|_2^2 = \tau(b^*(a^* a) b) \leq \|a^* a\| \, \tau(b^* b) = \|a\|^2 \left\|\hat{b}\right\|_2^2.$$

Standard form (contd.)

We have a faithful (i.e. 1-1) representation π of $\mathscr{M}$ on H_τ.
It is a fact that $\pi(\mathscr{M})$ is WOT-closed, hence a von Neumann algebra.
We say $\mathscr{M}$ is in standard form on H_τ.
For example, $\mathscr{M}_\mu$ is in standard form on $L^2(X,\mu)$ and $\mathscr{L}(G)$ is in standard form on $\ell^2(G)$. But M_n is not in standard form on $\mathbb{C}^n$; it is in standard form on $H_\tau = M_n$ equipped with the (normalised) Hilbert-Schmidt norm.
With $\xi := \hat{\mathbf{1}}$ we have $\tau(a) = \langle \pi(a)\xi, \xi \rangle_\tau$. Thus (π, H_τ, ξ) is the GNS triple for $(\mathscr{M}, \tau)$.

Standard form (contd.)

The densely defined map $J_0 : \widehat{\mathscr{M}} \to \widehat{\mathscr{M}} : \hat{a} \to \widehat{a^*}$ is antilinear and has the magical property:

$$\left\langle J_0(\hat{a}), J_0(\hat{b}) \right\rangle_\tau = \left\langle \hat{b}, \hat{a} \right\rangle_\tau$$

for all $a, b \in \mathscr{M}$. Indeed, since τ is a trace (!),

$$\left\langle J_0(\hat{a}), J_0(\hat{b}) \right\rangle_\tau = \left\langle \widehat{a^*}, \widehat{b^*} \right\rangle_\tau = \tau(ba^*) \overset{!}{=} \tau(a^* b) = \left\langle \hat{b}, \hat{a} \right\rangle_\tau .$$

Therefore J_0 is $\|\cdot\|_2$-isometric; also obviously $J_0^2 \hat{a} = \hat{a}$ for all $a \in \mathscr{M}$, so J_0 has dense range. Hence J_0 extends to an antilinear bijection J on H_τ which satisfies

$$\langle J\eta, J\zeta \rangle = \langle \zeta, \eta \rangle \quad \text{for all } \eta, \zeta \in H_\tau. \qquad (*)$$

Standard form (contd.)

In the representation π, the algebra $\pi(\mathscr{M})$ has 'the same size' as its commutant $\pi(\mathscr{M})'$:

Theorem

$J\pi(\mathscr{M})J = \pi(\mathscr{M})'.$

Proof The trivial direction: For $a \in \mathscr{M}$, need to show $J\pi(a)J \in \pi(\mathscr{M})'$ i.e. that $J\pi(a)J\pi(b) = \pi(b)J\pi(a)J$ for all $b \in \mathscr{M}$. It suffices to verify this on the dense set $\{\hat{x} : x \in \mathscr{M}\}$. Indeed:

$$
\begin{aligned}
&& J\pi(a)J\pi(b)\hat{x} &= \pi(b)J\pi(a)J\hat{x} \\
&\iff & J\pi(a)J\widehat{bx} &= \pi(b)J\pi(a)\widehat{x^*} \\
&\iff & J\pi(a)\widehat{(bx)^*} &= \pi(b)J\widehat{ax^*} \\
&\iff & J\pi(a)\widehat{x^*b^*} &= \pi(b)\widehat{xa^*} \\
&\iff & J\widehat{ax^*b^*} &= \widehat{bxa^*}
\end{aligned}
$$

Standard form (contd.)

We have shown that $J\pi(\mathscr{M})J \subseteq \pi(\mathscr{M})'$. [1]
For the reverse inclusion, first notice that

$$JT\hat{\mathbf{1}} = T^*\hat{\mathbf{1}} \quad \text{for each } T \in \pi(\mathscr{M})'.$$

Indeed, for all $a \in \mathscr{M}$,

$$\begin{aligned}\left\langle JT\hat{\mathbf{1}}, \hat{a}\right\rangle &= \left\langle J\hat{a}, T\hat{\mathbf{1}}\right\rangle = \left\langle \widehat{a^*}, T\hat{\mathbf{1}}\right\rangle = \left\langle \pi(a^*)\hat{\mathbf{1}}, T\hat{\mathbf{1}}\right\rangle \\ &= \left\langle \hat{\mathbf{1}}, \pi(a)T\hat{\mathbf{1}}\right\rangle = \left\langle \hat{\mathbf{1}}, T\pi(a)\hat{\mathbf{1}}\right\rangle = \left\langle T^*\hat{\mathbf{1}}, \pi(a)\hat{\mathbf{1}}\right\rangle \\ &= \left\langle T^*\hat{\mathbf{1}}, \hat{a}\right\rangle\end{aligned}$$

and so $JT\hat{\mathbf{1}} - T^*\hat{\mathbf{1}}$ is orthogonal to a dense set.

[1] In particular it follows that $\pi(\mathscr{M})'\hat{\mathbf{1}}$ is dense in H_τ; for if some $\eta \in H_\tau$ is orthogonal to $\pi(\mathscr{M})'\hat{\mathbf{1}}$, then for all $a \in \mathscr{M}$ we will have $\langle J\eta, \hat{a}\rangle = \left\langle J\eta, \pi(a)\hat{\mathbf{1}}\right\rangle = \left\langle J\eta, \pi(a)J\hat{\mathbf{1}}\right\rangle = \left\langle J\pi(a)J\hat{\mathbf{1}}, \eta\right\rangle = 0$ since $J\pi(a)J \in \pi(\mathscr{M})'$; thus $J\eta = 0$ hence $\eta = 0$.

Standard form (contd.)

Now let $T \in \pi(\mathscr{M})'$. To show that $JTJ \in \pi(\mathscr{M})$, it suffices, since $\pi(\mathscr{M})$ is a von Neumann algebra, to verify that JTJ commutes with $\pi(\mathscr{M})'$, i.e. that $JTJS = SJTJ$ for all $S \in \pi(\mathscr{M})'$. For this, it suffices (since $\pi(\mathscr{M})'\hat{\mathbf{1}}$ is dense in H_τ), to prove the equality

$$JTJSR\hat{\mathbf{1}} = SJTJR\hat{\mathbf{1}} \quad \text{for all } R \in \pi(\mathscr{M})'$$

We have, using the last remark repeatedly,

$$JTJSR\hat{\mathbf{1}} = JTJ(SR)\hat{\mathbf{1}} = JT(SR)^*\hat{\mathbf{1}} = (T(SR)^*)^*\hat{\mathbf{1}} = SRT^*\hat{\mathbf{1}}.$$

On the other hand,

$$SJTJR\hat{\mathbf{1}} = SJTR^*\hat{\mathbf{1}} = SJ(TR^*)\hat{\mathbf{1}} = SRT^*\hat{\mathbf{1}}$$

which proves the equality.

Standard form (contd.)

Remark

When $\mathscr{M} \curvearrowright H$ has a cyclic vector $\xi_0 \in H$ such that $\langle a\xi_0, \xi_0 \rangle = \tau(a)$ for all $a \in \mathscr{M}$, then the identity representation is unitarily equivalent to the representation (π, H_τ) via the map

$$U : a\xi_0 \to \pi(a)\hat{\mathbf{1}} = \hat{a} \quad (a \in \mathscr{M}).$$

The group - measure space construction

Recall two constructions:

• From a probability space (X, μ) we constructed the multiplication algebra $\mathscr{M}_\mu = \{M_f : f \in L^\infty(X, \mu)\}$.

• From a group G we constructed the group von Neumann algebra $\mathscr{L}(G)$, which is the WOT-closure of polynomials $\sum\limits_{t \in G} c_t \lambda_t$ in the group elements (represented as operators λ_t) with scalar coefficients c_t.

We now combine the two constructions:
If a group G acts on some space (X, μ), we will construct a von Neumann algebra whose elements are limits of polynomials in the group elements, but with coefficients from the multiplication algebra $\mathscr{M}_\mu$. The multiplication of such polynomials is 'twisted' to take into account the action $G \curvearrowright (X, \mu)$.

Measure-preserving actions

A measure preserving automorphism of a probability space (X,μ) is a measurable bijection $\theta : X \to X$ with measurable inverse which preserves the measure, i.e. $\mu(\theta^{-1}(E)) = \mu(E)$ for every Borel set $E \subseteq X$. Let $\mathrm{Aut}(X,\mu)$ be the group of all such automorphisms (modulo μ-null sets).

The map θ induces an automorphism of $L^\infty(X,\mu)$,by $f \to f \circ \theta$, preserving the integral, hence a WOT-continuous *-automorphism $\alpha : \mathscr{M}_\mu \to \mathscr{M}_\mu$ by $\alpha(M_f) = M_{f\circ\theta}$ which preserves the trace:

$\tau(\alpha(M_f)) = \int f \circ \theta d\mu = \int f d\mu = \tau(M_f)$ for all $M_f \in \mathscr{M}_\mu$.

Definition

A measure preserving action $G \curvearrowright (X,\mu)$ of a (countable, discrete) group G is a group homomorphism $G \to Aut(X,\mu)$.

Measure-preserving actions

Example (Bernoulli shift)

Consider the discrete set $\{0,1\}$ with measure $\nu(\{0\}) = p, \nu(\{1\}) = 1-p$ and let $X = \{0,1\}^G = \{x : G \to \{0,1\}\}$ with product measure μ. Define $G \curvearrowright (X,\mu)$ by $(\theta_s x)(t) = x(s^{-1}t)$.

Construction of the crossed product

Start with a measure preserving action $G \curvearrowright (X, \mu)$. This induces a unitary group $\{U_s : s \in G\}$ on $L^2(X, \mu)$ given by

$$(U_s f)(x) = f(s^{-1}x).$$

The restriction of U_s to $L^\infty(X, \mu)$ induces a *-automorphism α_s of $\mathscr{M} := \mathscr{M}_\mu$ given by $\alpha_s(M_f) = M_{f \circ s^{-1}}$ (identify G with its image in $Aut(X, \mu)$).
The pair satisfies the covariance relation

$$U_s M_f U_s^{-1} = \alpha_s(M_f) \quad \text{or} \quad U_s M_f = \alpha_s(M_f) U_s.$$

Construction of the crossed product (contd.)

Represent both $L^\infty(X,\mu)$ and G on $H := \ell^2(G, L^2(X,\mu)) = L^2(X,\mu) \otimes \ell^2(G)$. This consists of all functions $f : s \to f_s : G \to L^2(X,\mu)$ such that

$$\|f\|^2 := \sum_{s\in G} \|f_s\|_2^2 = \sum_{s\in G} \int |f_s(x)|^2 d\mu(x) < \infty.$$

Representation:

$$\pi(f) = M_f \otimes I,\ f \in L^\infty \quad \text{and} \quad W_t = U_t \otimes \lambda_t,\ t \in G$$

Define the crossed product $\mathscr{A} = L^\infty(\mu) \rtimes G$ by

$$\mathscr{A} = \{\pi(f), W_t : f \in L^\infty(\mu), t \in G\}'' = \overline{\{\sum_k \pi(f_k) W_{t_k} : f_k \in L^\infty, t_k \in G\}}^{wot}$$

Note: $\mathscr{A}$ contains an isomorphic copy of $\mathscr{M}$, since $\pi(\mathscr{M}) \subseteq \mathscr{A}$ and also an isomorphic copy of the group: $\{W_t = U_t \otimes \lambda_t : t \in G\} \subseteq \mathscr{A}$.

Construction of the crossed product (contd.)

Write any $g \in H$ as a sum $g = \sum_{s \in G} g_s u_s$ (convergent in H) where $u_s = \mathbf{1} \otimes \delta_s$ is the function taking the values $u_s(t) = 0$ when $t \neq s$ and $u_s(s) = \mathbf{1} \in L^2(X, \mu)$.
Note that

$$(\pi(f)W_s)(\mathbf{1} \otimes \delta_e) = (M_f \otimes I)(U_s \otimes \lambda_s)(\mathbf{1} \otimes \delta_e) = (M_f \otimes I)(\mathbf{1} \otimes \delta_s) = f \otimes \delta_s$$

and so a finite sum $\sum_k \pi(f_k) W_{t_k}$ may be identified with the function $\sum_k f_k \otimes \delta_{t_k} = \sum_k f_k u_{t_k}$ in H
(actually in $c_{00}(G, L^\infty(X, \mu)) := \mathscr{M}[G]$).
It can be shown that u_e is a separating vector for the whole crossed product $\mathscr{A}$, i.e. that the map $a \to a u_e : \mathscr{A} \to H$ is injective. This means that every $a \in \mathscr{A}$ can be identified with the element $\hat{a} := a u_e \in H$ which has a 'Fourier series'

$$\hat{a} = \sum_{s \in G} a_s u_s \quad (a_s \in L^\infty(X, \mu))$$

which converges in the norm of H.

Construction of the crossed product (contd.)

Define a *-algebra structure on $\mathscr{M}[G]$ by

$$(fu_s)(gu_t) = fU_s(g)u_{st} \quad \text{and} \quad (fu_s)^* = U_{s^{-1}}(\bar{f})u_{s^{-1}}$$

and a tracial functional τ by

$$\tau(f) = \langle fu_e, u_e \rangle = \int_X f_e \, d\mu \quad \text{for } f = \sum_{t \in G} f_t u_t.$$

The formula for $\tau(f)$ makes sense for all $f \in \mathscr{A}$. The fact that τ is a trace follows from the fact that μ is measure preserving. Also

$$\tau(f^* f) = \sum_{t \in G} \int_X |f_t|^2 \, d\mu$$

so that τ is faithful. It follows that the WOT closure of $\mathscr{M}[G]$, namely $\mathscr{A} = L^\infty(\mu) \rtimes G$, is in standard form on H.

Measure-preserving actions: free actions, ergodic actions

Definition

A measure-preserving action $G \curvearrowright (X,\mu)$ is said to be (essentially) free if for all $s \in G$ with $s \neq e$ the set of fixed points $X_s := \{x \in X : \theta_s(x) = x\}$ is μ-null.
The action $G \curvearrowright (X,\mu)$ is said to be ergodic if the action has no non-trivial (essentially) invariant sets, equivalently if the only $f \in L^\infty(X,\mu)$ with $\alpha_s(M_f) = M_f$ for all $s \in G$ are the (μ-a.e.) constant functions.

(For abelian groups, ergodic $\Rightarrow$ free.)

Proposition

(i) The action is free iff $\pi(\mathscr{M})$ is maximal abelian in $\mathscr{A}$, i.e. iff $\pi(\mathscr{M})' \cap \mathscr{A} = \pi(\mathscr{M})$.
(ii) Assume G acts freely. The action is ergodic iff $\mathscr{A}$ is a factor, i.e. iff the centre $\mathscr{A}' \cap \mathscr{A} = \mathscr{A}$ is trivial ($= \mathbb{C}I$).

Some proofs

To prove the last Proposition we will use:

Lemma

$\theta \in Aut(X,\mu)$ acts freely.

$\iff$

Every non-null $Y \subseteq X$ has $Z \subseteq Y$ with $\mu(Z) > 0$ such that $\theta(Z) \cap Z = \emptyset$.

$\iff$

If $a \in L^\infty(X,\mu)$ satisfies $aU_\theta(x) = xa$ for all $x \in L^\infty(X,\mu)$, then $a = 0$.

(The latter condition can be taken as a definition for a free action when $L^\infty(X,\mu)$ is replaced by a non-abelian von Neumann algebra.)

Examples

Let X be a compact group with Haar measure μ. Let $G \subseteq X$ be a countable dense subgroup. Now $G \curvearrowright (X, \mu)$ by left multiplication. This is measure preserving (Haar measure). The action is obviously free. It is ergodic since every $f \in L^2(X, \mu)$ which is invariant under translation by all elements of G is also invariant under all elements of X, hence must be constant. For example we may take $X = \mathbb{T}$ and $G = \{e^{2\pi i n\theta} : n \in \mathbb{Z}\}$ where θ is irrational.

Also, the Bernoulli shift is free and ergodic.

Example: the irrational rotation

Consider $(X, \mu) = (\mathbb{T}, m)$ and $\theta(z) = \omega z$ where $\omega = e^{2\pi i \phi}$, ϕ irrational. This gives an ergodic action of $\mathbb{Z}$, hence free (abelian group).

The multiplication algebra $\mathscr{M}_m$ is generated by the unitary $V = M_v$ where $v(z) = z$. The action is given by $\{U^n : n \in \mathbb{Z}\}$ where $(Uf)(z) = f(\omega z)$, $f \in L^2(\mathbb{T})$.

So the crossed product acting on $L^2(\mathbb{T}) \otimes \ell^2(\mathbb{Z})$ is generated (as a WOT closed algebra) by

$$(V \otimes I) \quad \text{and} \quad (U \otimes \lambda_1).$$

But the covariance relations are also satisfied by U and V; indeed, $UV = \omega VU$. However, $\{U, V\}''$ is not even algebraically isomorphic to the crossed product: it is $\mathscr{B}(L^2(\mathbb{T}))$. For example, it has no finite trace and it contains the unilateral shift, whereas the crossed product cannot contain a non-unitary isometry.

By contrast, the norm - closed *-algebra $\mathscr{A}_\phi$ generated by two unitaries satisfying $UV = \omega VU$ is unique, whether represented on $L^2(\mathbb{T}) \otimes \ell^2(\mathbb{Z})$ or on $L^2(\mathbb{T})$. It is the irrational rotation algebra.