Harmonic Analysis and Operator Algebra Theory

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Classical spectral synthesis

Example: $G = \widehat{\mathbb{R}}$.

$$A(\widehat{\mathbb{R}}) = \{\hat{f} : f \in L^1(\mathbb{R})\} \subseteq C_0(\widehat{\mathbb{R}})$$

This is a (selfadjoint) algebra under pointwise operations and is complete in the norm $\|\hat{f}\|_A = \|f\|_1$.

$$\begin{array}{ccc} L^1(\mathbb{R}) & \xrightarrow{\text{dual}} & L^\infty(\mathbb{R}) \\ \mathcal{F}\big\downarrow & & \big\downarrow \\ A(\widehat{\mathbb{R}}) & \xrightarrow{\text{dual}} & A(\widehat{\mathbb{R}})^* \end{array}$$

(Notice that since $L^\infty(\mathbb{R})$ is w*-generated by exponentials $e_x(t) = e^{ixt},\ x \in \mathbb{R}$, the space $A(\widehat{\mathbb{R}})^*$ is w*-generated by evaluations (characters) δ_x given by $\left\langle \delta_x, \hat{f} \right\rangle = \hat{f}(x)$.)

Now consider any locally compact abelian group G in place of $\widehat{\mathbb{R}}$.

Spectral synthesis

Given a closed set $E \subseteq G$, we say that a $\tau \in A(G)^*$ is *supported in E* if $\langle \tau, g \rangle = 0$ for all $g \in A(G)$ whose closed support $\operatorname{supp} g$ is (compact and) disjoint from E.
We say that E is *a set of synthesis (S-set)* if every $\tau \in A(G)^*$ which is supported in E is 'synthesisable' from characters in E:

$$E \text{ S-set:} \qquad \operatorname{supp} \tau \subseteq E \;\Rightarrow\; \tau \in \overline{\{\delta_x : x \in E\}}^{w*}.$$

Equivalently (Hahn-Banach), E is a set of synthesis if, for $\tau \in A(G)^*$ and $g \in A(G)$,

$$\operatorname{supp} \tau \subseteq E \subseteq \operatorname{null} g \;\Rightarrow\; \langle \tau, g \rangle = 0.$$

It was discovered by L. Schwartz (1948) that the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ does not satisfy synthesis.

Spectral synthesis

Formulation in terms of $A(G)$: Given a closed set $E \subseteq G$, we consider

$$I(E) = \{g \in A(G) : g|_E = 0\} \lhd A(G).$$

$$J(E) = \overline{\{g \in A(G) : \operatorname{supp} g \cap E = \emptyset\}}.$$

Then:

$$E \ \text{S-set} \quad \Longleftrightarrow \quad J(E) = I(E).$$

Support of an operator

An operator $S \in \mathcal{B}(\ell^2(\mathbb{Z}))$ vanishes on a rectangle $A \times B \subseteq \mathbb{Z} \times \mathbb{Z}$ if the matrix entries $s_{i,j} = \left(Se_j, e_i\right)$ of S are 0 for all $(j, i) \in A \times B$.
This is equivalent to $P(B)SP(A) = 0$, where $P(A)$ is the projection onto the space spanned by the basis elements $\{e_j : j \in A\}$.

Toeplitz operators and invariance

An $S \in \mathcal{B}(\ell^2(\mathbb{Z}))$ is called a Toeplitz operator if it is constant along diagonals, i.e. $s_{i,j} = s_{i+k,j+k}$ for all k.
(Invariant under the action of $\mathbb{Z}$)

Then $s_{i,j} = f(j-i)$ for some single-variable f.

Toeplitz operators have invariant supports:

$$\exists E \subseteq \mathbb{Z} \ : \ \operatorname{supp} S = \{(i,j) \in \mathbb{Z} : j - i \in E\} := E^*.$$

Supports of operators and masa bimodules

Say $T : L^2(X, \mu) \to L^2(Y, \nu)$ vanishes in a Borel rectangle $A \times B$ whenever $P(B)TP(A) = 0$.

Say T is supported in a set $\Omega \subseteq X \times Y$ if $(A \times B) \cap \Omega \simeq_\omega \emptyset$ whenever $P(B)TP(A) = 0$.

This means $\Omega \cap (A \times B) \subseteq M \times Y \cup X \times N$ where $\mu(M) = 0 = \nu(N)$ (marg. null). draw

Fix $\Omega \subseteq X \times Y$. If T is supported in Ω then $M_f T M_g$ is supported in Ω for all $f, g \in L^\infty$.

The set $\mathcal{M} = \mathcal{M}_{max}(\Omega)$ of all T which are supported in Ω is a w*-closed masa bimodule: $\mathcal{D}_x \mathcal{M} \mathcal{D}_y \subseteq \mathcal{M}$.

Supports of masa bimodules

Given a w*-closed masa bimodule $\mathcal{M}$, ‘the support’ ought to be the complement of the union of all Borel rectangles on which every $T \in \mathcal{M}$ vanishes. Measurability?

There is a countable family $\mathcal{E}$ of Borel rectangles whose union ω-contains (i.e. up to a mrarg. null set) every Borel $A \times B$ s.t. $P(B)\mathcal{M}P(A) = \{0\}$.

The complement of this union (such a set is called ω-closed) is called the ω-support $\operatorname{supp}\mathcal{M}$ of $\mathcal{M}$.

Predual formulation

(Shulman-Turowska, 2004)
The *predual* $T(X, Y)$ of $\mathcal{B}(L^2(X), L^2(Y))$ can be identified with the space of all functions of the form

$$h(x, y) = \sum_i f_i(x) g_i(y)$$

where $f_i, g_i \in L^2$ and $\sum_i \|f_i\|_2 \|g_i\|_2 < \infty$ (identify functions differing on a marginally null set). i.e. agreeing on $N^c \times N^c$ with N null. (draw)

$$\text{Duality} \quad \langle T, h \rangle = \sum_i (T f_i, \bar{g}_i) .$$

Failure of operator synthesis: Arveson's example

If $\Omega \subseteq X \times Y$ is ω-closed, and $\mathcal{M}$ is a w*-closed masa bimodule with $\operatorname{supp} \mathcal{M} = \Omega$, does it follow that every T which is supported in Ω) must lie in $\mathcal{M}$?

Arveson (1974): No! Take $\Omega = \{(s, t) \in \mathbb{R}^3 \times \mathbb{R}^3 : t - s \in \mathbb{S}^2\}$ where $\mathbb{S}^2 \subseteq \mathbb{R}^3$.

Operator synthesis

An ω-closed $\Omega \subseteq X \times Y$ is calllled a set of operator synthesis (OS-set) if, for $T \in \mathcal{B}(L^2(X), L^2(Y))$ and $h \in T(X \times Y)$,

$$\mathrm{supp}\, T \subseteq \Omega \subseteq \mathrm{null}\, h \ \Rightarrow\ \langle T, h \rangle = 0.$$

Equivalently, if $\mathcal{M}_1, \mathcal{M}_2$ are w*-closed masa bimodules with $\mathrm{supp}\, \mathcal{M}_i = \Omega$, then $\mathcal{M}_1 = \mathcal{M}_2$.

Synthesis and $\Sigma\upsilon\nu\theta\epsilon\sigma\iota\varsigma$

Theorem

Let G be locally compact second countable. Assume $A(G)$ has the approximation property: $u \in \overline{A(G)u}\ \ \forall u \in A(G)$.
Let $E \subseteq G$ be closed.

E *is an S-set* $\iff$ $E^* = \{(s,t) \in G \times G : ts^{-1} \in E\}$ *is an OS-set*

Due to : Froelich (1988) for abelian G,
Spronk-Turowska (2002) for compact G,
Ludwig-Turowska (2006) for general G but with local synthesis.

• Are there any groups s.t. $A(G)$ fails the approximation property?

NB Various sets of operator multiplicity are also studied (Shulman-Todorov-Turowska)

• What is $A(G)$?

The Fourier algebra $A(G)$ for non abelian groups

Represent G on $L^2(G)$ by $(\lambda_s f)(t) = f(s^{-1}t)$, $f \in L^2(G)$.

Definition (Eymard, 1964)

The Fourier algebra $A(G)$ is the set of all functions $u : G \to \mathbb{C}$ of the form $u(s) = (\lambda_s f, g)$ with $f, g \in L^2(G)$.

- This is a linear space, in fact an algebra of functions on G, complete in the norm is given by $\|u\|_A = \inf \|f\|_2 \|g\|_2$.
- Its dual is (isom. & w*-homeo.) to the von Neumann algebra of G:

$$\mathrm{VN}(G) = \text{w*-span}\{\lambda_s : s \in G\}.$$

Duality: $\langle \lambda_s, u \rangle_a := u(s)$.

Our approach (w. Anoussis & Todorov)

For $E \subseteq G$ closed, recall the ideals of $A(G)$

$$I(E) = \{g \in A(G) : g|_E = 0\}$$
$$J(E) = \overline{\{g \in A(G) : \mathrm{supp}\, g \cap E = \emptyset\}}^{\|\cdot\|_A}.$$

They are largest (resp. smallest) J with $\mathrm{null}(J) = E$.
For a closed ideal $J \lhd A(G)$, consider $J^\perp \subseteq \mathrm{VN}(G) \subseteq \mathcal{B}(L^2(G))$ and 'saturate it' to get

$$\mathrm{Bim}(J^\perp) := \text{w*-span}\{M_f T M_g : T \in J^\perp, f, g \in L^\infty(G)\}.$$

Theorem

Let $E \subseteq G$ closed. If $\mathcal{M} \subseteq \mathcal{B}(L^2(G))$ w-closed masa bimodule with $\mathrm{supp}\,\mathcal{M} = E^*$, then*

$$\mathrm{Bim}(I(E)^\perp) \subseteq \mathcal{M} \subseteq \mathrm{Bim}(J(E)^\perp).$$

Corollary E S-set $\Rightarrow$ E^* OS-set: Immediate.
E^* OS-set $\Rightarrow$ E S-set : when G has approx. property, have $\mathrm{Bim}(J^\perp) \cap \mathrm{VN}(G) = J^\perp$.

Harmonic functionals, Harmonic operators

- Harmonic functions: $k_t * f = f$ for $t > 0$.
- μ-harmonic functions: $\mu * f = f$.

Define (the non-abelian analogue of $\{\hat{\mu} : \mu \in M(\widehat{G})\}$):
$MA(G) = \{\sigma : G \to \mathbb{C} : \sigma u \in A(G)\ \forall u \in A(G)\}$.
We actually need the completely bounded multipliers $M^{\mathrm{cb}}A(G)$

Chu and Lau define σ-harmonic functionals (on $A(G)$)

$$\mathcal{H}_\sigma = \{T \in \mathrm{VN}(G) : \sigma \cdot T = T\}$$

where $\langle \sigma \cdot T, u \rangle = \langle T, \sigma u \rangle$ for all $u \in A(G)$.

Harmonic functionals, Harmonic operators

Neufang and Runde define σ-harmonic operators

$$\widetilde{\mathcal{H}}_\sigma = \{T \in \mathcal{B}(L^2(G)) : \sigma \bullet T = T\}$$

using an action $T \to \sigma \bullet T$ of $M^{\mathrm{cb}}A(G)$ on $\mathcal{B}(L^2(G))$ which extends $T \to \sigma \cdot T$.

Define $T \to \sigma \bullet T$: $\langle \sigma \bullet T, h\rangle = \langle T, (N\sigma)h\rangle$ for all $h \in T(G)$, where $(N\sigma)(s,t) = \sigma(ts^{-1})$ (Varopoulos / Toeplitz-like).

Theorem

Let $\Sigma \subseteq M^{\mathrm{cb}}A(G)$. Then $\widetilde{\mathcal{H}}_\Sigma = \mathrm{Bim}(\mathcal{H}_\Sigma)$.

Special case: Let $P^1(G) = \{\sigma \in C(G) : \text{+ive defn}, \sigma(1) = 1\}$.

Theorem

Let $\Sigma \subseteq P^1(G)$. The space $\widetilde{\mathcal{H}}_\Sigma$ is a the von Neumann subalgebra of $\mathcal{B}(L^2(G))$ generated by the multiplication algebra $\mathcal{D}$ and $\mathcal{H}_\Sigma$.

Jointly invariant subspaces

We call a weak* closed subspace $\mathcal{U} \subseteq \mathcal{B}(L^2(G))$ jointly invariant if it is simultaneously invariant under
(i) all left and right multiplications M_f by $f \in L^\infty(G)$ and
(ii) all $\mathrm{Ad}\rho_r : T \to \rho_r T \rho_r^*$, $r \in G$. (ρ: right regular rep.)

Theorem

Let $\mathcal{U} \subseteq \mathcal{B}(L^2(G))$ be a weak closed subspace. The following are equivalent:*

(i) the space $\mathcal{U}$ is jointly invariant;

(ii) there exists a closed ideal $J \subseteq A(G)$ such that $\mathcal{U} = \mathrm{Bim}(J^\perp)$;

(iii) there exists a subset $\Sigma \subseteq M^{cb}A(G)$ such that $\mathcal{U} = \widetilde{\mathcal{H}}(\Sigma)$.