During the last two decades, several authors have identified several classes of queueing networks (with batch movements, negative customers, signals etc.) that have product-form stationary distributions. On the contrary, there is a very slow progress in the analytic investigation of queueing networks with retrials and there does not exist a single example of such a network with a product-form stationary distribution. In this note we prove the non-existence of product-form solutions for queueing networks with retrials, using a recent characterization of product-form networks.

Keywords: Queueing Networks; Retrials; Product Form Stationary Distributions

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1. Introduction

After the pioneering paper of Jackson (1957) who proved that networks of Markovian queues with single transitions and state-independent Markovian routing have product-form stationary distributions, several authors tried to extend the product-form methodology to investigate several important classes of networks.

Baskett et al. (1975) extended the result to include single transition networks with general service times under specific queueing disciplines that appear in some application areas (processor sharing, LCFS-PR). A significant number of authors published results in this direction. Most of them are summarized in the books of Kelly (1979) and van Dijk (1993).

In the late 80s and early 90s the product form methodology was extended to include networks with batch movements (see e.g. Henderson and Taylor (1990)). Later on, Gelenbe (1991) introduced the notion of a negative customer and proved that simple Markovian networks with negative customers and independent routing do possess product-form stationary distributions. Starting from this work, several authors extended the results to include networks with triggers, state-dependent signalling, batch services, customer coalescence and string
transitions. They also studied network structures with blocking and rerouting
(see e.g. Economou and Fakinos (1998)). Overviews of the above advances un-
der different perspectives are presented in the recent books of Chao et al. (1999)
and Serfozo (1999).

On the contrary, there are very few papers dealing with queueing networks
with retrials as it can be seen in the recent book of Falin and Templeton (1997)
and in the classified bibliography by Artalejo (1999). The available exact re-
results for queueing networks with retrials are very limited and concern very sim-
ple network topologies such as two tandem queues (see e.g. Moutzoukis and
Langaris (2001)). Most authors use approximation and heuristic methods to
study queueing networks with retrials (see e.g. Pourbabai (1990) and Takahara
(1996)).

It seems that the product-form methodology is not applicable for queueing
networks with retrials. Van Dijk (1993) has also noticed this fact but up to now
there exist only intuitive explanations of this remark. In this note we consider
networks of single-server queues with the linear retrial policy introduced by Ar-
talejo and Gomez-Corral (1997), which encompasses as special cases both the
usual and the constant retrial policies. Using the recent product-form charac-
terization of Chao et al. (1998), we prove that networks of single-server queues
with the linear retrial policy and Markovian routing cannot have product-form
stationary distributions.

2. Preliminaries

We will prove the non-existence of product-form solutions for queueing net-
works with retrials by applying a general characterization of Chao et al. (1998).
For completeness we summarize their framework below (for details see Chao et
al. (1999)).

We consider a queueing network with \( N \) nodes denoted as \( 1, 2, \ldots, N \). The
exterior of the network is represented as an additional node \( 0 \). This enables
the treatment of more complicated arrival schemes than the Poisson arrival
process. Every node \( j \) is described by a state variable \( x_j \) taking values in the
state space \( S_j \). Moreover, it is subject to three types of transitions referred to as
arrival, departure or internal transitions that are parametrized by the functions
\( p^A_j(x_j, y_j) \), \( q^D_j(x_j, y_j) \) and \( q^I_j(x_j, y_j) \), \( (x_j, y_j \in S_j) \) respectively. Every node is
characterized by the following dynamics: Suppose that node \( j \) is in state \( x_j \).
Then

(i) An arrival at node \( j \) changes the node’s state from \( x_j \) to \( y_j \) with proba-
bility \( p^A_j(x_j, y_j) \), \( (y_j \in S_j) \).

(ii) The departure rate that changes the state from \( x_j \) to \( y_j \) is \( q^D_j(x_j, y_j) \),
\( (y_j \in S_j) \).

(iii) The internal transition rate that changes the state from \( x_j \) to \( y_j \) is
\( q^I_j(x_j, y_j) \), \( (y_j \in S_j) \).
Thus the overall transition rates \( q_j(x_j, y_j) \) at node \( j \) when the node is fed by a Poisson arrival stream at rate \( \alpha_j \) is

\[
q_j(x_j, y_j) = \alpha_j p_j^A(x_j, y_j) + q_j^D(x_j, y_j) + q_j^I(x_j, y_j) \quad x_j, y_j \in S_j. \tag{1}
\]

This decomposition of the overall rates becomes operative when the nodes are put in a network with Markovian routing. Then we assume that a departure from node \( j \) becomes an arrival at node \( i \) with probability \( r_{ji}, j, i = 0, 1, ..., N \), where \( R = (r_{ji}) \) is a known stochastic matrix.

For any node \( j \), let \((\pi_j(x_j) : x_j \in S_j)\) be the stationary distribution of the node when it is in isolation, fed with a Poisson arrival process with rate \( \alpha_j \). Then, for every \( x_j \in S_j \), let

\[
\tilde{p}_j^A(x_j) = \sum_{y_j} \frac{\pi_j(y_j) p_j^A(y_j, x_j)}{\pi_j(x_j)}, \quad x_j \in S_j, \tag{2}
\]

\[
\tilde{q}_j^D(x_j) = \sum_{y_j} \frac{\pi_j(y_j) q_j^D(y_j, x_j)}{\pi_j(x_j)}, \quad x_j \in S_j \tag{3}
\]

and

\[
\beta_j = \sum_{x_j} \sum_{y_j} \pi_j(x_j) q_j^D(x_j, y_j). \tag{4}
\]

Of course \( \beta_j \) are functions of \( \alpha_j \) (due to their definition through the stationary distribution \((\pi_j(x_j))\) which depends on \( \alpha_j \)). Using the above quantities Chao et al. (1998) proved the following characterization of product-form networks.

**Theorem 1** The network has the product form stationary distribution

\[
\pi(x) = \prod_{j=0}^{N} \pi_j(x_j), \quad x = (x_0, x_1, ..., x_N) \in S, \tag{5}
\]

if and only if each \( \pi_j \) is the stationary distribution of \( q_j \) given by (1) with coefficients (arrival rates) \( \alpha_j \) satisfying the traffic equations

\[
\alpha_j = \sum_{k \neq j} \beta_k(\alpha_k) r_{kj}, \quad j = 0, 1, ..., N \tag{6}
\]

and

\[
(\tilde{q}_j^D(x_j) - \beta_j) r_{jk}(\tilde{p}_k^A(x_k) - 1) + (\tilde{q}_k^D(x_k) - \beta_k) r_{kj}(\tilde{p}_j^A(x_j) - 1) = 0, \tag{7}
\]

for all \( j, k \) with \( j \neq k \) and \( x_j \in S_j, x_k \in S_k \).

3. The result

We consider a network of \( N \) service stations. Every station has a single server with exponential service times and operates as a retrial queue under the linear retrial policy. More specifically we assume the following framework:
(i) Customers arrive from the outside of the network according to a Poisson process at rate $\lambda$. A newly arriving customer is routed to station $j$ with probability $r_{0j}$.

(ii) Station $j$ has a single server and no waiting room. Any customer who finds the server idle upon his arrival, begins to be served. The service times at station $j$ are exponentially distributed with rate $\nu_j$. Any customer who finds the server busy upon his arrival leaves the service area and joins the retrial pool of customers at station $j$.

(iii) When there are $n_j$ customers in the retrial pool at station $j$, the time till the next retrial is exponentially distributed with rate $\mu_j(n_j) = \mu_j(1 - \delta_{nj}) + n_j\mu^*_j$, where $\mu_j, \mu^*_j \geq 0$, $\mu_j + \mu^*_j > 0$ and $\delta_{ij}$ is the Kronecker’s symbol ($\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$). This constitute the so-called linear retrial policy. According to this policy the customers in the retrial pool reattempt for service with independent exponential retrial times with rates $\mu_j^*$. Moreover, the server seeks for customers in the retrial pool with exponential retrial times with rate $\mu_j$ whenever there exists at least one customer in the retrial pool.

(iv) After finishing service at station $j$, a customer goes to station $i$ with probability $r_{ji}$ or leaves the network with probability $r_{j0}$. For simplicity we assume that $r_{jj} = 0$.

(v) The arrival, service and retrial processes are all independent.

To have a Markovian description for the network, every station $j$ is described by a 2-dimensional state $x_j = (c_j, n_j)$ where $c_j$ is the state of the server (1 = busy, 0 = idle) and $n_j$ is the number of customers in the retrial pool. Hence $S_j = \{0, 1\} \times \{0, 1, 2, \ldots\}$, $j = 1, 2, \ldots, N$. The rates of the above retrial queues can be expressed in the general framework of Section 2. For every $j = 1, 2, \ldots, N$ consider two generic states $x_j = (c_j, n_j)$ and $y_j = (c'_j, n'_j)$. Then

$$p^D_j(x_j, y_j) = \begin{cases} 1 & \text{if } (c_j, n_j) = (0, n_j), \ (c'_j, n'_j) = (1, n_j), \ n_j \geq 0 \\ 1 & \text{if } (c_j, n_j) = (1, n_j), \ (c'_j, n'_j) = (1, n_j + 1), \ n_j \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$q^D_j(x_j, y_j) = \begin{cases} \nu_j & \text{if } (c_j, n_j) = (1, n_j), \ (c'_j, n'_j) = (0, n_j), \ n_j \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$q^*_j(x_j, y_j) = \begin{cases} \mu_j(n_j) & \text{if } (c_j, n_j) = (0, n_j), \ (c'_j, n'_j) = (1, n_j - 1), \ n_j \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the stationary distribution of the above station when it is fed with a Poisson arrival process at rate $\alpha_j$ is given by (see Artalejo and Gomez-Corral (1997))

$$\pi_j(0, n_j) = \pi_j(0, 0) \left( \frac{\alpha_n}{\nu_j} \right)^{n_j} \frac{\alpha_j}{\alpha_j + \mu_j} \left( \frac{\alpha_j + \mu_j}{\mu_j + \nu_j + 1} \right)^{n_j}, \ n_j \geq 1,$$

$$\pi_j(1, n_j) = \pi_j(0, 0) \left( \frac{\alpha_n}{\nu_j} \right)^{n_j+1} \frac{\alpha_j + \mu_j}{\mu_j + \nu_j + 1} \left( \frac{\mu_j}{\nu_j + 1} \right)^{n_j}, \ n_j \geq 0, \quad (9)$$
where \((x)_n = x(x + 1)...(x + n - 1)\), \((n \geq 1)\), \((x)_0 = 1\) and the probability \(\pi_j(0,0)\) is obtained by the normalization equation (see (4.3) in Artalejo and Gomez-Corral (1997)).

Hence, the network under consideration cannot have a product-form distribution.

The quantities \(\tilde{\beta}_j^A(x_j)\), \(\tilde{q}^D_j(x_j)\) and \(\beta_j\) are now easily computed by plugging (8) and (9) in (2), (3) and (4) above and considering the various cases. After some algebraic manipulations we have that

\[
\tilde{\beta}_j^A((1,0)) = \frac{\pi_j(0,0)}{\pi_j(1,0)} = \frac{\nu_j}{\alpha_j}, \\
\tilde{\beta}_j^A((1,n_j)) = \frac{\pi_j(0,n_j) + \pi_j(1,n_j-1)}{\pi_j(1,n_j)} = \frac{\nu_j}{\alpha_j}, \quad n_j \geq 1, \\
\tilde{\beta}_j^A((0,n_j)) = 0, \quad n_j \geq 0
\]

and

\[
\tilde{q}_j^D((1,n_j)) = 0, \quad n_j \geq 0, \\
\tilde{q}_j^D((0,0)) = \frac{\pi_j(1,0)\nu_j}{\pi_j(0,0)} = \alpha_j - \mu_j(0), \\
\tilde{q}_j^D((0,n_j)) = \frac{\pi_j(1,n_j)\nu_j}{\pi_j(0,n_j)} = \alpha_j - \mu_j(n_j), \quad n_j \geq 1.
\]

Moreover, (using Artalejo and Gomez-Corral (1997) Equations (4.3) and (4.8)), we obtain \(\beta_j = \sum_{n_j=0}^{\infty} \pi_j(1,n_j)\nu_j = \alpha_j\), \(j = 1, 2, ..., N\). Node 0 has a single state \(x_0\) and we have \(p_0^A(x_0, x_0) = 1\), \(q_0^A(x_0, x_0) = 0\) and \(q_0^D(x_0, x_0) = \lambda\) to model the Poisson process of the network exterior arrivals. Therefore we have \(\tilde{p}_0^A(x_0) = 1\), \(\tilde{q}_0^D(x_0) = \lambda\) and \(\beta_0 = \lambda\). The parameters \(\alpha_j\) that we need to check the conditions (7) of Theorem 1 are the solutions to the traffic equations \(\alpha_j = \lambda r_{0j} + \sum_{k=1}^{N} \alpha_k r_{kj}\), \(j = 1, 2, ..., N\). The network has product-form stationary distribution if and only if the equations (7) hold. Consider two nodes \(j, k\) with \(j \neq k\) and \(x_j = (c_j, n_j)\) and \(x_k = (c_k, n_k)\). We have to distinguish four different cases according to \(c_j\) and \(c_k\) being 0 or 1, because of the form of \(\tilde{p}_j^A(x_j)\) and \(\tilde{q}_j^D(x_j)\). In the first case where \(c_j = c_k = 0\), equations (7) reduce after some simplifications to

\[
\mu_j(n_j)r_{jk} + \mu_k(n_k)r_{kj} = 0.
\]

Clearly, these equations do not hold in general. For example by choosing nodes \(k, j\) with \(k \neq j\) and \(r_{jk} > 0\) and \(n_j > 0\) we have \(\mu_j(n_j)r_{jk} + \mu_k(n_k)r_{kj} > 0\). Hence, the network under consideration cannot have a product-form distribution.

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