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# The restrictive *H*-coloring problem☆

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**Abstract**

We define a variant of the *H*-coloring problem where the number of preimages of certain vertices is predetermined as part of the problem input. We consider the decision and the counting version of the problem, namely the *restrictive H-coloring* and the *restrictive #H-coloring* problems, and we provide a dichotomy theorem determining the *H*'s for which the restrictive *H*-coloring problem is either NP -complete or polynomial time solvable. Moreover, we prove that the same criterion discriminates the #P -complete and the polynomially solvable cases of the *restrictive #H-coloring* problem. Finally, we show that both our results apply also for the list versions and other extensions of the problems.


*Keywords:* Restrictive H-coloring; NP-completeness; P-completeness

## 1. Introduction

Let us consider the following processing setting: we have a host network *H* of processors with communication links between them, and a set of jobs with communication demands between them, where these jobs and their restrictions in their concurrent execution are modeled by a graph *G*. We may have further restrictions, for instance in many practical cases, several qualitative restrictions are imposed by the guest network concerning the types of processors that are able to carry out each of the jobs. In this situation, each job may be accompanied by a list of the processors that are allowed to perform the task. In real systems, the host network wants to keep bounded (or fixed) the load of some of its processors. Thus, some processors may have the *number* of jobs assigned to them as an additional quantitative restriction. The goal is to make a suitable assignment of jobs to processors satisfying all the communication load and all the preference constrains. Historically, the *H*-coloring problem has been a good model to simulate these problems of assignation of paper we propose a model for the full generality problem, incorporating all the above restrictions. In the best of our knowledge, it is the first time, such a model is proposed.

Given two graphs *G* and *H*, a *homomorphism* from *G* to *H* is any function mapping the vertices in *G* to vertices in *H*, in such a way that the image of an edge is also an edge. In the case that *H* is fixed, such a homomorphism is called an *H*-coloring of *G*. For a given graph *H*, the *H-coloring problem* asks whether there exists an *H*-coloring of the input graph *G*, while the *#H-coloring* asks for the number of the *H*-colorings of the input graph *G*. The complexity of both problems depends on the choice of the

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particular graph *H*. It is known that the *H*-coloring problem is polynomial time solvable if *H* is bipartite or it contains a loop; otherwise it is NP -complete [12]. Its counting version, the #*H*-coloring problem, is polynomially solvable if all the connected components of *H* are either complete reflexive graphs or complete irreflexive bipartite graphs [7], otherwise the problem is #P -complete.

The more general version in which a list of allowed processors is given for each job is known as the *list H-coloring* problem. The complexity of the *list H-coloring* problem has been studied in [8–10], and for its counting version, the *list #H-coloring* problem, has been studied in [13,4].

Variants of the *H*-coloring problem in which some quantitative restrictions are fixed independently of the graph have been studied. Bačík considers the *equitable H-coloring* problem [1]. An *H*-coloring is equitable if all the vertices in *H* have approximately the same number of preimages. The problem was also extended by pre-fixing the minimum proportion of vertices to be map into a given vertex. In the case that the graph *H* is *irreflexive*, without loops, it is shown that the equitable coloring problem can be solved in polynomial time when all the connected components of *H* are complete bipartite graphs, otherwise the problem is NP -complete.

The $(H, C, K)$*-coloring* problem was considered in [5,4]. In this variant, the number of pre-images is fixed, independently of the input graph, for a subset of the vertices in *H* is fixed. The complexity of the problem and of its list and counting versions was studied in [5,4]. See [6] and [3] for surveys on different problems based on *H*-colorings.

In this paper we consider the case in which the additional restriction depends on the graph *G*, and thus form part of the input. We call this new problem the *restrictive H-coloring* problem. We examine the complexity of the restrictive *H*-coloring and its variants (see definitions later). We prove that all these problems are polynomial time solvable if all the connected components of the host graph *H* are either complete reflexive graphs or complete irreflexive bipartite graphs. Moreover, we prove that in any other case, the decision problems are NP -complete and the counting problems are #P -complete. Observe that, in contrast to the non restrictive problems, the dichotomy result attained for this problem is the same for both list and non list problems, as well as, for counting and decision problems.

## 2. Definitions

All the graphs in this paper are finite, undirected, and cannot have multiple edges but can have loops. A graph with all its vertices looped is called *reflexive*. If none of the vertices of a graph is looped then we call it irreflexive. We use the notations $V(G)$ and $E(G)$ for the vertex and the edge set of a graph *G*. Trough all the paper let $n = |V(G)|$ be the number of vertices. For a connected bipartite graph *G*, we use the notation $V_1(G)$, $V_2(G)$ to denote the corresponding (unique) partition, with $n_1 = |V_1(G)|$ and $n_2 = |V_1(G)|$. For a given graph *G* and a vertex subset $S \subseteq V(G)$, the subgraph induced by *S* is the graph $G[S] = (S, E(G) \cup S \times S)$. We use standard notation for graphs: $K_n^r$ is a reflexive clique on *n* vertices and $K_{n,m}$ is the complete irreflexive bipartite graph, with partitions of size *n* and *m*.

For a given graph *G*, a function $\mathsf{w} : V(G) \to \{0, \ldots, |V(G)|, \infty\}$ is called a *weight assignment* of *G*. Given a *weight assignment* of *G*, let $n = |V(G)|$, define the set of *bounded functions*

$$\mathsf{B}(\mathsf{w}) = \{f : V(H) \to \{0, \ldots, n\} \mid \text{ for all } a \in V(H) \;\; f(a) \leqslant \mathsf{w}(a)\}$$

and the set of *acceptable functions* as

$$\mathsf{A}(\mathsf{w}) = \{f \in \mathsf{B}(\mathsf{w}) \mid \mathsf{w}(a) = f(a) \text{ for all } a \in H \text{ with } \mathsf{w}(a) \neq \infty\}.$$

Given two graphs *G* and *H*, an *homomorphism* from *G* to *H* is any function $\sigma : V(G) \to V(H)$, where for any edge $\{v, u\} \in E(G)$, $\{\sigma(v), \sigma(u)\}$ is also an edge of *H*. For a fixed graph *H*, we say that $\sigma$ is an *H-coloring* of *G*.

For a fixed graph *H*, the *H-coloring problem* asks for the existence of an *H*-coloring of the input graph *G*, while the *#H-coloring* asks for the number of the *H*-colorings of the input graph *G*.

For a fixed graph *H*, and given a graph *G*, a *list of preferences* is a function $L : V(G) \to 2^{V(H)}$. Given the pair $(G, L)$ a *list H-coloring* of $(G, L)$ is an homomorphism $\sigma$ from *G* to *H* such that for any $v \in V(G)$, $\sigma(v) \in L(v)$.

For a fixed graph *H*, given an input formed by a graph *G* and an associated list of preferences *L*, the *list H-coloring* problem asks for the existence of a list *H*-coloring of the input, while the *list #H-coloring* asks for the number of list *H*-colorings of the input.

For a fixed graph *H*, given an input graph *G* with *n* vertices and a weight assignment w of *H*, a *restrictive H-coloring* of $(G, \mathsf{w})$ is an *H*-coloring $\sigma$ of *G* such that for all $a \in V(H)$ with $\mathsf{w}(a) \neq \infty$, $|\sigma^{-1}(a)| = \mathsf{w}(a)$. When $\mathsf{w}(a) = \infty$, $\sigma^{-1}(a)$ can have any number of vertices. Notice that $\infty$ is used to represent the lack of restrictions on the number of preimages of a vertex, as usual we write $n < \infty$ for any natural *n*. Given a graph *G*, a preference list *L*, and a weight assignment w, a *restrictive list H-coloring*

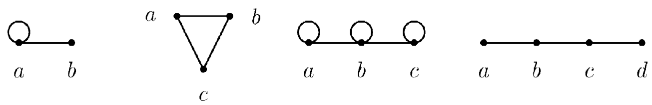


Fig. 1. The four forbidden subgraphs of Lemma 2.

of the triple $(G, L, \mathsf{w})$ is a list $H$-coloring $\sigma$ of $(G, L)$ such that $\sigma$ is also a $(G, \mathsf{w})$ restrictive $H$-coloring. The problems we will treat in this paper are the following:

**Name** : Restrictive $H$-coloring problem
**Input** : A graph $G$ and a weight assignment $\mathsf{w}$ on $H$
**Question** : Does $(G, \mathsf{w})$ have a restrictive $H$-coloring?

**Name** : Restrictive $H$-coloring problem
**Input** : A graph $G$, a list $L$ on $G$, and a weight assignment $\mathsf{w}$ on $H$
**Question** : Does $(G, L, \mathsf{w})$ have a restrictive list $H$-coloring?

Note that, by setting $\mathsf{w}(a) = \infty$ for all $a \in H$, the restrictive $H$-coloring problems solves the corresponding $H$-coloring problem, therefore we can translate all the hardness results to the restrictive problem versions. In particular, the $\#\mathsf{P}$ -hardness results in [7,13,4] translates in the following result.

**Theorem 1.** *If H has a connected component that is not a complete irreflexive bipartite graph or a complete reflexive clique, then the restrictive #$H$-coloring and the restrictive list #$H$-coloring problems both are $\#\mathsf{P}$ -hard.*

In the remaining of the paper we will show that the condition in Theorem 1 discriminates the $\mathsf{P}$ and hard cases for the four restrictive problems.

## 3. NP-completeness results

In this section we show that when $H$ has a connected component which is not a complete irreflexive bipartite graph or a complete reflexive clique, the restrictive $H$-coloring problem, and therefore the restrictive list $H$-coloring decision problem, are $\mathsf{NP}$ -complete. As the two problems are clearly in $\mathsf{NP}$ , we provide only the hardness proofs.

The following characterization of connected graphs is well known [14].

**Lemma 2.** *All the connected components of a graph H are either a complete reflexive graph or a complete irreflexive bipartite graph iff H does not contain as induced subgraphs any of the graphs given in Fig. 1.*

We will take advantage of the previous characterization to show $\mathsf{NP}$ -hardness. Now we can state the $\mathsf{NP}$ -completeness result in this section. Some of the $\mathsf{NP}$ -hardness proofs can also be obtained by a Turing reduction from the equitable coloring problem, using the hardness results in [1]. However, we present simpler *many-to-one* reductions for all the cases.

**Theorem 3.** *If H contains any of the graphs in Fig.* 1 *as an induced subgraph, then the restrictive H-coloring problem is* $\mathsf{NP}$ *-complete.*

**Proof.** We will distinguish four cases, depending on which of the graphs in Fig. 1 appears as an induced subgraph of $H$. Observe that we can select a particular induced subgraph of $H$ by setting to zero the number of tasks that a processor can perform.

*Case* 1: If $\{a, b\}$ is an edge in $H$ where $a$ is looped and $b$ is unlooped then we define

$$\mathsf{w}(v) = \begin{cases} \infty & \text{if } v = a, \\ k & \text{if } v = b, \\ 0 & \text{otherwise.} \end{cases}$$

In this case $(G, \mathsf{w})$ has a restrictive $H$-coloring iff $G$ has an independent set of size at least $k$.

*Case* 2: If $\{a, b, c\}$ form a triangle in $H$ then we set

$$\mathsf{w}(v) = \begin{cases} \infty & \text{if } v \in \{a, b, c\}, \\ 0 & \text{otherwise.} \end{cases}$$

In this case $(G, \mathsf{w})$ has a restrictive *H*-coloring iff *G* is 3-colorable.

*Case* 3: Let now $\{a, b, c\}$ be an induced reflexive path in *H*. We will reduce the following problem to the restrictive *H*-coloring problem:

**Name** : Balanced Separator
**Input** : Graph *G* and positive integer $k \leqslant n$.
**Question** : Is there a partition of $V(G)$ in three sets $A, B, C$, with $|C| = k$, and such that the removal of *C* leaves a graph with no edges between *A* and *B*, and $\max\{|A|, |B|\} \leqslant |V|/2$.

By a slight variation of the NP -hardness proof given in [2] for the *minimum B-vertex separator* problem, the above problem can be shown to be NP -complete

Let *G* be an input of the above problem, we construct a new graph $\tilde{G}$ with $k+1$ new vertices, $V(\tilde{G}) = V(G) \cup \{u_0, \ldots, u_k\}$, and with edge set $E(\tilde{G}) = E(G) \cup \{\{u_0, x\} \mid x \in V(G)\} \cup \{\{u_0, u_i\} \mid 1 \leqslant i \leqslant k\}$.

For any $v \in V(H)$, we set

$$\mathsf{w}(v) = \begin{cases} n/2 & \text{if } v = a, \\ k+1 & \text{if } v = b, \\ n/2 & \text{if } v = c, \\ 0 & \text{otherwise.} \end{cases}$$

**Claim.** *G has a balanced separator if and only if* $(\tilde{G}, \mathsf{w})$ *has a restrictive H-coloring.*

**T.** o prove the if part of the claim, assume that *G* has a balanced separator, then map all the vertices in *A* to *a*, all the vertices in *B* to *b* and all the vertices in *C* to *c*. The vertex $u_0$ is mapped to *b*, an the remaining vertices in $\tilde{G}$ are splitted between *a* and *c* to attain the demanded sizes.

For the only if part, in the case that $(\tilde{G}, \mathsf{w})$ has a restrictive *H*-coloring $\sigma$, by construction all the vertices must be mapped to *a, b* or *c*. Define $A = \sigma^{-1}(a) \cap V(G)$, $B = \sigma^{-1}(b) \cap V(G)$ and $C = \sigma^{-1}(c) \cap V(G)$. As *a* and *c* are not connected, then *B* is a balanced separator. However, *B* might have less than *k* vertices. In such a case, we can move vertices from *A* and/or *B* to pad *C* to the demanded size. This completes the proof of the claim.

*Case* 4: Let now $\{a, b, c, d\}$ be an induced irreflexive path in *H*. We consider the following NP -complete problem [11]:

**Name** : Balanced Complete Bipartite Subgraph
**Input** : Bipartite connected graph $G = (V_1, V_2, E)$ and positive integer *k*, such that $k \leqslant |V_1| + |V_2|$.
**Question** : Does *G* contain $K_{k,k}$ as an induced subgraph?

Let $(G, k)$ be an input of the above problem. Let $u_1$ and $u_2$ be two new vertices not in $V(G)$. We construct a new bipartite graph $\tilde{G} = (W_1, W_2, F)$ with $W_1 = V_1(G) \cup \{u_1\}$ and $W_2 = V_2(G) \cup \{u_2\}$, and with edge set

$$\begin{aligned} F = {} & \{\{u_1, x\} \mid x \in V_2(G)\} \cup \{\{x, u_2\} \mid x \in V_1(G)\} \cup \{\{u_1, u_2\}\} \\ & \cup \{\{x, y\} \mid x \in V_1(G), y \in V_2(G), \text{ and } \{x, y\} \notin E(G).\} \end{aligned}$$

Notice that $\tilde{G}$ is the bipartite complement of *G* with two new adjacent vertices $u_1$ and $u_2$, such that $u_1$ is connected with all the vertices in one part and $u_2$ with all the vertices in the other.

For all $v \in V(H)$, we set

$$\mathsf{w}(v) = \begin{cases} k & \text{if } v = a, \\ \infty & \text{if } v = b, \\ \infty & \text{if } v = c, \\ k & \text{if } v = d, \\ 0 & \text{otherwise.} \end{cases}$$

**Claim.** *G contains* $K_{k,k}$ *as a subgraph if and only if* $\tilde{G}$ *has a restrictive H-coloring.*

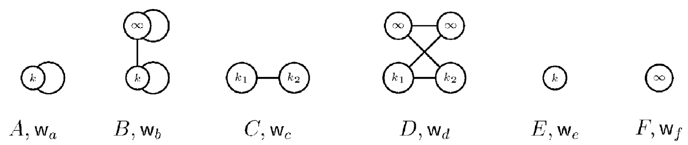


Fig. 2. The six basic cases for counting *H*-colorings.

**T.** o prove the if part of the claim, suppose that $G$ contains $K_{k,k}$ as a subgraph and let, $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ be the partition producing the $K_{k,k}$. Define the function $\sigma : V(G) \to V(H)$ as

$$\sigma(u) = \begin{cases} a & \text{if } u \in U_1, \\ d & \text{if } u \in U_2, \\ c & \text{if } u \in V_1 - U_1, \\ b & \text{if } u \in V_2 - U_2, \\ b & \text{if } u = u_1, \\ c & \text{if } u = u_2. \end{cases}$$

It is straightforward to verify that $\sigma$ is a restrictive $H$-coloring of $(\tilde{G}, \mathsf{w})$.

For the only if part, suppose now that $\sigma : V(G) \to V(H)$ is a restrictive $H$-coloring of $\tilde{G}$. First we prove that $\sigma(\{u_1, u_2\}) = \{b, c\}$. Indeed, if one, say $u_1 \in \{u_1, u_2\}$, is mapped to one, say $a \in \{a, d\}$, then none of the vertices of $\tilde{G}$ can be mapped to $d$, as all the vertices of $\tilde{G}$ are of distance at most 2 from $u_1$. W.l.o.g. assume that $\sigma(u_1) = b$ and $\sigma(u_2) = c$.

Let $U_2$ be the set formed with the neighbors of $u_1$ that are mapped to $a$. Notice that $U_2 \subseteq V_2$ and $|U_2| = k$. Similarly let $U_1$ be the subset with the neighbors of $u_2$ that are mapped to $d$. Observe that $U_1 \subseteq V_1$ and $|U_1| = k$. As $\sigma$ is a $H$-coloring and $\{a, d\}$ is not an edge of $H$, there is no edge in $\tilde{G}$ connecting a vertex in $U_1$ with a vertex in $U_2$. Therefore, in $G$ all the vertices of $U_1$ are connected with all the vertices in $U_2$ which implies that $G[U_1 \cup U_2]$ is $K_{k,k}$. This completes the proof of the claim, and the proof of the theorem. □

Using the fact that the restrictive list $H$-coloring problem can solve the restrictive $H$-coloring problem, we obtain the following NP -hardness result.

**Theorem 4.** *If H has a connected component that is neither a complete irreflexive bipartite graph nor a complete reflexive clique then the restrictive H-coloring and the restrictive list H-coloring problems are* NP *-complete.*

## 4. Restrictive *H*-coloring: the connected case

In this section we solve in polynomial time the counting version of the restrictive $H$-coloring problem in the case that $H$ does not contain as a subgraph any of the forbidden graphs in Fig. 1 and, furthermore, $G$ is connected.

Let us first show that for any of the different graphs and weight assignments shown in Fig. 2, the number of restrictive $H$-colorings of a graph $G$ can be computed in polynomial time.

Given two graphs $G, H$ and a weight assignment $\mathsf{w}$ on $V(H)$, let $\mathsf{H}(G, H, \mathsf{w})$ will denote the number of restrictive $H$-colorings of $(G, \mathsf{w})$. We set $n = |V(G)|$, and for a connected bipartite graph $G$, we set $n_1, n_2$ to be the sizes of the two partitions. We start solving the counting problem for the six graphs depicted in Fig. 2. For each one of them we show a formula that allows to compute in polynomial time the number of restrictive colorings. For sake of simplicity, let $\binom{n}{k} = 0$ whenever $n < k$.

**Lemma 5.** *Given a graph G,* $\mathsf{H}(G, H, \mathsf{w}_H)$ *can computed in polynomial time for any* $(H, \mathsf{w}_H) \in \{(A, \mathsf{w}_a), (B\mathsf{w}_b), (C\mathsf{w}_c), (D\mathsf{w}_d), (E\mathsf{w}_e), (F\mathsf{w}_f)\}$ *(given in Fig. 2).*

**Proof.** For the graph $A$, the unique restriction is the number of allowed pre-images, therefore $\mathsf{H}(G, A, \mathsf{w}_a) = 1$ when $n = k$, otherwise $\mathsf{H}(G, A, \mathsf{w}_a) = 0$.

For the graph $B$, given the pair $(G, \mathsf{w}_b)$, the situation is similar to the previous one. The $B$-colorings must map $k$ vertices of $G$ to the vertex with weight $k$ and the remaining vertices to the other vertex. Therefore, $\mathsf{H}(G, B, \mathsf{w}_b) = \binom{n}{k}$.

For the graph $C$, given the pair $(G, \mathsf{w}_c)$, as $C$ is bipartite it is required that $G$ is bipartite and that $n_1 + n_2 = k_1 + k_2$. To accommodate $G$, we have to control the sizes of its partitions that must fill the allowed number of preimages. Therefore,

$$\mathsf{H}(G, C, \mathsf{w}_c) = \begin{cases} 0 & G \text{ is not bipartite and } n_1 + n_2 \neq k_1 + k_2, \\ 2 & n_1 = n_2 = k_1 = k_2, \\ 1 & \text{otherwise.} \end{cases}$$

For the graph $D$, given the pair $(G, \mathsf{w}_d)$, the situation is similar to the previous one. The $D$-colorings must map $k_1$ vertices in one partition to the vertex with weight $k_1$ and the remaining vertices in the same partition to the unbounded vertex. Therefore,

$$\mathsf{H}(G, D, \mathsf{w}_d) = \begin{cases} 0 & G \text{ is not bipartite and } \\ & n_1 + n_2 \leqslant k_1 + k_2, \\ \binom{n_1}{k_1}\binom{n_2}{k_2} + \binom{n_2}{k_1}\binom{n_1}{k_2} & \text{otherwise.} \end{cases}$$

For the graphs $E$ and $F$ the situation is simpler. For the existence of a coloring the graph $G$ must be an isolated vertex. Furthermore, it is needed that $k = n$ $(k = 1)$. □

The particular cases treated in the previous lemma are the main ingredient in the polynomial time algorithm to compute the number of restrictive $H$-colorings, when $G$ is a connected graph.

**Lemma 6.** *Let $H$ be a reflexive clique, given a connected graph $G$ and a weight assignment $\mathsf{w}$ on $H$, then $\mathsf{H}(G, H, \mathsf{w})$ can be computed in polynomial time.*

**Proof.** Let $C = \{a \in V(H) \mid \mathsf{w}(a) \neq \infty\}$, let $k = \sum_{a \in C} \mathsf{w}(a)$, and let $\alpha = |V(H) - C|$. We will consider two cases.

*Case* 1: $C = V(H)$. In this case, collapsing all the vertices in $H$ into a single vertex and assigning weight $k$ to it, we get the graph $A$ in Fig. 2, with a weight assignment $\mathsf{w}_a$. Observe that any restrictive $A$-coloring of $(G, \mathsf{w}_a)$ can be extended in $k!$ ways to obtain a valid restrictive $H$-coloring of $(G, \mathsf{w})$, and any valid $H$-coloring of $(G, \mathsf{w})$ can be contracted to provide a valid restrictive $A$-coloring of $(G, \mathsf{w}_a)$. Therefore, $\mathsf{H}(G, H, \mathsf{w}) = k!$ when $n = k$, otherwise $\mathsf{H}(G, H, \mathsf{w}) = 0$.

*Case* 2: $C \neq V(H)$. In this case by collapsing all the vertices in $C$ to a vertex with weight $k$ and all the remaining vertices in $V(H) - C$ to a vertex with weight $\infty$, we obtain the graph $B$ in Fig. 2, with a weight assignment $\mathsf{w}_b$. Observe that any restrictive $B$-coloring of $(G, \mathsf{w}_b)$ can be extended in $k!\alpha^{n-k}$ ways to obtain a valid restrictive $H$-coloring of $(G, \mathsf{w})$, and that any valid $H$-coloring of $(G, \mathsf{w})$ can be contracted to provide a valid restrictive $B$-coloring of $(G, \mathsf{w}_b)$. Therefore, $\mathsf{H}(G, H, \mathsf{w}) = k!\,\alpha^{n-k}\binom{n}{k}$. □

**Lemma 7.** *Let $H$ be a complete irreflexive bipartite graph with more than one vertex. Then, given a connected graph $G$ and a weight assignment $\mathsf{w}$ on $H$, $\mathsf{H}(G, H, \mathsf{w})$ can be computed in polynomial time.*

**Proof.** Let $H = (V_1, V_2, E)$. For $i = 1, 2$, let $C_i = \{a \in V_i \mid \mathsf{w}(a) \neq \infty\}$, let $k_i = \sum_{a \in C_i} \mathsf{w}(a)$, and let $\alpha_i = |V_i - C_i|$. We will consider two cases.

*Case* 1: $C_1 = V_1$ and $C_2 = V_2$. In this case collapsing all the vertices in $V_1$ to a vertex with weight $k_1$ and collapsing all the vertices in $V_2$ to a vertex with weight $k_2$ we obtain the graph $C$ in Fig. 2 and a weight assignment $\mathsf{w}_c$. Observe that any restrictive $C$-coloring of $(G, \mathsf{w}_c)$ can be extended in $k_1!\,k_2!$ ways to obtain a valid restrictive $H$-coloring of $(G, \mathsf{w})$, and any valid $H$-coloring of $(G, \mathsf{w})$ can be contracted to provide a valid restrictive $C$-coloring of $(G, \mathsf{w}_c)$. Therefore,

$$\mathsf{H}(G, H, \mathsf{w}) = \begin{cases} 0 & G \text{ is not bipartite and } n_1 + n_2 \neq k_1 + k_2, \\ 2\,k_1!\,k_2! & n_1 = n_2 = k_1 = k_2, \\ k_1!\,k_2! & \text{otherwise.} \end{cases}$$

*Case* 2: $C_1 \neq V_1$ or $C_2 \neq V_2$. In this case by collapsing all the vertices in $C_i$ to a vertex with weight $k_i$ and all the remaining vertices in $V_i$ to an unbounded vertex, we obtain the graph $D$ in Fig. 2 with a weight assignment $\mathsf{w}_d$. Observe that any restrictive $D$-coloring of $(G, \mathsf{w}_d)$ can be extended in $k_1!\,k_2!\,\alpha_1^{n_1-k_1}\,\alpha_2^{n_2-k_2}$ ways to obtain a valid restrictive $H$-coloring of $(G, \mathsf{w})$, and that any valid $H$-coloring of $(G, \mathsf{w})$ can be contracted to provide a valid restrictive $D$-coloring of $(G, \mathsf{w}_d)$. Therefore, $\mathsf{H}(G, H, \mathsf{w})$ is 0 when $G$ is not bipartite or $n_1 + n_2 \leqslant k_1 + k_2$, otherwise

$$\mathsf{H}(G, H, \mathsf{w}) = k_1!\,k_2!\,\alpha_1^{n_1-k_1}\,\alpha_2^{n_2-k_2}\left(\binom{n_1}{k_1}\binom{n_2}{k_2} + \binom{n_2}{k_1}\binom{n_1}{k_2}\right). \qquad \square$$

In the case that $H$ is an isolated vertex, $G$ must also be an isolated vertex and we can compute $\mathsf{H}(G, E, \mathsf{w}_e)$ and $\mathsf{H}(G, F, \mathsf{w}_f)$ in polynomial time.

Now we are ready to prove the main result in this section.

**Theorem 8.** *If all the connected components of H are either a complete irreflexive bipartite graph or a complete reflexive clique, then the restrictive #H-coloring problem can be solved in polynomial time.*

**Proof.** Assume that $H$ has $l$ connected components. Given a weight assignment w on $H$, let $\mathsf{w}_j$ denote the restriction of w to the vertices in $H_j$. As the given graph $G$ is connected, it can be mapped only to one connected component of $H$, therefore we only have to count the number of restrictive $H_j$ colorings of $(G, \mathsf{w}_j)$ that fulfill the weight bounds with an empty assignment of vertices in $G$ to the remaining components.

We classify the connected components of $H$ as follows: $H_j$ is *forbidden* if $\mathsf{w}(H_j) = \{0\}$; $H_j$ is *free* if $\mathsf{w}(H_j) = \{\infty\}$; otherwise $H_j$ is *restricted*. Therefore, we have

$$\mathsf{H}(G, H, \mathsf{w}) = \begin{cases} \sum_{j=1}^{l} \mathsf{H}(G, H_j, \mathsf{w}_j) & \text{if all the components are free or forbidden,} \\ \mathsf{H}(G, H_j, \mathsf{w}_j) & \text{if } H_j \text{ is the unique restricted component,} \\ 0 & \text{if more than one component is restricted.} \end{cases}$$

The last formula can be evaluated in polynomial time by Lemmas 6 and 7. □

Notice that counting in polynomial time implies deciding in polynomial time, so we get the same result for the decision versions.

**Corollary 9.** *If all the connected components of H are either a complete irreflexive bipartite graph or a complete reflexive clique, then the restrictive H-coloring problem can be solved in polynomial time.*

## 5. *H*-coloring: the general case

Now we show how to compute the number of restrictive $H$-colorings for the general case where the graph $G$ might not be connected. Observe that in a restrictive $H$-coloring of $G$ a connected component of $G$ may only provide a part of the demanded number of preimages. Due to this fact, we are forced to take into consideration $H$-colorings of components of $G$ that are $H$-colorings but that fill only part of the number of preimages required by w.

**Theorem 10.** *If all the connected components of H are either a complete irreflexive bipartite graph or a complete reflexive clique, then the restrictive #H-coloring problem can be solved in polynomial time.*

**Proof.** In order to keep an uniform notation, we assume that all the weight assignments are defined over $V(H)$. To fulfill this goal, any weight assignment of a connected component $H_j$ is extended to $H$ by assigning the weight 0 to all the vertices outside $V(H_j)$. We say that a weight assignment w defined over $H$ is *proper* for $H_j$ if for all $u \in V(H) - V(H_j)$, $\mathsf{w}(u) = 0$. We will represent by $\mathsf{P}(j)$ the set of proper functions for the component $G_j$, $1 \leqslant j \leqslant l$.

We assume that $G$ has $m$ connected components $G_1, \ldots, G_m$, and use the notation $G^i$ to denote the graph formed by the disjoint union of $G_1, \ldots, G_i$. For given $G$ and w, to compute $\mathsf{H}(G, H, \mathsf{w})$, we construct initially a table, $T[i, j, f]$, such that for any $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant l$ and $f \in \mathsf{B}(\mathsf{w})$ we have

$$T[i, j, f] = \begin{cases} \mathsf{H}(G_i, H_j, f) & \text{if } f \in \mathsf{P}(j), \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 8, $T[i, j, f]$ can be computed in polynomial time, for any $f$. As $G$ has $n$ vertices, the size of $\mathsf{B}(\mathsf{w})$ is at most $n^h$ and therefore polynomial in the input size, so the whole table can be computed in polynomial time.

Using dynamic programming we can compute a table $S[i, f]$, for $1 \leqslant i \leqslant m$ and $f \in (\mathsf{Bw})$, where $S[i, f]$ keeps the number of restrictive $H$-colorings of $(G^i, f)$. To get the equation, we have only to take into account that a connected component of $G$ must be mapped entirely to a unique connected component of $H$. So, for any $f \in (\mathsf{Bw})$, we get

$$S[1, f] = \sum_{1 \leqslant j \leqslant l} T[1, j, f]$$

and, for any $1 < j \leqslant m$, we get

$$S[j, f] = \sum_{1 \leqslant j \leqslant l, f_1 + f_2 = f} S[j-1, f_1] \cdot T[i, j, f_2].$$

As the size of $\mathsf{B}(\mathsf{w})$ is polynomial, table $S$ can be computed in polynomial time.

Finally, we have,

$$\mathsf{H}(G, H, \mathsf{w}) = \sum_{f \in A(\mathsf{w})} S[m, f],$$

which again can be computed in polynomial time. □

Putting together Theorems 1, 4 and 10, we get the dichotomy result.

**Theorem 11.** *If all connected components of H are either a complete irreflexive bipartite graph or a complete reflexive clique, then the restrictive H-coloring and the restrictive #H-coloring problems can be solved in polynomial time, otherwise they are* NP *-complete or* #P *-complete, respectively.*

## 6. The restrictive list *H*-coloring problem

Now we will show how to extend the previous result to counting restrictive list *H*-colorings. The main difficulty here is that the vertices in a connected component of *H* cannot be collapsed to a single vertex, because this may put together vertices that are not in the same vertex list. Once we have solved the connected case the second step is identical to the disconnected case for the restrictive *H*-coloring.

We will consider the two main types of connected components and show that a dynamic programming approach allow us to compute the number of restrictive list *H*-colorings. Making an abuse of notation we will represent by $\mathsf{H}(G, H, \mathsf{w}, L)$ the number of restrictive list *H*-colorings of a triple $(G, \mathsf{w}, L)$.

**Lemma 12.** *Let H be a reflexive clique. Then, given a connected graph G a weight assignment* w *on H and a list selection L for G,* $\mathsf{H}(G, H, \mathsf{w}, L)$ *can be computed in polynomial time.*

**Proof.** As *H* is a reflexive clique we can assign a vertex of *G* to any vertex in *H* provided the additional restrictions are fulfilled. Let $V(G) = \{u_1, \ldots, u_n\}$. For any $a \in H$ define $f_a$ by

$$f_a(b) = \begin{cases} 1 & \text{if } b = a, \\ 0 & \text{otherwise.} \end{cases}$$

We want to compute a table $R[i, f], 1 \leqslant i \leqslant n, f \in \mathsf{B}(\mathsf{w})$, which counts the number of restrictive list *H*-colorings for the triple $(G[\{u_1, \ldots, u_i\}], f, L)$. The recurrence is the following: for any $f \in \mathsf{B}(\mathsf{w})$

$$R[1, f] = \begin{cases} 1 & \text{if } \exists a \in L(u_1) \;\; f = f_a, \\ 0 & \text{otherwise} \end{cases}$$

and, for any $1 < j \leqslant m$,

$$R[i, f] = \sum_{\substack{\exists\, a \in L(u_1) \;\; f_2 = f_a \\ f_1 + f_2 = f}} R[j-1, f_1].$$

As the size of $\mathsf{B}(\mathsf{w})$ is polynomial, we can fill the table $R$ in polynomial time.

Finally,

$$\mathsf{H}(G, H, \mathsf{w}, L) = \sum_{f \in \mathsf{A}(\mathsf{w})} R[n, f]. \qquad \square$$

**Lemma 13.** *Let H be a complete irreflexive bipartite graph with more than one vertex. Then, given a connected graph G, a weight assignment* w *on H, and a list selection L for G,* $\mathsf{H}(G, H, \mathsf{w}, L)$ *can be computed in polynomial time.*

**Proof.** If $H$ is bipartite, then $G$ must be bipartite. In this case, we can work separately with the two possible assignments of partitions of $G$ with partitions of $H$. Notice that once the global assignment is set, any vertex can be mapped to any one in the assigned partition, thus working as in the previous lemma we can compute $\mathsf{H}(G, H, \mathsf{w})$ in polynomial time. □

Using the same technique of Section 5, we can obtain the polynomial time result, this together with Theorems 1 and 4 give the dichotomy for the list version of the problem.

**Theorem 14.** *If all the connected components of H are either a complete irreflexive bipartite graph or a complete reflexive clique, then the restrictive list H-coloring and the restrictive list #H-coloring problems can be solved in polynomial time, otherwise they are* NP *-complete and* #P *-complete, respectively.*

## 7. Further variations and conclusions

We can consider a variation of the restrictive $H$-coloring problem, in which the weight of each element is replaced by a list of weight ranges. In the most general version the *loosely restrictive list H-coloring*, the input is a triple $(G, L, \mathsf{W})$, where $G$ is a graph $G$, $L$ is a preference list for $G$, and $\mathsf{W}$ is a range weights list for $H$. This problem acts as a generic mark for solving the equitable $H$-coloring problem, the $(H, C, K)$-coloring problem and the restrictive $H$-coloring problem, in their plain and list versions. Its hardness follows from Theorem 1. Using the algorithm described in the proof of Theorem 10, we can compute the number of restrictive list $H$-colorings of $(G, f)$, for any weight assignment $f$, therefore we can compute the number of "loosely restrictive" list $H$-colorings in polyomial time. And we attain the same dichotomy result as for the corresponding restrictive version.

Two variants of the list $H$-coloring problem have been considered in the literature: the *connected list H-coloring* problem, in which any set in the list must induce a connected subgraph; and the *one-all list H-coloring*, in which all the sets in the list either contain all the vertices in $G$ or exactly one vertex. The two problems were introduced in [8]. As the $H$-coloring problem can be reduced to both variants the dichotomy results presented in this paper also hold for the restrictive and loosely restrictive decision and counting version of both problems.

The running time of the algorithms in this paper has the form $\mathrm{O}(n^{h+c})$, where $c$ is a constant independent of $h$. It is an open problem to characterize the family of graphs for which a time bound of the form $\mathrm{O}(f(h)n^c)$ can be achieved, where $f$ depends on $h$ but it is independent of $n$.

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