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DOMINATING SETS IN PLANAR GRAPHS: BRANCH-WIDTH AND EXPONENTIAL SPEED-UP*

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Abstract. We introduce a new approach to design parameterized algorithms on planar graphs which builds on the seminal results of Robertson and Seymour on graph minors. Graph minors provide a list of powerful theoretical results and tools. However, the widespread opinion in the graph algorithms community about this theory is that it is of mainly theoretical importance. In this paper we show how deep min-max and duality theorems from graph minors can be used to obtain exponential speed-up to many known practical algorithms for different domination problems. Our use of branch-width instead of the usual tree-width allows us to obtain much faster algorithms. By using this approach, we show that the k-dominating set problem on planar graphs can be solved in time $O(2^{15.13\sqrt{k}} + n^3)$.

Key words. branch-width, tree-width, dominating set, planar graph, fixed-parameter algorithm

AMS subject classifications. 05C35, 05C69, 05C83, 05C85, 68R10

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1. Introduction.

DOMINATING SET is a classic NP-complete graph problem which fits into the broader class of *domination* and *covering* problems on which hundreds of papers have been written. (The book of Haynes, Hedetniemi, and Slater [32] is a nice source for further references on the dominating set problem.) The problem PLANAR DOMINATING SET asks, given a planar graph G and a positive k, whether G has a dominating set of size at most k. It is well known that the PLANAR DOMINATING SET (as well as several variants of it) is NP-complete. In this paper we design exact *fixed-parameter* algorithms (which run fast provided that the parameter k is small). The theory of fixed-parameter algorithms and parameterized complexity has been thoroughly developed over the past few years; see, e.g., [1, 3, 4, 8, 12, 13, 21, 23, 24].

The last six years have seen dramatic developments and improvements to the design of subexponential algorithms with running times of $2^{O(\sqrt{k})}n^{O(1)}$ for different planar graph problems; see, e.g., [1, 4, 8, 9, 13, 14, 22, 31, 35]. For example, the first algorithm for the PLANAR DOMINATING SET appeared in [2], with running time $O(8^k n)$. The first algorithm with a *sublinear* exponent was given by Alber et al. in [1] and its running time was $O(2^{69.98\sqrt{k}}n)$. A time $O(2^{49.88\sqrt{k}}n)$ algorithm was obtained in [4], and Kanj and Perković [35] announced an algorithm of running time $O(2^{27\sqrt{k}}n)$.

A common method for solving PLANAR DOMINATING SET is to prove that every planar graph with a dominating set of size at most k has tree-width at most $c\sqrt{k}$, where c is a constant. With some work (sometimes very technical) a tree decomposition of width at most $c\sqrt{k} + O(1)$ is constructed, and standard dynamic programming techniques on graphs of bounded tree-width are implemented. Currently, the fastest

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dynamic programming algorithm for a dominating set on graphs of tree-width at most t runs in $O(2^{2t}n)$ steps and was given by Alber et al. in [1]. This implies an $O(2^{2c\sqrt{k}}n)$ step algorithm for the PLANAR DOMINATING SET. Let

$$c_{\mathrm{tw}} = \min\{c \mid \text{if } G \text{ is planar and dominated by } k \text{ vertices, then } \mathbf{tw}(G) \leq c\sqrt{k} + O(1)\}.$$

The challenge in this approach is that a small bound for c_{tw} is required for most practical applications. The first bound for c_{tw} appeared in [1] and was $c_{\mathrm{tw}} < 6\sqrt{34} = 34.98$, while the next improvement was given by Kanj and Perković in [35], who proved that $c_{\mathrm{tw}} < 16.5$.

The main tool of this paper is the branch-width of a graph. Branch-width was introduced by Robertson and Seymour in their graph minors series of papers several years after tree-width. These parameters are rather close, but surprisingly many theorems of the graph minors series are easier to prove when one uses branch-width instead of tree-width. Nice examples of the use of branch-width in proof techniques can be found in [38] and [39]. Another powerful property of branch-width is that it can be naturally generalized for hypergraphs and matroids. A good example of generalization of Robertson and Seymour theory for matroids by using branch-width is the paper by Geelen, Gerards, and Whittle [29]. Algorithms for problems expressible in monadic second-order logic on matroids of bounded branch-width are discussed by Hliněný [34]. Alekhnovich and Razborov [5] use branch-width of hypergraphs to design algorithms for SAT.

From a practical point of view, branch-width is also promising. For some problems, branch-width is more suitable for actual implementations. Cook and Seymour [10, 11] used branch decompositions to solve the ring routing problem, related to the design of reliable cost-effective SONET networks and to solving TSP (see also [7, 19]). In theory, there is not a big difference between tree-width and branch-width based algorithms. However in practice, branch-width is sometimes easier to use. The question due to Bodlaender (private communication) is the following: Are there examples where the constant factors for branch-width algorithms are significantly smaller than for their tree-width counterparts? This paper is partially motivated[1] by this question.

Our results. In this paper we introduce a new approach for solving the PLANAR DOMINATING SET problem. Our approach is based on branch-width and provides an algorithm of running time $O(2^{15.13\sqrt{k}} + n^3)$, which is a significant step toward a practical algorithm. Instead of constructing a tree decomposition and proving that the width of the obtained tree decomposition is upper bounded by $c\sqrt{k}$, we prove a combinatorial result relating the branch-width with the domination number of a planar graph. The proof of the combinatorial bounds is complicated and is based on nice properties of branch-width, which follow from deep results of the graph minors series.

Our proof is not constructive, in the sense that it cannot be turned into a polynomial algorithm that *constructs* the corresponding branch decomposition. Fortunately, there is a well-known algorithm due to Seymour and Thomas for computing an optimal branch decomposition of a planar graph in $O(n^4)$ steps. We stress that this algorithm does not have the so-called enormous hidden constants and is really practical.

[1] One of the challenges that appeared during the workshop "Optimization Problems, Graph Classes and Width Parameters" (Centre de Recerca Matemàtica, Bellaterra, Spain, November 15–17, 2001), was the following question: *Is it possible, using bounded branch-width instead of bounded tree-width, to obtain more efficient solutions for* PLANAR DOMINATING SET *and other parameterized problems?*

(We refer to the work of Hicks [33] on implementations of the Seymour and Thomas algorithm; see also [30] for a recent algorithm that runs in $O(n^3)$ steps.)

Our main combinatorial result is that for every planar graph G with a dominating set of size $\leq k$, the branch-width of G is at most $3\sqrt{4.5}\sqrt{k} < 6.364\sqrt{k}$. We combine this bound with the following algorithmic results: (i) the algorithm of Seymour and Thomas for planar branch-width, (ii) the results of Alber, Fellows, and Niedermeier [3] on a linear problem kernel for PLANAR DOMINATING SET, and (iii) a new dynamic programming algorithm for solving the dominating set problem on graphs of bounded branch-width (see subsection 4.2). As a result, we obtain an algorithm of running time $O(2^{15.13\sqrt{k}} + n^3)$.

According to Robertson and Seymour [36], for any graph G with at least three edges, the tree-width of G is always bounded by $\frac{3}{2}$ times its branch-width. This result, in combination with our bound, implies that $c_{\mathrm{tw}} < 9.546$. To our knowledge, this gives an improvement on any other bound for the tree-width of planar graphs dominated by k vertices.

Organization of the paper. In section 2, we give basic definitions and state some known theorems. We also present how a theorem of Robertson, Seymour, and Thomas can be directly used to prove that every planar graph with a dominating set of size $\leq k$ has branch-width at most $\leq 12\sqrt{k} + 9$. This observation (combined with the results discussed in section 4) already implies an algorithm for the PLANAR DOMINATING SET problem with running time $O(2^{28.56\sqrt{k}} + n^3)$, where n is the number of vertices of G. This is already a strong improvement (for large k) on the result of Alber et al. in [1] and is close to the running time $O(2^{27\sqrt{k}}n)$ of the algorithm of Kanj and Perković in [35].

Section 3 is devoted to the proof of Theorem 3.22, the main combinatorial result of the paper. The proof of this result is complicated, and we split it into several parts. In subsection 3.1, we give technical results about branch decompositions. These results are based on the powerful theorem of Robertson and Seymour on the branch-width of dual graphs. We emphasize that these results are crucial for our proof. In subsection 3.2, we define the notion of the *extension* of a graph and prove that the branch-width of an extension is at most three times the branch-width of the original graph. In section 3.3 we introduce the notion of *nicely dominated graphs*, which is a suitable "normalization" of the structure of the dominated planar graphs. In subsection 3.4, we explain how nicely dominated graphs can be gradually decomposed into simpler ones so that the branch-width of the original graph is bounded by the branch-width of some "enhanced version" of the simpler ones. In subsection 3.5 we introduce the *prime nicely dominated graphs* as those that are "the simplest possible" with respect to the decomposition of subsection 3.4. In subsection 3.6, we prove that any prime nicely dominated graph G is "contained" in the extension of a simpler planar graph denoted as $\mathbf{red}(G)$. In subsection 3.7 we use this fact along with the results of subsections 3.2, 3.4, and 3.6 to prove that $\mathbf{bw}(G) \leq 3 \cdot \mathbf{bw}(\mathbf{red}(G))$. By its construction, all the vertices of $\mathbf{red}(G)$ are vertices of the dominating set D. The result follows because, according to [28], $\mathbf{bw}(\mathbf{red}(G)) \leq \sqrt{4.5 \cdot |D|}$.

Section 4 contains discussions on algorithmic consequences of the combinatorial result. Subsection 4.1 describes the general algorithmic scheme that we follow. Subsection 4.2 contains a dynamic programming algorithm for the solving dominating set problem on graphs of branch-width $\leq \ell$ and m edges, in time $O(3^{1.5 \cdot \ell} m)$.

In section 5 we discuss the optimality of our results (subsection 5.1) and provide some concluding remarks and open problems (subsection 5.2).

2. Definitions and preliminary results. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For every nonempty $W \subseteq V(G)$, the subgraph of G induced by W is denoted by $G[W]$. A vertex $v \in V(G)$ of a connected graph G is called a *cut vertex* if the graph $G - \{v\}$ is not connected. A connected graph on at least three vertices without a cut vertex is called 2-*connected*. Maximal 2-connected subgraphs of a graph G or induced edges whose two endpoints are cut vertices are called 2-*connected components*.

Let Σ be a sphere. By Σ-*plane* graph G we mean a planar graph G with the vertex set $V(G)$ and the edge set $E(G)$ drawn in Σ. To simplify notations, we usually do not distinguish between a vertex of the graph and the point of Σ used in the drawing to represent the vertex, or between an edge and the open line segment representing it. If $\Delta \subseteq \Sigma$, then $\overline{\Delta}$ denotes the *closure* of Δ, and the boundary of Δ is $\widehat{\Delta} = \overline{\Delta} \cap \overline{\Sigma - \Delta}$. We denote the set of the faces of the drawing by $R(G)$. (Every face is an open set.) An edge e (a vertex v) is incident to a face r if $e \subseteq \bar{r}$ ($v \subseteq \bar{r}$). We do not distinguish between a boundary of a face and the subgraph of the drawing induced by edges incident to the face. The *length* of a face r is the number of edges incident to r. $\Delta \subseteq \Sigma$ is an open disc if it is homeomorphic to $\{(x, y) : x^2 + y^2 < 1\}$. Let C be a cycle in a Σ-plane graph G. By the Jordan curve theorem, C bounds exactly two discs. For a vertex $x \in V(G)$, we call a disc Δ bounded by C x-*avoiding* if $x \notin \overline{\Delta}$. We call a face $r \in R(G)$ *square face* if $\widehat{r}$ is a cycle of length four.

A set $D \subseteq V(G)$ is a *dominating set* in a graph G if every vertex in $V(G) - D$ is adjacent to a vertex in D. Graph G is D-*dominated* if D is a dominating set in G.

For a hypergraph $\mathcal{G}$ we denote by $V(\mathcal{G})$ its vertex (ground) set and by $E(\mathcal{G})$ the set of its hyperedges. A *branch decomposition* of a hypergraph $\mathcal{G}$ is a pair (T, τ), where T is a tree with vertices of degree one or three and τ is a bijection from $E(\mathcal{G})$ to the set of leaves of T. The *order function* $\omega : E(T) \rightarrow 2^{V(\mathcal{G})}$ of a branch decomposition maps every edge e of T to a subset of vertices $\omega(e) \subseteq V(\mathcal{G})$ as follows. The set $\omega(e)$ consists of all vertices $v \in V(\mathcal{G})$ such that there exist edges $f_1, f_2 \in E(\mathcal{G})$ with $v \in f_1 \cap f_2$, and such that the leaves $\tau(f_1)$, $\tau(f_2)$ are in different components of $T - \{e\}$.

The *width* of (T, τ) is equal to $\max_{e \in E(T)} |\omega(e)|$, and the *branch-width* of $\mathcal{G}$, $\mathbf{bw}(\mathcal{G})$, is the minimum width over all branch decompositions of $\mathcal{G}$.

Given an edge $e = \{x, y\}$ of a graph G, the graph G/e is obtained from G by contracting the edge e; that is, to get G/e we identify the vertices x and y and remove all loops and duplicate edges. A graph H obtained by a sequence of edge contractions is said to be a *contraction* of G. H is a *minor* of G if H is a subgraph of a contraction of G. We use the notation $H \preceq G$ (resp., $H \preceq_c G$) when H is a minor (a contraction) of G. It is well known that $H \preceq G$ or $H \preceq_c G$ implies $\mathbf{bw}(H) \leq \mathbf{bw}(G)$. Moreover, the fact that G has a dominating set of size k and $H \preceq_c G$ imply that H has a dominating set of size $\leq k$ (which is not true for $H \preceq G$).

For planar graphs the branch-width can be bounded in terms of the domination number by making use of the following result of Robertson, Seymour, and Thomas (Theorems 4.3 in [36] and 6.3 in [38]).

THEOREM 2.1 (see [38]). *Let $k \geq 1$ be an integer. Every planar graph with no (k, k)-grid as a minor has branch-width $\leq 4k - 3$.*

To give an idea on how results from graph minors can be used on the study of dominating sets in planar graphs, we present the following simple consequence of Theorem 2.1.

LEMMA 2.2. *Let G be a planar graph with a dominating set of size $\leq k$. Then $\mathbf{bw}(G) \leq 12\sqrt{k} + 9$.*

Proof. Suppose that $\mathbf{bw}(G) > 12\sqrt{k}+9$. By Theorem 2.1, there exists a sequence of edge contractions or edge/vertex removals reducing G to a (ρ,ρ)-grid where $\rho = 3\sqrt{k}+3$. We apply to G only the contractions from this sequence and call the resulting graph J. J contains a (ρ,ρ)-grid as a subgraph. As $J \preceq_c G$, J also has a dominating set D of size $\leq k$. A vertex in D cannot dominate more than nine internal vertices of the (ρ,ρ)-grid. Therefore, $k \geq (\rho-2)^2/9$, which implies $\rho \leq 3\sqrt{k}+2 = \rho - 1$, a contradiction. ◻

In the remaining part of the paper we show how the above upper bound for the branch-width of a planar graph in terms of its dominating set number can be strongly improved. Our results will use as a basic ingredient the following theorem, which is a direct consequence of the Robertson and Seymour min-max Theorem 4.3 in [36] relating tangles and branch-width and Theorem 6.6 in [37] establishing relations between tangles of dual graphs. Since the result is not mentioned explicitly in the articles of Robertson and Seymour, we provide here a short explanation of how it can be derived.

We denote as K_2^2 the graph consisting of two vertices connected by a double edge. Notice that K_2^2 is a dual of itself; therefore, if G contains K_2^2 as a minor, then its dual G^* also contains K_2^2 as a minor.

THEOREM 2.3. *Let G be a Σ-plane graph that contains K_2^2 as a minor and let G^d be its dual. Then $\mathbf{bw}(G) = \mathbf{bw}(G^d)$.*

Proof. A *separation* of a graph G is a pair (A,B) of subgraphs with $A \cup B = G$ and $E(A \cap B) = \emptyset$, and its *order* is $|V(A \cap B)|$. A *tangle of order* $\theta \geq 1$ is a set $\mathcal{T}$ of separations of G, each of order less than θ, such that

1. for every separation (A,B) of G of order less than θ, $\mathcal{T}$ contains one of (A,B) and (B,A);
2. if $(A_1,B_1),(A_2,B_2),(A_3,B_3) \in \mathcal{T}$, then $A_1 \cup A_2 \cup A_3 \neq G$; and
3. if $(A,B) \in \mathcal{T}$, then $V(A) \neq V(G)$.

The *tangle number* $\theta(G)$ of G is the maximum order of tangles in G. By the result of Robertson and Seymour [36, Theorem 4.3], for any graph G of branch-width at least two, $\theta(G) = \mathbf{bw}(G)$. Since $\mathbf{bw}(K_2^2) = 2$ and $K_2^2 \preceq G$, we have that $\theta(G) = \mathbf{bw}(G)$. By similar arguments, $\theta(G^d) = \mathbf{bw}(G^d)$.

Let G be a graph 2-cell embedded in a connected surface Σ. A subset of Σ meeting the drawing only at vertices of G is called *G-normal*. The *length* of a G-normal arc is the number of vertices it meets. A tangle $\mathcal{T}$ of order θ is *respectful* if, for every homeomorphic to a circle G-normal arc N in Σ of length less than θ, there is a closed disk $\Delta \subseteq \Sigma$ with $\widehat{\Delta} = N$ such that the separation $(G \cap \Delta, G \cap \overline{\Sigma - \Delta}) \in \mathcal{T}$. By the first tangle property, every tangle $\mathcal{T}$ of a graph embedded in a *sphere* is respectful.

By [37, Theorem 6.6], for every 2-cell embedded graph G on a connected surface Σ, G has a respectful tangle of order θ if and only if its dual G^d does. This implies that $\theta(G) = \theta(G^d)$ and the theorem follows. ◻

For our bounds, we need an upper bound on the size of branch-width of a planar graph in terms of its size. The best published bound for the branch-width that we were able to find in the literature is $\mathbf{bw}(G) \leq 4\sqrt{|V(G)|} - 3$ which follows directly from Theorem 2.1. An improvement of this inequality can be found in [28]. This proof is based on a relation between slopes and majorities, the two notions introduced by Robertson and Seymour in [36] and Alon, Seymour, and Thomas in [6], respectively.

THEOREM 2.4 (see [28]). *For any planar graph G, $\mathbf{bw}(G) \leq \sqrt{4.5 \cdot |V(G)|}$.*

3. Bounding branch-width of *D*-dominated planar graphs. This section is devoted to the proof of the main combinatorial result of this paper: The branch-width of any planar graph with a dominating set of size k is at most $3\sqrt{4.5}\sqrt{k}$. The idea of the proof is to show that for every planar graph G with a dominating set of size k there is a graph H on at most k vertices such that $\mathbf{bw}(G) \leq 3 \cdot \mathbf{bw}(H)$. Then Theorem 2.4 will do the rest of the job.

The construction of the graph H and the proof of $\mathbf{bw}(G) \leq 3 \cdot \mathbf{bw}(H)$ is not direct. First we prove that every planar graph with a dominating set D is a minor of some graph with a nice structure. We call these "structured" graphs nicely D-dominated. For a nicely D-dominated planar graph G we show how to define a graph $\mathbf{red}(G)$ on $|D|$ vertices. The most complicated part of the proof is the proof that $\mathbf{bw}(G) \leq 3 \cdot \mathbf{bw}(\mathbf{red}(G))$ (clearly this implies the main combinatorial result). The proof of this inequality is based on a more general result about isomorphism of special hypergraphs obtained from G and $\mathbf{red}(G)$ (Lemma 3.16) and the structural properties of nicely D-dominated graphs.

3.1. Auxiliary results. In this section we obtain some useful technical results about branch-width.

LEMMA 3.1. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be hypergraphs with one hyperedge in common, i.e., $V(\mathcal{G}_1) \cap V(\mathcal{G}_2) = f$ and $\{f\} = E(\mathcal{G}_1) \cap E(\mathcal{G}_2)$. Then* $\mathbf{bw}(\mathcal{G}_1 \cup \mathcal{G}_2) \leq \max\{\mathbf{bw}(\mathcal{G}_1), \mathbf{bw}(\mathcal{G}_2), |f|\}$. *Moreover, if every vertex $v \in f$ has degree ≥ 2 in at least one of the hypergraphs (i.e., v is contained in at least two edges in $\mathcal{G}_1$ or in at least two edges in $\mathcal{G}_2$), then* $\mathbf{bw}(\mathcal{G}_1 \cup \mathcal{G}_2) = \max\{\mathbf{bw}(\mathcal{G}_1), \mathbf{bw}(\mathcal{G}_2)\}$.

Proof. Clearly, $\mathbf{bw}(\mathcal{G}_1 \cup \mathcal{G}_2) \geq \max\{\mathbf{bw}(\mathcal{G}_1), \mathbf{bw}(\mathcal{G}_2)\}$.

For $i = 1, 2$, let (T_i, τ_i) be a branch decomposition of $\mathcal{G}_i$ of width $\leq k$ and let $e_i = \{x_i, y_i\}$ be the edge of T_i having as endpoint the leaf $\tau_i(f) = x_i$. We construct tree T as follows. First we remove the vertices x_i and add edge $\{y_1, y_2\}$. Then we subdivide $\{y_1, y_2\}$ by introducing a new vertex y. Finally we add vertex x and make it adjacent to y.

We set $\tau(f) = x$. For any other edge $g \in E(\mathcal{G}_1) \cup E(\mathcal{G}_2)$ we set $\tau(g) = \tau_1(g)$ if $g \in E(\mathcal{G}_1)$ and $\tau(g) = \tau_2(g)$ otherwise.

Because $|\omega(\{y_1, y\})| = |\omega(\{y_2, y\})| = |\omega(\{x, y\})| \leq |f|$ and for any other edge of T, its order is equal to the order of the corresponding edge in one of the T_i's, we have that (T, τ) is a branch decomposition of width $\leq \max\{k, |f|\}$.

If every vertex v of f has degree ≥ 2 in one of the hypergraphs, then $|f| \leq \max\{|\omega(e_1)|, |\omega(e_2)|\} \leq k$. Thus in this case, (T, τ) is a branch decomposition of width $\leq k$. ∎

Let G be a connected Σ-plane graph with all vertices of degree at least two. For a vertex x of G and a pair (z, y) of two of its neighbors, we call (z, y) a *pair of consecutive neighbors of* x if edges $\{x, z\}$, $\{x, y\}$ appear consecutively in the cyclic ordering of the edges incident to x. (Notice that if x has only two neighbors y and z, then both (y, z) and (z, y) are pairs of consecutive neighbors of x.)

LEMMA 3.2. *Let G be a planar graph. Then G is the minor of a planar 2-connected graph H such that* $\mathbf{bw}(H) = \max\{\mathbf{bw}(G), 2\}$.

Proof. We use induction on the number of vertices in G. Every graph on at most three vertices is the minor of a complete graph on three vertices, which is 2-connected and has branch-width two. Suppose that the lemma is correct for every planar graph on at most n vertices.

Let G be a graph on $n + 1$ vertices.

Case A. *G is 2-connected.* In this case the lemma trivially holds.

Case B. *G is connected (but not 2-connected).* Then G has a cut vertex x. Let $V_1, V_2, \ldots, V_k$ be the vertex sets of the connected components of $G - \{x\}$. Let G_1 be the subgraph of G induced by $V_1 \cup \{x\}$ and let G_2 be the subgraph of G induced by $V_2 \cup V_3 \cup \cdots \cup V_k \cup \{x\}$.

By the induction assumption, there are 2-connected planar graphs H_i, $i = 1, 2$, such that $\mathbf{bw}(H_i) = \max\{\mathbf{bw}(G_i), 2\}$, and $G_i \prec H_i$.

Planar graphs H_1 and H_2 have only one common vertex x, and thus the graph $H_1 \cup H_2$ is also planar. Let H be a Σ-plane graph which is a drawing of $H_1 \cup H_2$. Let a and b be two consecutive neighbors of x in H (i.e., vertices such that the edges $\{a, x\}$, $\{b, x\}$ are incident to the same face), where $a \in V(H_1)$ and $b \in V(H_2)$. We denote by H' the graph obtained from H by drawing the edge $\{a, b\}$ so that it does not intersect other edges of H (this is possible because $\{a, x\}$, $\{b, x\}$ are incident to the same face). Let us remark that H' is 2-connected and contains H (and therefore G) as a minor.

The complete graph K on three vertices $\{a, b, x\}$ has one common edge $\{a, b\}$ with H_1. The degrees of a and x in K are two, and at least two in H_1 (H_1 is 2-connected). By Lemma 3.1, we have that

$$\mathbf{bw}(H_1 \cup K) = \max\{\mathbf{bw}(H_1), 2\} = \max\{\mathbf{bw}(G_1), 2\}.$$

By applying Lemma 3.1, for $H_1 \cup K$ and H_2, we arrive at

$$\mathbf{bw}(H') = \mathbf{bw}(H_1 \cup H_2 \cup K) = \max\{\mathbf{bw}(G_1), \mathbf{bw}(G_2), 2\} \leq \max\{\mathbf{bw}(G), 2\}.$$

Since G is the minor of H', we have that $\mathbf{bw}(H') = \max\{\mathbf{bw}(G), 2\}$.

Case C. *G is not connected.* Let F be the graph obtained from G by adding an edge connecting two connected components. By making use of Lemma 3.1, it is easy to show that $\mathbf{bw}(F) \leq \max\{\mathbf{bw}(G), 2\}$, and this case can be reduced to Case B. □

A graph G is *weakly triangulated* if all its faces are of length two or three. A graph is $(2, 3)$-regular if all its vertices have degree two or three. Notice that the dual of a weakly triangulated graph is $(2, 3)$-regular and vice versa.

LEMMA 3.3. *Every* 2*-connected* Σ*-plane graph* G *has a weak triangulation* H *such that* $\mathbf{bw}(H) = \mathbf{bw}(G)$.

Proof. Because G is 2-connected every face of G is bounded by a cycle. Suppose that there is a face r of G bounded by a cycle $C = (x_0, \ldots, x_{s-1})$, $s \geq 4$. We show that there are vertices x_i and x_j that are not adjacent in C such that the graph G' obtained from G by adding the edge $\{x_i, x_j\}$ has $\mathbf{bw}(G') = \mathbf{bw}(G)$. By applying this argument recursively, one obtains a weak triangulation of G of the same branch-width.

If there are vertices x_i and x_j that are adjacent in G and are not adjacent in C, then we can draw a chord joining x_i and x_j in r. Because G is 2-connected it holds that $\mathbf{bw}(G) \geq 2$ and, therefore, the addition of multiple edges does not increase the branch-width. Suppose now that the cycle C is chordless. Let (T, τ) be a branch decomposition of G and let ω be its order function. Every edge f of T corresponds to the partition of $E(G)$ into two sets. One of these sets contains at least $\lceil |C|/2 \rceil \geq 2$ edges of C. By induction on the number of edges in G, it is easy to show that there is always an edge f of T such that for the corresponding partition (E_1, E_2) of $E(G)$, the set E_1 contains exactly two edges of C. Let e_1, e_2 be such edges. Because C is chordless and its length is at least four, we have that $\omega(f)$ contains at least two vertices, say x_i and x_j, of C that are not adjacent. Then adding edge $\{x_i, x_j\}$ does not increase the branch-width. (The decomposition can be obtained from T by subdividing f and adding the leaf corresponding to $\{x_i, x_j\}$ to the vertex subdividing f.) □

In the next lemma we use powerful duality results of Robertson and Seymour. Moreover, the implications of these results play the crucial role in our proof.

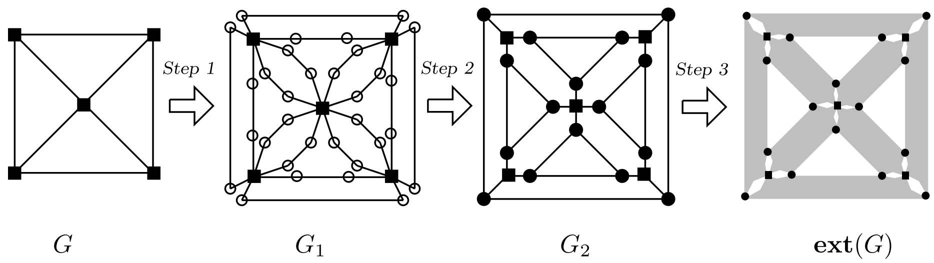

FIG. 1. *The steps* 1, 2, *and* 3 *of the definition of the function* **ext**.

LEMMA 3.4. *Every* 2*-connected* Σ*-plane graph* G *is the contraction of a* $(2,3)$*-regular* Σ*-plane graph* H *such that* $\mathbf{bw}(H) = \mathbf{bw}(G)$.

Proof. Let G^d be the dual graph of G. By Theorem 2.3, $\mathbf{bw}(G^d) = \mathbf{bw}(G)$ (the dual of a 2-connected graph is 2-connected, and any 2-connected graph contains K_2^2 as a minor). By Lemma 3.3, there is a weak triangulation H^d of G^d such that $\mathbf{bw}(H^d) = \mathbf{bw}(G^d)$. The dual of H^d, which we denote by H, contains G as a contraction (each edge removal in a planar graph corresponds to an edge contraction in its dual and vice versa). Applying Theorem 2.3 the second time, we obtain that $\mathbf{bw}(H) = \mathbf{bw}(H^d)$. Hence, $\mathbf{bw}(H) = \mathbf{bw}(G)$. Since H^d is weakly triangulated, we have that H is $(2,3)$-regular. ∎

3.2. Extensions of Σ-plane graphs. Let G be a connected Σ-plane graph where all the vertices have degree at least two. We define the *exension* of G, $\mathbf{ext}(G)$, as the hypergraph obtained from G by making use of the following three steps (see Figure 1 for an example).

Step 1. For each edge $e \in E(G)$, duplicate e and then subdivide each of its two copies twice. That way, each edge $e = \{x, y\}$ of G is replaced by a cycle denoted as $C_{x,y} = (x, x^+_{x,y}, y^-_{x,y}, y, y^+_{x,y}, x^-_{x,y}, x)$ (indexed in clockwise order). Let G_1 be the resulting graph.

Step 2. For each vertex $x \in V(G)$ and each pair (y, z) of consecutive neighbors of x (in G), identify the edges $\{x, x^-_{x,y}\}$ and $\{x, x^+_{x,z}\}$ in G_1. Let G_2 be the resulting graph.

Step 3. The hypergraph $\mathbf{ext}(G)$ is defined by setting $\mathbf{ext}(G) = (V(G_2), \{C_{x,y} \mid \{x, y\} \in E(G)\})$.

From the above construction, if $\mathcal{H} = \mathbf{ext}(G)$, then there exists a bijection $\theta : E(G) \to E(\mathcal{H})$ mapping each edge $e = \{x, y\}$ to the hyperedge formed by the vertices of $C_{x,y}$. See Figure 1 for an example of the definition of **ext**.

LEMMA 3.5. *For any* $(2,3)$*-regular* Σ*-plane graph* G, $\mathbf{bw}(\mathbf{ext}(G)) \leq 3 \cdot \mathbf{bw}(G)$.

Proof. Let (T, τ) be a branch decomposition of G of width $\leq k$. By the definition of $\mathbf{ext}(G)$, there is a bijection $\theta : E(G) \to E(\mathbf{ext}(G))$ defining which edge of G is replaced by which hyperedge of $\mathbf{ext}(G)$. Let L be the set of leaves in T. For $\mathbf{ext}(G)$ we define a branch decomposition (T, τ') with a bijection $\tau' : E(\mathbf{ext}(G)) \to L$ such that $\tau'(t) = \theta(\tau(t))$. We use the notations ω and ω' for the order functions of (T, τ) and (T, τ'), respectively.

We claim that (T, τ') is a branch decomposition of $\mathbf{ext}(G)$ of width $\leq 3k$. To prove the claim we show that for any $f \in E(T)$, $|\omega'(f)| \leq 3 \cdot |\omega(f)|$. In other words, we need to show that it is possible to define a function σ_f mapping each vertex $v \in \omega(f)$ to a set of three vertices of $\omega'(f)$ such that every vertex $y \in \omega'(f)$ is contained in $\sigma_f(x)$ for some $x \in \omega(f)$.

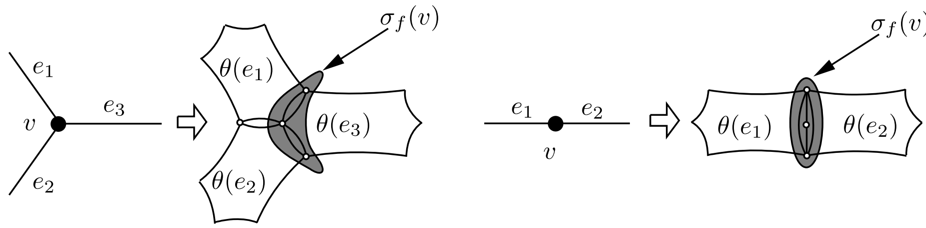

FIG. 2. *The construction of the value of $\sigma_f(v)$ in the proof of Lemma* 3.5.

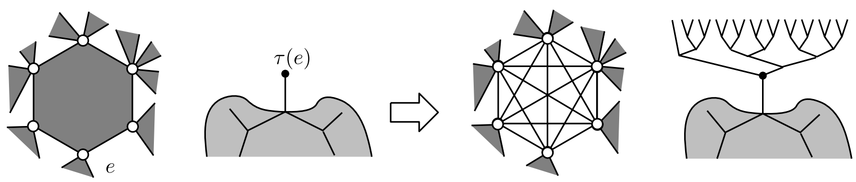

FIG. 3. *The construction of the branch decomposition of $\mathbf{cl}_E(H)$ in the proof of Lemma* 3.6.

Let T_1 and T_2 be the components of $T - \{f\}$. We construct σ_f by distinguishing two cases.

• *The degree of v is three in G.* We can assume that two of its incident edges, say e_1, e_2, are images of leaves of T_1 and one, say e_3, is an image of a leaf in T_2. We define $\sigma_f(v) = (\theta(e_1) \cap \theta(e_3)) \cup (\theta(e_2) \cap \theta(e_3))$. (This process is illustrated in the left half of Figure 2.)

• *The degree of v is two in G.* We can assume that one of its incident edges, say e_1, is an image of some leaf of T_1 and the other, say e_2, is an image of a leaf in T_2. We define $\sigma_f(v) = \theta(e_1) \cap \theta(e_2)$ (this is illustrated in the right half of Figure 2).

Note that in both cases $|\sigma_f(v)| = 3$. Suppose now that y is a vertex in $\omega'(f)$. Then y should be an endpoint of at least two hyperedges α and β of $\mathbf{ext}(G)$ and without loss of generality we assume that $\tau'(\alpha)$ is a leaf of T_1 and $\tau'(\beta)$ is a leaf of T_2. By the definition of τ', this means that $\tau(\theta^{-1}(\alpha))$ is a leaf of T_1 and $\tau(\theta^{-1}(\beta))$ is a leaf of T_2. By the construction of $\mathbf{ext}(G)$, $\theta^{-1}(\alpha)$ and $\theta^{-1}(\beta)$ have a vertex x in common; therefore $x \in \omega(f)$. From the definition of σ_f, we get that $y \in \sigma_f(x)$. This proves the relation $|\omega'(f)| \leq 3 \cdot |\omega(f)|$, and the lemma follows. ∎

Let $\mathcal{H}$ be a planar hypergraph and let $E \subseteq E(\mathcal{H})$. We set $\mathbf{cl}_E(\mathcal{H}) = (V(\mathcal{H}), E_{\mathcal{H}})$, where $E_{\mathcal{H}} = E(\mathcal{H}) - E \cup \{\{x,y\} \subseteq V(\mathcal{H}) \mid \exists_{e \in E(\mathcal{H})} : \{x,y\} \in e\}$ (in other words, we replace each hyperedge $e \in E$ by a clique formed by connecting each pair of endpoints of e).

LEMMA 3.6. *Let $\mathcal{H}$ be a hypergraph with every vertex of degree at least two. Then for any $E \subseteq E(\mathcal{H})$, $\mathbf{bw}(\mathbf{cl}_E(\mathcal{H})) \leq \mathbf{bw}(\mathcal{H})$.*

Proof. If (T, τ) is a branch decomposition of $\mathcal{H}$, then we construct a branch decomposition of $\mathbf{cl}_E(\mathcal{H})$ by identifying each leaf t where $\tau(t) \in E$ with the root of a binary tree T_t that has $\binom{|\tau(t)|}{2}$ leaves. The leaves of T_t are mapped to the edges of the clique made up by pairs of endpoints in $\tau(t)$ (see also Figure 3). ∎

LEMMA 3.7. *Let G and H be connected Σ-plane graphs with all vertices of minimum degree at least two and such that $G \preceq H$. Then $\mathbf{bw}(\mathbf{ext}(G)) \leq \mathbf{bw}(\mathbf{ext}(H))$.*

Proof. Let E' (resp., E'') be the set of edges that one should contract (resp., remove) in H in order to obtain G (clearly, we can assume that $E' \cap E'' = \emptyset$). Let

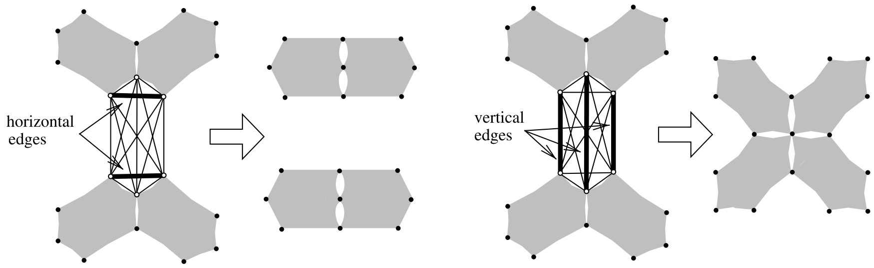

FIG. 4. *The construction of the branch decomposition of* $\mathbf{cl}_E(H)$ *in the proof of Lemma* 3.7.

θ be the bijection mapping edges of G to hyperedges of $\mathbf{ext}(G)$. If we prove that $\mathbf{ext}(G)$ is a minor of $\mathbf{cl}_{\theta(E'\cup E'')}(\mathbf{ext}(H))$, then the result will follow from Lemma 3.6. To see this, for each $e = \{x,y\} \in E'$, we separate the edges of the clique replacing $\theta(e) = (x, x^+_{x,y}, y^-_{x,y}, y, y^+_{x,y}, x^-_{x,y}, x)$ into two categories: We call $\{x^+_{x,y}, y^-_{x,y}\}$, $\{x,y\}$, and $\{y^+_{x,y}, x^-_{x,y}\}$ *horizontal* and we call the rest *unimportant*. Moreover, for any edge $e = \{x,y\} \in E''$, we separate the edges of the clique replacing $\theta(e) = (x, x^+_{x,y}, y^-_{x,y}, y, y^+_{x,y}, x^-_{x,y}, x)$ into two categories: We call $\{x^+_{x,y}, x^-_{x,y}\}$ and $\{y^+_{x,y}, y^-_{x,y}\}$ *vertical* and the rest *useless*. To obtain $\mathbf{ext}(G)$ from $\mathbf{cl}_{E'}(\mathbf{ext}(H))$ we first remove the useless and the unimportant edges and then contract all the horizontal and vertical ones (see Figure 4). ∎

We are ready to state the main property of **ext**.

LEMMA 3.8. *Let G be a connected Σ-plane graph with all vertices of degree at least two. Then* $\mathbf{bw}(\mathbf{ext}(G)) \leq 3 \cdot \mathbf{bw}(G)$.

Proof. The branch-width of G is at least two, and by Lemma 3.2, G is the minor of a 2-connected Σ-plane graph G' such that $\mathbf{bw}(G') = \mathbf{bw}(G)$. By Lemma 3.4, G' is the contraction of a $(2,3)$-regular Σ-plane graph H where $\mathbf{bw}(H) \leq \mathbf{bw}(G')$. Notice that G is a minor of H and both G and H are Σ-plane and connected and have all vertices of degree at least two. By Lemma 3.7, $\mathbf{bw}(\mathbf{ext}(G)) \leq \mathbf{bw}(\mathbf{ext}(H))$. Note also that H is $(2,3)$-regular. By Lemma 3.5, $\mathbf{bw}(\mathbf{ext}(H)) \leq 3 \cdot \mathbf{bw}(H)$, and the result follows. ∎

3.3. Nicely D-dominated Σ-plane graphs. An important tool spanning all of our proofs is the concept of unique D-domination. We call a D-dominated graph G *uniquely dominated* if there is no path of length < 3 connecting two vertices of D. Let us remark that this implies that each vertex $x \in V(G) - D$ has exactly one neighbor in D (i.e., is uniquely dominated).

We call a multiple edge $\{a,b\}$ represented by lines $l_1, l_2, \ldots, l_r$ of a D-dominated Σ-plane graph G *exceptional* if

- $a, b \notin D$;
- a and b are both adjacent to the same vertex in D;
- for any i, j, $i \neq j$, each of the open discs bounded by $l_i \cup l_j$ contains at least one vertex of D.

For example, all the multiple edges in the graphs in Figure 5 are exceptional.

LEMMA 3.9. *For every 2-connected D-dominated Σ-plane graph G without multiple edges, there exists a Σ-plane graph H such that the following hold:*

(a) *G is a minor of H.*
(b) *H is uniquely D-dominated.*
(c) *All multiple edges of H are exceptional.*

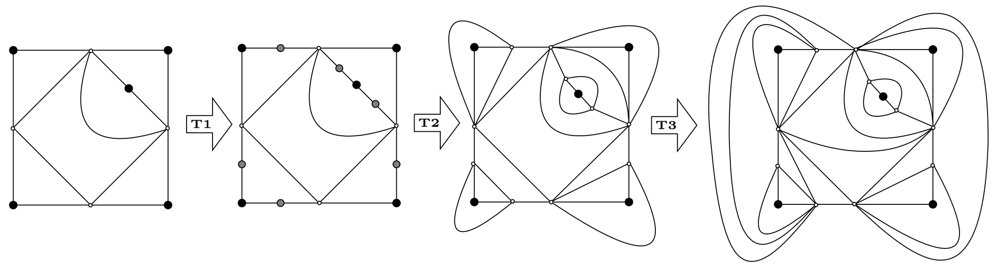

FIG. 5. *Example of the transformations* **T1**, **T2**, *and* **T3** *in the proof of Lemma* 3.9.

(d) *For any face r of H, $\widehat{r}$ is either a triangle or a square.*
(e) *If the distance between vertices $x, y \in D$ in H is three, then there exist at least two distinct (x, y)-paths in H of length three.*
(f) *If a (closed) face r of H contains a vertex of D, then $\widehat{r}$ is a triangle.*
(g) *Every square face of H contains two edges $e_i, i = 1, 2$, without common vertices such that for each $i = 1, 2$, there exists a vertex $x_i \in D$ adjacent to both endpoints of e_i.*
(h) *If $x, y \in D$, then every two distinct (x, y)-paths of H of length three are internally disjoint.*

Proof. We construct a graph H, satisfying properties (a)–(f), by applying, one after the other, on G the following transformations:

• **T1**. As long as there exists in G a vertex x with more than one neighbor y in D, subdivide the edge $\{x, y\}$.

We call the resulting graph G_1.

As G_1 does not have multiple edges, properties (a), (c) are trivially satisfied. Moreover, notice that, if G_1 is not uniquely dominated, then **T1** can be further applied. Therefore, (b) holds for G_1. For an example of the application of **T1**, see the first step of Figure 5.

• **T2**. As long as G_1 has a face r bounded by a cycle $\widehat{r} = (x_0, \ldots, x_{q-1}), q \geq 4$, and such that $x_i \in D$ for some $i, 0 \leq i \leq q - 1$, add in G_1 the edge $\{x_{i-1}, x_{i+1}\}$ (indices are taken modulo q).

We call the resulting graph G_2.

Notice that the vertices of $\widehat{r}$ are distinct because G_2 is 2-connected. Clearly, G_2 satisfies property (a). Recall now that G_1 satisfies property (b). Therefore, if some vertex $x_i \in \widehat{r}$ is in D, then its neighbors x_{i-1} and x_{i+1} (the indices are taken modulo q) are not in D. Therefore, property (b) holds also for G_2. Notice that, if **T2** creates a multiple edge, then this can be only an exceptional multiple edge. Therefore, (c) holds for G_2. For an example of the application of **T2**, see the second step of Figure 5.

Finally note that none of the vertices of D is in a face of G_2 of length ≥ 4.

We call a square face that satisfies property (g) *solid*.

• **T3**. As long as G_2 has a face r that is not a solid square and such that $\widehat{r} = (x_0, \ldots, x_{q-1}), r \geq 4$, choose an edge in $\{\{x_1, x_3\}, \{x_0, x_2\}\}$ that is not already present in G_2 and add it to G_2.

We call the resulting graph G_3.

The above transformation can always be applied because it is impossible that both $\{x_1, x_3\}$ and $\{x_0, x_2\}$ are in the planar graph G_3. Therefore, property (c) is an invariant of **T3**. Clearly, G_3 satisfies property (a). Property (b) is an invariant of **T3** as the added edge has no endpoints in D. We have that all the faces of G_3 are either

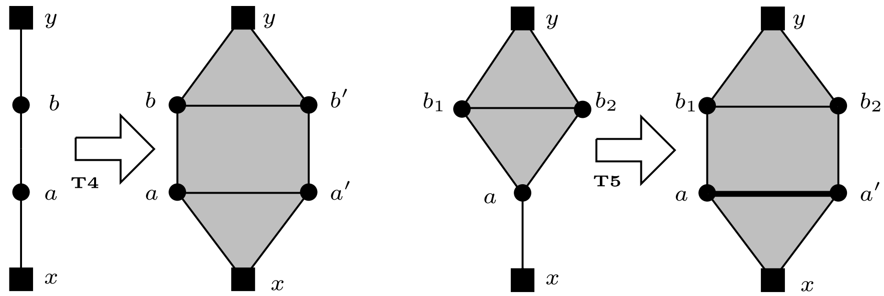

FIG. 6. *The transformations* **T4** *and* **T5** *in the proof of Lemma* 3.6.

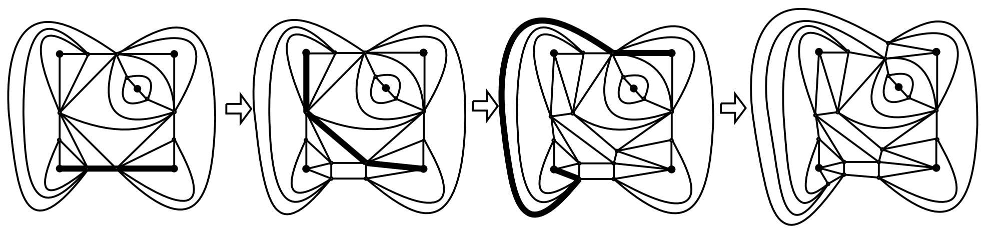

FIG. 7. *Example of the transformation* **T4** *in the proof of Lemma* 3.6.

triangles or solid squares and therefore G_3 also satisfies (d) and (g). For an example of the application of **T3**, see the third step of Figure 5.

• **T4**. As long as G_3 has a *unique* (x, y)-path $P = (x, a, b, y)$, where $x, y \in D$, apply the first transformation of Figure 6 on P.

We call the resulting graph G_4.

It is easy to verify that properties (a)–(d) are invariants of **T4**. Also, it is easy to see that the transformation of Figure 6 creates square faces with property (g) and does not alter property (g) for square faces that already have been created. Moreover, G_4 satisfies (e) because each time we apply the transformation of Figure 6 the number of pairs in D connected by unique paths decreases. Finally, none of the square faces appearing (because of **T4**) contains a vertex in D. Thus (f) holds. For an example of the application of **T4**, see Figure 7.

In order to give the transformation that enforces property (h) we need some definitions. Observe that if property (h) does not hold for G_4, this implies the existence of some pair of paths $P_i = (x, a, b_i, y), i = 1, 2$. We call the graph O defined by this pair an (h)-obstacle and we define its (h)-disc as the x-avoiding closed disc Δ_O bounded by the cycle (a, b_1, y, b_2, a). An (h)-obstacle is *minimal* if no (x, y)-path has vertices contained in its (h)-disc. Notice that if G_4 has an (h)-obstacle it also has a minimal (h)-obstacle and vice versa. We call an (h)-obstacle *hollow* if its (h)-disc contains no neighbor of a except b_1 and b_2. Notice that a hollow (h)-obstacle is always minimal. We claim that in any hollow (h)-disc, vertices b_1 and b_2 are adjacent. Indeed, by property (b), a is not adjacent to y in G_4. Therefore b_1, a, b_2 are in a face of G_4 that, from property (g), cannot be a square face (otherwise, property (b) would be violated). Therefore, (b_1, a, b_2) is a triangle and the claim follows.

• **T5**. As long as G_4 has a hollow (h)-obstacle O, apply the second transformation of Figure 6 on edge $\{a, x\}$ and the face bounded by (b_1, b_2, a).

We call the resulting graph G_5.

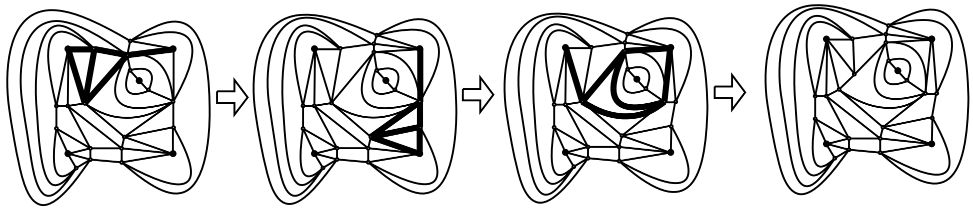

FIG. 8. *Example of the transformation* **T5** *in the proof of Lemma* 3.6.

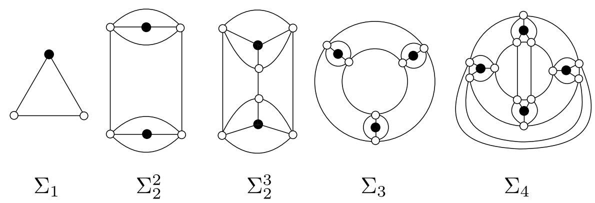

FIG. 9. *Simple examples of nicely D-dominated Σ-plane graphs.*

Notice that after **T5** none of the properties (a)–(g) is altered by the application of **T5** (the arguments are the same as those used for the previous transformations). Moreover, each time the second transformation of Figure 6 is applied, the number of hollow (h)-obstacles decreases and no new nonhollow (h)-obstacles appear. For an example of the application of **T5**, see Figure 8. To finish the proof, we show that **T5** is able to eliminate all the (h)-obstacles. It remains to prove the following claim.

Claim. If a 2-connected D-dominated Σ-plane graph satisfies properties (b)–(g) and contains a minimal (h)-obstacle, then it also contains a hollow (h)-obstacle.

Proof of claim. Let $O = (P_1, P_2)$ be a minimal nonhollow (h)-obstacle with (h)-disc Δ_O and let $\mathcal{O}$ be the set containing O along with of all the minimal (h)-obstacles that contain the edge $\{a, x\}$ and whose (h)-disc is a subset of Δ_O. If $O_1, O_2 \in \mathcal{O}$ and $\Delta_{O_1} \subset \Delta_{O_2}$, then we say that $O_1 < O_2$ (clearly, for any $O' \in \mathcal{O} - \{O\}$, $O' < O$). Let us remark that relation "<" is a partial order on $\mathcal{O}$ and that all its minimal elements are hollow (h)-obstacles. The claim follows and thus **T5** is able to enforce property (h). ◻

Let G be a connected D-dominated Σ-plane graph satisfying properties (b)–(h) of Lemma 3.9. We call such graphs *nicely* D-dominated Σ-plane graphs. For example, the graphs of Figure 9 and the last graph in Figure 8 are nicely D-dominated Σ-plane graphs (see also Figure 10 and all the graphs of Figure 11).

Given a nicely D-dominated Σ-plane graph G, we define $\mathcal{T}(G)$ as the set of all the triangles (cycles of length three) containing a vertex of D. By property (f), for every face r with $\widehat{r} \cap D \neq \emptyset$, $\widehat{r} \in \mathcal{T}(G)$. (The inverse is not always correct; i.e., not every triangle in $\mathcal{T}(G)$ bounds a face.) We call the triangles in $\mathcal{T}(G)$ D-triangles.

We also define $\mathcal{C}(G)$ as the set of all cycles consisting of two distinct paths of length three connecting two vertices of D (these are indeed cycles because of property (h) of nicely dominated graphs). Thus each cycle C in $\mathcal{C}(G)$ is of length six and is the union of two length-three paths connecting its two dominating vertices.

We call the cycles in $\mathcal{C}(G)$ *D-hexagons*. The *poles* of a cycle $C \in \mathcal{C}(G)$ are the vertices in $D \cap C$. We call a D-triangle T (D-hexagon C) *empty* if one of the open discs bounded in Σ by T (C) does not contain vertices of G. Notice that all empty

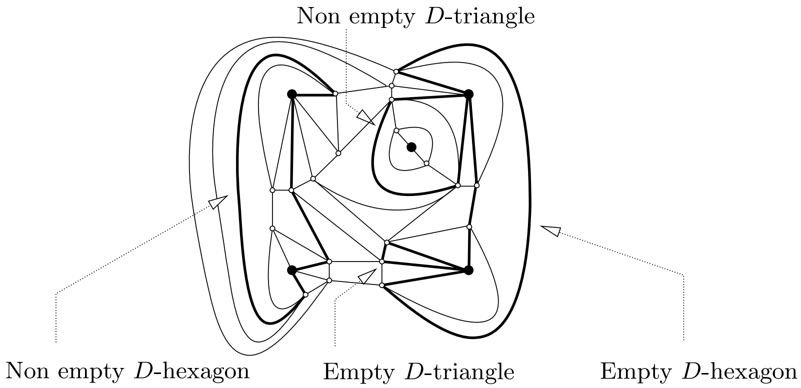

FIG. 10. *D-triangles and D-hexagons of the last graph of Figure* 8.

D-triangles are boundaries of faces of G. For some examples of the above definitions see Figure 10.

3.4. Decomposing nicely D-dominated Σ-planar graphs. In this subsection we show how nicely D-dominated planar graphs can be simplified. The idea is based on the structure imposed by properties (b)–(h): Any nicely D-dominated planar graph can be seen as the result of gluing together two simpler structures of the same type. This is described by the following two lemmata.

LEMMA 3.10. *Let G be a nicely D-dominated Σ-plane graph G and let $T \in \mathcal{T}(G)$ be a nonempty D-triangle bounding the closed discs Δ_1, Δ_2. Let also G_i, $i = 1, 2$, be the subgraph of G containing all vertices and edges included in Δ_i. Then $G_i, i = 1, 2$, is a nicely D_i-dominated graph for some $D_i \subseteq D$ and G_i has fewer vertices than G.*

Proof. Let $D_i = D \cap \Delta_i, i = 1, 2$. Clearly, $D_i \subseteq D$. Moreover, as T is nonempty, we have that $|V(G_i)| < |V(G)|$. Let us verify that properties (b)–(h) hold for $G_i, i = 1, 2$. First of all we observe that, by the construction of G_i, two vertices in G_i are adjacent if and only if they are adjacent in G. We will refer to this fact saying that G_i *preserves the adjacency of* G. (Note that since G can have multiple edges, G_i is not necessary an induced subgraph of G.)

To prove property (b), we show first that G_i is D_i-dominated. For the sake of contradiction, suppose that there exists a vertex $a \in V(G_i)$ that is not dominated by D_i. As property (b) holds for G, there exists a vertex $w \in D - D_i$ so that a is uniquely dominated by w in G. This means that $w \in \Sigma - \Delta_i$ and $a \in \Delta_i$. Therefore, a is a vertex of T. Because T is a D-triangle, there is some $x \in D \cap T$. Since a is adjacent in G_i to x and $x \neq w$, we have a contradiction to the property (b) on G. Now it remains to prove that G_i is uniquely D'-dominated and that this is a direct consequence of the fact that G_i preserves the adjacency of G.

For property (c), let $e = \{v, u\}$ be some multiple edge in G_i represented by edges $l_1, \dots, l_r$, and suppose that x is the dominating vertex of T. As e is an exceptional multiple edge in G and because of property (b), none of its endpoints is in D and also $x \notin e$. Let Δ_l, Δ_l^* be the two closed discs defined by some pair l_h, l_j of edges representing e. By the definition of G_i, $l_h \cup l_j \subseteq \Delta_i$, therefore one, say Δ_l, of Δ_l, Δ_l^* includes T. As $x \notin e$, we have that $x \notin \hat{\Delta}_l$ and $\Delta_l - \hat{\Delta}_l$ contains some vertex of D. Observe now that $\Delta_l^* \subseteq \Delta_i$. Therefore, if $\Delta_l^* - \hat{\Delta}_l^*$ does not contain vertices of D in G_i, then the same holds also for G, which is a contradiction as e is exceptional in G. It remains now to prove that v and u are adjacent to the same vertex of D in G_i.

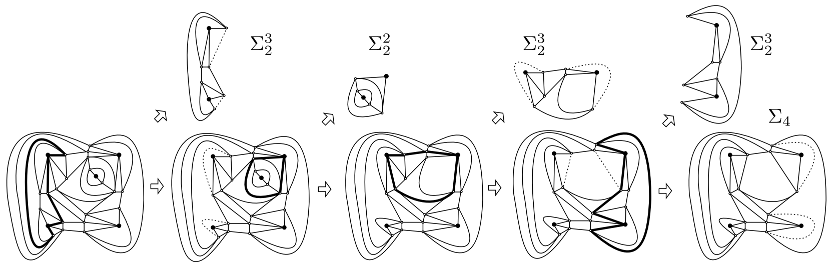

FIG. 11. *Examples of the application of Lemmata* 3.10 *and* 3.11.

Indeed, this is the case for G, and we let w be this vertex. If $w \notin \Delta_i$, then both v, u should be vertices of T, which contradicts property (b). Therefore, $w \in V(G_i)$ and property (c) holds for G_i.

For (d), we stress that all the faces of G_i that are in Δ_i are also the faces of G. Therefore, property (d) holds for all these faces. Also, it holds for the unique new face $r = \Sigma - \Delta_i$ of G_i because $\widehat{r}$ is a triangle.

For property (e), let x, y be two vertices in D_i of distance three in G_i. Let P_i^1 and P_i^2 be two internally disjoint paths connecting x and y in G (these paths exist because of properties (e) and (h) in G). Notice that (e) holds if we prove that both P_i^j, $j = 1, 2$, are paths of G_i, $i = 1, 2$, as well. Suppose to the contrary that one, say $P_i^1 = (x, a, b, y)$, of P_i^j, $j = 1, 2$, is not a path in G_i. This means that at least one of a, b is in $(\Sigma - \Delta_i) \cap V(G)$. It follows that two nonconsecutive vertices of P_i^1 are vertices of T. Therefore, the distance between x and y in G is at most two, a contradiction to property (b) for G.

Suppose now that (f) does not hold for G_i. As (d) holds for G_i we have that there exists a square in G_i containing a vertex of D. As G_i preserves the adjacency of G, this square also should exist in G, a contradiction to (f) for G.

To prove (g), suppose that (a, b, c, d) is a square of G_i. As G_i preserves the adjacency of G, (a, b, c, d) is also a square of G; therefore we may assume that there are vertices $z, w \in D$ where (z, a, b) and (w, c, d) are triangles of G. It is enough to prove that $\{z, a\}, \{z, b\}, \{w, c\}$, and $\{w, d\}$ are edges of G_i. Suppose to the contrary that one of them, say $\{a, z\}$, is not an edge of G_i. As G_i preserves the adjacency of G, this means that $z \notin V(G_i)$. In other words, we have that (z, a, b) is a triangle of G where $z \in (\Sigma - \Delta_i) \cap V(G)$ and $\{a, b\} \in \Delta_i \cap V(G)$. If this is true, then a, b should be vertices of T; therefore the distance in G between z and the dominating vertex belonging in T is at most two, a contradiction to property (b).

Finally, if there exist two paths violating (h) in G_i the same also should happen in G as G_i preserves the adjacency of G. □

For an example of the application of Lemma 3.10, see the second step of Figure 11.

LEMMA 3.11. *Let G be a nicely D-dominated Σ-plane graph G and let $C = (x, a, b, y, c, d, x)$ be a nonempty D-hexagon with poles x, y bounding the closed discs Δ_1, Δ_2. Let also $G_i, i = 1, 2$, be the graph containing all the edges and vertices included in Δ_i and extended by adding the edges $\{b, c\}$ and $\{a, d\}$ (edges $\{b, c\}$ and $\{a, d\}$ are placed outside D_i to ensure planarity of G_i). Then $G_i, i = 1, 2$, is a nicely D_i-dominated graph for some $D_i \subseteq D$ and $G_i, i = 1, 2$, has fewer vertices than G.*

Proof. Let G_i^- be a graph where $V(G_i^-) = \Delta_i \cap V(G)$ and $E(G_i^-) = \{e \in E(G) \mid e \text{ is included in } \Delta_i\}$; i.e. G_i^-, contains all edges and vertices included in Δ_i. Set $D_i = D \cap \Delta_i, i = 1, 2$. Therefore, G_i can be seen as the graph with $V(G_i) = V(G_i^-)$

and $E(G_i) = E(G_i^-) \cup \{\{b,c\} \cup \{a,d\}\}$. As in the proof of Lemma 3.10, we will say that G_i^- preserves the adjacency of G in the sense that two vertices in G_i^- are adjacent if and only if they are adjacent in G. We also have that $D_i \subseteq D$ and $|V(G_i)| < |V(G)|$.

Let us verify properties (b)–(h) for G_i, $i = 1, 2$.

To prove (b) we first claim that G_i is D_i-dominated. If some vertex $\alpha \in V(G_i) - D_i$ is not dominated by D_i, then it is dominated by some vertex $w \in D - D_i$ (property (b) for G). This means that $w \in \Sigma - \Delta_i$ implying $\alpha \in C$. Thus $\alpha \in \{a, b, c, d\}$. But this means that the distance between $w, x \in D$ or the distance between $w, y \in D$ in G is ≤ 2, which also violates (b) for G. Therefore G_i is D_i-dominated. Clearly, as G_i preserves the adjacency of G, G_i should be uniquely dominated and (b) holds for G_i.

For property (c), we will first prove that it holds for G_i^-. Let $e = \{v, u\}$ be some multiple edge in G_i^- represented by edges $l_1, \dots, l_r$. As e is an exceptional multiple edge in G and because of property (b), none of its endpoints is in D and also $x, y \notin e$. Let Δ_l, Δ_l^* be the two closed discs defined by some pair l_h, l_j of edges representing e. By the definition of G_i^-, $l_h \cup l_j \subseteq \Delta_i$, therefore one of Δ_l, Δ_l^*, say Δ_l, includes C. As $x, y \notin e$, we have that $x, y \notin \hat{\Delta}_l$ and $\Delta_l - \hat{\Delta}_l$ contains some vertex of D. Observe now that $\Delta_l^* \subseteq \Delta_i$. Therefore, if $\Delta_l^* - \hat{\Delta}_l^*$ does not contain vertices of D, then the same holds also for G, which is a contradiction, as e is exceptional in G. It remains now to prove that v and u are adjacent to the same vertex of D in G_i^-. Since this holds for G, we have that there exists a vertex $w \in D$ such that $\{u, w\}, \{v, w\} \in E(G)$. If $w \notin \Delta_i$, then both v, u should be vertices of C, which contradicts property (b). Therefore, $w \in V(G_i^-)$ and property (c) holds for G_i^-. If now the addition of any, say $\{b, c\}$, of $\{b, c\}$, $\{a, d\}$ creates a multiple edge, then $\{b, c\}$ should already be an edge in G_i^-. Suppose then that $l_{\text{old}}, l_{\text{new}}$ are two lines in G_i, representing $\{b, c\}$, and l_{new} is the newly added one. As $l_{\text{new}} \not\subseteq D_i$ and $l_{\text{old}} \subseteq D_i$, it follows that the one of the open discs defined by $l_{\text{old}} \cup l_{\text{new}}$ contains y and the other contains x. Therefore, (c) holds also for G_i.

Notice that all the faces of G_i that are included in Δ_i are also faces of G_i. The boundaries of the new faces are the cycles (y, a, b), (a, b, c, d), and (x, c, b) that are all either triangles or squares. Therefore, (d) holds for G_i.

If property (e) holds for G_i^-, then it also holds for G_i. Let P be a (w, v)-path in G_i^- of length three. Property (e) holds trivially for G_i^- if $\{w, v\} = \{x, y\}$. So suppose that it is violated for some pair $\{w, v\} \neq \{x, y\}$. Because (e) holds for G, we can find a $\{w, v\}$-path $P' = (w, \alpha, \beta, v)$ of length three in G that is not a path in G_i^-. As $\{w, v\} \neq \{x, y\}$, only one, say α, of α, β can be outside Δ_i. This means that w and β are vertices of C. Since $\beta \in \{a, b, c, d\}$, we have that v is adjacent in G to a vertex in $\{a, b, c, d\}$. This contradicts property (b) for G, as it implies the existence of a path of length ≤ 2 connecting $v \in D$ and one of the vertices $x, y \in D$.

It is easy to verify (f) for the new faces (x, a, d), (a, b, c, d), and (y, c, d) of G_i. Suppose now that (f) is violated for some face of G_i that is also a face of G. As (d) holds for G_i, we have that there exists a square in G_i containing a vertex of G_i. As G_i preserves the adjacency of G, this square should exist also in G, a contradiction to (f) for G.

Property (g) is trivial for the new square face of G_i bounded by (a, b, c, d). Let us prove that (g) also holds for all the square faces of G_i^-. Let $\widehat{r} = (\alpha, \beta, \gamma, \delta)$ be the boundary of some square face r of G_i^-. As G_i^- preserves the adjacency of G, $(\alpha, \beta, \gamma, \delta)$ is also the boundary of some square face of G. Therefore, we may assume that there are vertices $z, w \in D$ where (z, α, β) and (w, γ, δ) are triangles of G. It is enough to prove that $\{z,\alpha\},\{z,\beta\},\{w,\gamma\}$, and $\{w,\delta\}$ are all edges of G_i^-. Suppose, to the contrary, that one of them, say $\{a,z\}$, is not an edge of G_i^-. As G_i^- preserves the adjacency of G, this means that $z \notin V(G_i^-)$. In other words, we have that (z,α,β) is a triangle of G, where $z \in (\Sigma - \Delta_i) \cap V(G)$ and $\{\alpha,\beta\} \in \Delta_i \cap V(G)$. Then α,β should be vertices of C different from x and y. Therefore, either z,x or z,y are at distance at most two in G, contradicting property (b).

For (h), we observe that no path of length three in G_i connecting two vertices of D can use the edges $\{a,d\}$ and $\{b,c\}$ in G_i. Indeed, if this is possible for one, say $\{a,d\}$, of the edges $\{a,d\}$ and $\{b,c\}$, then such a path would have extremes in distance two from x, a contradiction to property (b) for G_i. Therefore, if there exist two paths violating (h) in G_i, they should be paths of G_i^- and also paths of G as G_i^- preserves the adjacency of G, a contradiction to property (b). ∎

For an example of the application of Lemma 3.10, see steps 1, 3, and 4 of Figure 11.

3.5. Prime *D*-dominated Σ-plane graphs. A nicely D-dominated Σ-plane graph G is a *prime D-dominated Σ-plane graph* (or just *prime*) if all its D-triangles and D-hexagons are empty. For example, all the graphs in Figure 9 are prime.

LEMMA 3.12. *Let G be a prime D-dominated Σ-plane graph. If G contains two vertices $x,y \in D$ connected by three paths of length three, then $V(P_1)\cup V(P_2)\cup V(P_3) = V(G)$.*

Proof. By property (h), the paths $P_i, i = 1,2,3$, are mutually internally disjoint. Then $\Sigma - (P_1 \cup P_2 \cup P_3)$ contains three connected components that are open discs. We call them $\Delta_{1,2}$, $\Delta_{2,3}$, and $\Delta_{1,3}$ assuming that they do not contain vertices of P_3, P_1, and P_2, respectively. Let i,j,h be any three distinct indices of $\{1,2,3\}$. As $P_i \cup P_j$ forms an empty D-hexagon, all the vertices of G should be contained in one, say Δ, of the closed discs bounded by the cycle $P_i \cup P_j$. Notice that P_h should be entirely included in $\overline{\Delta}_{i,j}$ because of its internal vertices. Therefore, $\Delta = \overline{\Delta}_{i,j}$ and thus $V(G) = V(G) \cap \overline{\Delta}_{i,j}$. Resuming, we have that $V(G) = V(G) \cap (\overline{\Delta}_{1,2} \cap \overline{\Delta}_{2,3} \cap \overline{\Delta}_{1,3})$ and the lemma follows as $\overline{\Delta}_{1,2} \cap \overline{\Delta}_{2,3} \cap \overline{\Delta}_{1,3}$ contains exactly the vertices of the paths $P_i, i = 1,2,3$. ∎

The graph Σ_2^3 of Figure 11 is a graph satisfying the conditions of Lemma 3.12.

Let us recall that $\mathcal{C}(G)$ is the set of all cycles consisting of two distinct paths of length three connecting two vertices of D. For a nicely D-dominated Σ-plane graph G, we define its *reduced* graph, $\mathbf{red}(G)$, as the graph with vertex set D and where two vertices $x,y \in D$ are adjacent in $\mathbf{red}(G)$ if and only if the distance between x and y in G is three. Let us stress that $\mathbf{red}(G)$ is a connected graph. The main idea of our proof is that $\mathbf{red}(G)$ expresses a "good" part of the structure of a nicely D-dominated graph G.

An important relation of a prime graph and its reduced graph is provided by the following lemma.

LEMMA 3.13. *Let G be a prime D-dominated Σ-plane graph with $|D| \geq 3$. Then the mapping*

$$\phi : E(\mathbf{red}(G)) \to \mathcal{C}(G), \text{ where } \phi(e) = C \text{ if and only if the endpoints of } e \text{ are in } D \cap C,$$

is a bijection.

Proof. Clearly, any D-hexagon C with poles x and y implies the existence of a (x,y)-path in G and therefore C is the image of $\{x,y\} \in E(\mathbf{red}(G))$. In order to show that ϕ is a bijection, we have to show that for every $e = \{x,y\} \in E(\mathbf{red}(G))$, there exists a *unique* D-hexagon C with poles x and y. By the definition of $\mathbf{red}(G)$, x and y are within distance three in G. By properties (e) and (h) of nicely

D-dominated Σ-plane graphs, there are at least two internally disjoint paths connecting x and y. Suppose to the contrary that G has at least three (x, y)-paths P_1, P_2, P_3. As $|D| \geq 3$, G contains vertices that are not in $V(P_1) \cup V(P_2) \cup V(P_3)$, a contradiction to Lemma 3.12. ∎

Let G be a prime D-dominated Σ-plane graph with $|D| \geq 3$ and let ϕ be the bijection defined in Lemma 3.13. For every edge $e = \{x, y\} \in E(\mathbf{red}(G))$, we choose a vertex $w \in D - \{x\} - \{y\}$ and define $\Delta(e)$ as the w-avoiding open disc bounded by $\phi(e)$ (because G is prime, the definition does not depend on the choice of w). Observe that for any two different $e_1, e_2 \in E(\mathbf{red}(G))$, it holds that $\Delta(e_1) \cap \Delta(e_2) = \emptyset$.

Some of the properties of prime D-dominated Σ-plane graphs are given by the next two lemmata.

LEMMA 3.14. *Let G be a prime D-dominated Σ-plane graph with $|D| \geq 2$. For any D-triangle $T = (x, a, b)$ with $x \in D$, the edges $\{x, a\}$ and $\{x, b\}$ are also the edges of some D-hexagon of G with poles x and $y \in D$. Moreover, if $|D| \geq 3$, the edge $\{a, b\}$ is in $\Delta(\{x, y\})$.*

Proof. Because G is a prime graph, one of the open discs bounded by T is a face of G. Let r_x, $\widehat{r}_x = T = (x, a, b)$, be such a face. Let r, $r \neq r_x$, be the (unique) face incident to $\{a, b\}$, i.e., $\{a, b\} \subseteq \widehat{r}$. By (d), r is either a triangle or a square face.

We claim that it is a square face. Suppose to the contrary that $\widehat{r} = (a, b, c)$. Then, from property (b), $c \notin D$. Let $y \in D$ be the unique vertex dominating c. We distinguish two cases:

Case 1. $x = y$. In this case all vertices in $V(G) - \{x, a, b, c\}$ are covered (in Σ) by four open discs bounded by triangles $(x, a, b), (x, a, c), (x, b, c)$, and (a, b, c). Since G is prime, all D-triangles $(x, a, b), (x, a, c), (x, b, c)$ are empty. Therefore, all vertices in $V(G) - \{x, a, b, c\}$ are in the x-avoiding open disc Δ bounded by (a, b, c). As $\Delta = r$ is a face of G, we have that $V(G) - \{x, a, b, c\} = \emptyset$, a contradiction to the fact that $|D| \geq 2$.

Case 2. $x \neq y$. Then G contains the paths (x, a, c, y) and (x, b, c, y), a contradiction to property (h), and the claim holds.

As r is a square face, we assume that $\widehat{r} = (a, b, c, d)$. Property (g), together with the fact that a, b are adjacent to x, implies that either all vertices a, b, c, d are adjacent to x, or there is $y \in D$, $y \neq x$, that is adjacent to c and d.

We claim that the first case is impossible. Indeed, if a, b, c, d are adjacent to x, then all the vertices in $V(G) - \{x, a, b, c, d\}$ should be included in the five open discs bounded by triangles $(x, a, b), (x, a, c), (x, b, d)$, (c, d, x) and square (a, b, c, d). Four discs bounded by D-triangles are faces of G (G is prime); thus all the vertices of $V(G) - \{x, a, b, c, d\}$ are in the x-avoiding open disc r bounded by (a, b, c, d). Because r is a face of G, we conclude that $V(G) - \{x, a, b, c, d\} = \emptyset$. Since by property (b), $a, b, c, d \notin D$, we have a contradiction to the fact that $|D| \geq 2$, and the claim holds.

Therefore, there is $y \in D$, $y \neq x$, and y is adjacent to c and d. Because (y, c, d) is a D-triangle in a prime graph, one of the discs r_y bounded by (y, c, d) is the face of G. Hence $C = (x, a, c, y, d, b, x)$ is a D-hexagon containing edges $\{x, a\}$ and $\{x, b\}$, as required. Notice now that $\Delta = r_x \cup \{a, b\} \cup r \cup \{c, d\} \cup r_y$ is one of the open discs bounded by C (here an edge represents an open set). As $V(G) \cap \Delta = \emptyset$, we have that $\Delta(\{x, y\}) = \Delta$ and thus the edge $\{a, b\}$ is contained in $\Delta(\{x, y\})$. ∎

LEMMA 3.15. *Let G be a prime D-dominated Σ-plane graph with $|D| \geq 2$. Then the endpoints of each edge of G are the vertices of some D-hexagon.*

Proof. Let $e = \{x, y\}$ be an edge of G.

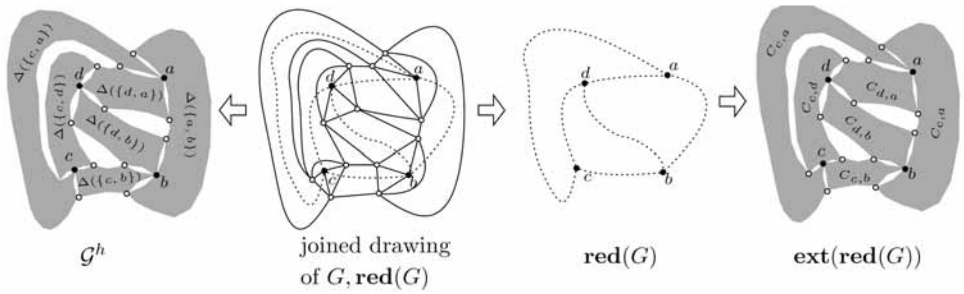

FIG. 12. *An example of the proof of Lemma* 3.17.

Case 1. $\{x, y\} \cap D = \{x\}$ (by property (b), $|\{x, y\} \cap D| \leq 1$). Let r be the face of G incident to $e = \{x, y\}$. From property (f), r is a D-triangle and the result follows from Lemma 3.14.

Case 2. $\{x, y\} \cap D = \emptyset$. Let d_x and d_y be the vertices of D-dominating x and y, respectively. If $d_x = d_y$, then e is incident to the D-triangle (d_x, x, y), and the result follows from Lemma 3.14. Suppose now that $d_x \neq d_y$. Then (d_x, x, y, d_y) is the path connecting two vertices in D. From property (e), $\{x, y\}$ belongs to the union of two distinct paths connecting d_x and d_y. Therefore, $\{x, y\}$ should be an edge of some D-hexagon and the lemma follows. □

3.6. On the structure of nicely D-dominated Σ-plane graphs. For a given nicely D-dominated Σ-plane graph G, we define hypergraph $\mathcal{G}^*$ with the vertex set $V(\mathcal{G}^*) = V(G)$ and edge set $E(\mathcal{G}^*) = E(G) \cup \mathcal{T}(G) \cup \mathcal{C}(G)$; i.e., $\mathcal{G}^*$ is obtained from G by adding all D-triangles and D-hexagons as hyperedges. We also define hypergraph $\mathcal{G}^h$ with the vertex set $V(\mathcal{G}^h) = V(G)$ and the edge set $E(\mathcal{G}^h) = \mathcal{C}(G)$; i.e., $\mathcal{G}^h$ has the vertices of G as vertices and each of its hyperedges contains the vertices of some D-hexagon of G. Observe that $\mathcal{G}^h$ can be obtained from $\mathcal{G}^*$ by removing all the (hyper)edges of size two and three.

LEMMA 3.16. *For any prime D-dominated Σ-plane graph G with $|D| \geq 2$,* $\mathbf{bw}(\mathcal{G}^*) \leq \max\{\mathbf{bw}(\mathcal{G}^h), 3\}$.

Proof. By Lemmata 3.14 and 3.15, we have that for each hyperedge in $\mathcal{G}^*$ there exists some D-hexagon containing all its endpoints. In other words, each hyperedge of $\mathcal{G}^*$ is a subset of some hyperedge of $\mathcal{G}^h$. By applying Lemma 3.1 recursively for every hyperedge f of $\mathcal{G}^*$ that is an edge or a triangle, we arrive at $\mathbf{bw}(\mathcal{G}^*) \leq \max\{\mathbf{bw}(\mathcal{G}^h), 3\}$. □

The following structural result will serve as a base for the recursive application of Lemmata 3.10 and 3.11 in the proof of Lemma 3.21.

LEMMA 3.17. *Let G be a prime D-dominated Σ-plane graph with $|D| \geq 3$. Then* $\mathbf{red}(G)$ *is a connected Σ-plane graph, all vertices of G have degree at least two, and $\mathcal{G}^h$ is isomorphic to* $\mathbf{ext}(\mathbf{red}(G))$.

Proof. We define the *joined drawing of G and* $\mathbf{red}(G)$ *in Σ* as follows:

Take a drawing of G on Σ and draw the vertices of $\mathbf{red}(G)$ identically to the vertices of G. For each edge $e_i = \{x, y\} \in E(\mathbf{red}(G))$ we draw $\{x, y\}$ as an I-arc connecting x and y and contained in $\Delta(e_i)$.

For an example of joined drawing, see the second drawing of Figure 12. The following three auxiliary propositions are used in the proof of the lemma.

PROPOSITION 3.18. *If G is a prime D-dominated Σ-plane graph, then* $\mathbf{red}(G)$ *is a Σ-plane graph.*

To prove the proposition, let us take the joined drawing of G and $\mathbf{red}(G)$ in Σ. Observe that, for any pair of edges $e_i, e_i \in E(\mathbf{red}(G))$, $\Delta(e_i) \cap \Delta(e_i) = \emptyset$. Therefore, if in this drawing we delete all the points that are not points of vertices or edges of $\mathbf{red}(G)$, what remains is a planar drawing of $\mathbf{red}(G)$.

PROPOSITION 3.19. *Let G be a prime D-dominated Σ-plane graph where $|D| \geq 3$ and let ϕ be the bijection defined in Lemma* 3.13. *In the joined drawing of G and $\mathbf{red}(G)$ in Σ, for any vertex $x \in D$, of degree at least three, two edges $\{x, y\}$ and $\{x, z\}$ are consecutive if and only if the D-hexagons $\phi(\{x, y\})$ and $\phi(\{x, z\})$ have exactly one edge in common. In the special case where $x \in D$ has degree two, the D-hexagons $\phi(\{x, y\})$ and $\phi(\{x, z\})$ have exactly two edges in common.*

In fact, let $\phi(\{x, y\})$ and $\phi(\{x, z\})$ be two hexagons sharing only x as a common vertex. By property (f), all faces of G incident to x are bordered by triangles that in turn are cyclically ordered according to the cyclic ordering of their edges incident to x. This ordering contains one triangle from $\phi(\{x, y\})$ and one from $\phi(\{x, z\})$. The removal of these triangles from the cyclic ordering breaks it into two nonempty subintervals, such that each of the subintervals contains one of the triangles T_1 and T_2. By Lemma 3.14, each of T_1, T_2 is a part of some D-hexagon $\phi(\{x, z_1\})$ and $\phi(\{x, z_2\})$, respectively, and this implies that the edges $\{x, y\}$ and $\{x, z\}$ cannot be consecutive in $\mathbf{red}(G)$. The inverse direction follows directly by the definition of the joined drawing of G and $\mathbf{red}(G)$.

PROPOSITION 3.20. *Let G be a prime D-dominated Σ-plane graph where $|D| \geq 3$. Then all vertices of $\mathbf{red}(G)$ have degree at least two.*

In fact, let $x \in D$ be a vertex of G incident to a face r. By property (f) of Lemma 3.9, the boundary of r is a triangle $\widehat{r} = (x, a_1, a_2)$. By Lemma 3.14, the edges $\{x, a_1\}$ and $\{x, a_2\}$ are also the edges of some D-hexagon with poles x and y. We distinguish the following cases:

Case 1. x has a neighbor a_3, distinct from a_1 and a_2. We choose a_3 so that a_2 and a_3 are consecutive in the cyclic ordering of the neighbors of x. Note also that the unique face whose boundary contains x, a_2, and a_3 should be a triangle (otherwise we have a contradiction to property (f)). By Lemma 3.14, the edges $\{x, a_2\}$ and $\{x, a_3\}$ are contained in some D-hexagon with poles x and w. Clearly $w \neq y$ (otherwise x and y are connected by three internally disjoint paths), and from Lemma 3.12 we have that $|D| = 2$, a contradiction. We conclude that $\{x, w\}$ is an edge of $\mathbf{red}(G)$, different from $\{x, y\}$.

Case 2. The only neighbors of x are the vertices a_1 and a_2. From property (f), $e = \{a_1, a_2\}$ is an exceptional edge; i.e., there are two lines l_1 and l_2, representing e, whose extremes are a_1 and a_2. Let T^1, T^2 be the triangles containing x and lines l_1 and l_2, respectively. For $i = 1, 2$, we apply Lemma 3.14 for T^i and derive that both $\{x, a_i\}, i = 1, 2$, belong to some D-hexagon C^i of G with poles x and y_i. Moreover, as $|D| \geq 3$, the line l_i is contained in $\Delta(\{x, y_i\})$. Therefore, for the case $y_1 = y_2$ we have that both lines l_1, l_2 are in $\Delta(\{x, y_i\})$, which is impossible. So, x has two neighbors in $\mathbf{red}(G)$, which completes the proof of Proposition 3.20.

Now we are in position to prove Lemma 3.17.

By Proposition 3.18, G is a Σ-plane graph. By Proposition 3.20, all vertices of $\mathbf{red}(G)$ have degree at least two. Therefore, the three transformation steps of $\mathbf{ext}$ can be applied on $\mathbf{red}(G)$. Consider now the joint drawing of G and $\mathbf{red}(G)$ in Σ. For each edge $e = \{x, y\} \in E(\mathbf{red}(G))$, we use the notation $\phi(x, y) = (x, x^+_{x,y}, y^-_{x,y}, y, x^+_{x,y}, x^-_{x,y}, x)$ (the ordering is clockwise). Apply Steps 1 and 2 of the definition of $\mathbf{ext}$ on $\mathbf{red}(G)$. During Step 2, identify vertices $x^-_{x,y}$, $x^+_{x,z}$ with the vertices of G that are denoted in the same way. This is possible because of Proposition 3.19 and because the graph G_2 created after Step 2 has exactly the same vertex set as the graph G. Let us recall that there exists a bijection $\theta : E(G) \rightarrow E(\mathbf{ext}(G))$ mapping each edge $e = \{x, y\}$ to the hyperedge formed by the vertices of $C_{x,y}$. Moreover, for any edge $e = \{x, y\} \in E(\mathbf{red}(G))$, the cycle $\theta(x, y) = C_{x,y}$ is identical to the D-hexagon $\phi(x, y)$. Notice now that the application of Step 3 of the definition of **red** on G_2 ignores the edges of G_2 and adds as edges all the cycles $\phi(e), e \in E(\mathbf{red}(G))$. As these cycles are exactly those added toward constructing $\mathcal{G}^h$, the graph $\mathcal{G}^h$ is also identical to the result of Step 3. Thus $\mathcal{G}^h$ is isomorphic to $\mathbf{ext}(\mathbf{red}(G))$. □

3.7. Main combinatorial result.

LEMMA 3.21. *For any nicely D-dominated Σ-plane graph G,* $\mathbf{bw}(G) \leq 3 \cdot \sqrt{4.5 \cdot |D|}$.

Proof. For $|D| = 1$, $G - D$ is outerplanar. It is well known that the branch-width of an outerplanar graph is at most two, implying $\mathbf{bw}(G) \leq 3$.

Suppose that $|D| \geq 2$. Clearly, $\mathbf{bw}(G) \leq \mathbf{bw}(\mathcal{G}^*)$, and to prove the lemma we show that $\mathbf{bw}(\mathcal{G}^*) \leq 3 \cdot \sqrt{4.5 \cdot |D|}$.

Prime case. We first examine the special case where G is a prime D-dominated Σ-plane graph. There are two subcases:

• If $|D| = 2$, then we set $D = \{x, y\}$. If there are only two (x, y)-paths in G, then $G = \Sigma_2^2$. If there are three (x, y)-paths in G, then $G = \Sigma_2^3$ (see Figure 9). Moreover, G cannot contain more than three (x, y)-paths; otherwise it would not be prime. Therefore, $|V(G)| \leq 8$ and thus $\mathbf{bw}(\mathcal{G}^*) \leq 8 \leq 3 \cdot \sqrt{4.5 \cdot 2} = 9$.

• Suppose now that G is a prime D-dominated Σ-plane graph and $|D| \geq 3$. By Theorem 2.4, $\mathbf{bw}(\mathbf{red}(G)) \leq \sqrt{4.5 \cdot |D|}$. By Lemma 3.17, all the vertices $\mathbf{red}(G)$ have degree ≥ 2. Therefore, we can apply Lemma 3.8 on $\mathbf{red}(G)$ (recall that $\mathbf{red}(G)$ is connected) and get $\mathbf{bw}(\mathbf{ext}(\mathbf{red}(G))) \leq 3 \cdot \mathbf{bw}(\mathbf{red}(G))$. By Lemma 3.17, $\mathbf{bw}(\mathcal{G}^h) = \mathbf{bw}(\mathbf{ext}(\mathbf{red}(G)))$ and by Lemma 3.16, $\mathbf{bw}(\mathcal{G}^*) \leq \max\{\mathbf{bw}(\mathcal{G}^h), 3\}$. Resuming, we conclude that if G is prime, then $\mathbf{bw}(\mathcal{G}^*) \leq 3 \cdot \sqrt{4.5 \cdot |D|}$.

General case. Suppose that G is a nicely D-dominated Σ-plane graph. We use induction on the number of vertices of G. If $|V(G)| = 3$, then G is a triangle (the graph Σ_1 of Figure 9) and $\mathbf{bw}(\mathcal{G}^*) = 3 \leq 3 \cdot \sqrt{4.5}$. Suppose that $\mathbf{bw}(\mathcal{G}^*) \leq 3 \cdot \sqrt{4.5 \cdot |D|}$ for every nicely D-dominated graph on $< n$ vertices. Let G be a nicely D-dominated Σ-plane graph where $|V(G)| = n$ and let q be a nonempty D-triangle or D-hexagon (if q does not exist, then the induction step follows by the prime case above). By Lemmata 3.10 and 3.11, we have that if Δ_1, Δ_2 are the discs bounded by q, then, for $i = 1, 2$, $G_i = G[V(G) \cap \Delta_i]$ is a subgraph of a nicely D_i-dominated Σ-plane graph for some $D_i \subseteq D$, $i = 1, 2$, and that $|V(G_i)| < n$ (we use the expression "subgraph" in order to capture the case when q is a D-hexagon). Applying the induction hypothesis, we get that $\mathbf{bw}(G_i^*) \leq 3 \cdot \sqrt{4.5 \cdot |D_i|}, i = 1, 2$. Notice also that $\mathcal{G}^* = \mathcal{G}_1^* \cup \mathcal{G}_2^*$ and that $V(\mathcal{G}_1^*) \cap V(\mathcal{G}_2^*) = q \in E(\mathcal{G}_1^*) \cap E(\mathcal{G}_2^*)$. Therefore, we can apply Lemma 3.1 and we get $\mathbf{bw}(\mathcal{G}^*) \leq 3 \cdot \sqrt{4.5 \cdot |D_i|}$ (recall that $|q| \leq 6$). □

For an example of the induction of the general case in the proof of Lemma 3.21, see Figure 11.

The following is the main combinatorial result of this paper.

THEOREM 3.22. *Let G be a D-dominated Σ-plane graph. Then* $\mathbf{bw}(G) \leq 3\sqrt{4.5 \cdot |D|}$.

Proof. If the branch-width of G is at most one, the theorem is trivial. Suppose that $\mathbf{bw}(G) \geq 2$. Then removing multiple edges does not decrease the branch-width of G, and we can assume that G is simple.

Let A be the set of cut vertices of G. Let G_i be the 2-connected components of G, $D_i = D \cap V(G_i)$, and $A_i = A \cap V(G_i)$, $1 \le i \le r$. Let also N_i be the vertices of G_i that are not dominated by D_i, $1 \le i \le r$.

Note also that each vertex of N_i is dominated in G by some vertex from $V(G) - V(G_i)$. Moreover, a vertex from $V(G) - V(G_i)$ cannot dominate more than one vertex in G_i. Therefore, $|N_i| \le |D - D_i|$. Thus for $D'_i = N_i \cup D_i$, we have that G_i is D'_i-dominated and $|D'_i| \le |D|$.

Consider now two cases for the graph $G_i, 1 \le i \le r$.

Case 1. G_i is a D'_i-dominated 2-connected planar graph. We take a drawing of this graph in a sphere Σ and apply Lemma 3.9. In this way, we construct a nicely D'_i-dominated Σ-plane graph H_i containing (property (a)) G_i as a minor. By Lemma 3.21, $\mathbf{bw}(H_i) \le 3 \cdot \sqrt{4.5 \cdot |D'_i|}$. Since G_i is a minor of H_i, we have that $\mathbf{bw}(G_i) \le 3\sqrt{4.5 \cdot |D'_i|} \le 3\sqrt{4.5 \cdot |D|}$.

Case 2. G_i is an induced edge. Clearly, in this case, $\mathbf{bw}(G_i) \le 3\sqrt{4.5 \cdot |D|}$.

Each graph G_i can be treated as a hypergraph with the ground set $V(G_i)$ and the edge set $E(G) \cup \{\{v\} \mid v \in V(G)\}$. As hypergraphs, graphs G_i have at most one edge (edge consisting of one vertex) in common, and by applying Lemma 3.1 recursively we obtain that $\mathbf{bw}(G) \le \max\{1, \max_{1 \le i \le r} \mathbf{bw}(G_i)\} \le 3\sqrt{4.5 \cdot |D|}$. ∎

4. Algorithmic consequences. In this section we discuss an algorithm that, given a planar graph G on n vertices and an integer k, decides whether G has a dominating set of size at most k.

4.1. The general algorithm. The algorithm runs in $O(2^{12.75\sqrt{k}} + n^3)$ steps and works in three phases as follows.

Phase 1. We use the known reduction of PLANAR DOMINATING SET problem to a linear problem kernel as a preprocessing procedure. Alber, Fellows, and Niedermeier [3] designed a procedure that, for a given integer k and planar graph G on n vertices, outputs a planar graph H on $\le 335k$ vertices such that G has a dominating set of size $\le k$ if and only if H has a dominating set of size $\le k$. Later, Chen, Fernau, Kanj, and Xia [9] improved this result, providing a reduction to a kernel of a size $\le 67k$. Each of the aforementioned reductions can be performed in $O(n^3)$ steps.

Phase 2. We compute an optimal branch decomposition of the graph H. For this step, one can use the algorithms due to Seymour and Thomas (algorithms 7.3 and 9.1 of sections 7 and 9 in [39]—for an implementation, see the work of Hicks in [33]). These algorithms need $O(n^2)$ steps for checking and $O(n^4)$ steps for constructing the branch decomposition for graphs on n vertices. We stress that there are no *large hidden constants* in the running time of these algorithms, which is important for practical applications. Thus a branch decomposition of H can be constructed in $O(k^4)$ steps. Check whether $\mathbf{bw}(H) \le (3\sqrt{4.5})\sqrt{k} < 6.364\sqrt{k}$. If the answer is "no," then by Theorem 3.22 we conclude that there is no dominating set of size k in G. If the answer is "yes," then we proceed with the next phase.

Phase 3. Here we use a dynamic programming approach to solve the PLANAR DOMINATING SET problem on graph H. Alber et al. [1] suggested a dynamic programming algorithm based on the so-called monotonicity property of the domination problem. For a graph G on n vertices with a given tree decomposition of width ℓ, the algorithm of Alber et al. can be implemented in $O(2^{2\ell}n)$ steps. There is a well known transformation due to Robertson and Seymour [36] that, given a branch decomposition of width $\le \ell$ of a graph with m edges, constructs a tree decomposition of width $\le (3/2)\ell$ in $O(m^2)$ steps. Thus the result of Alber et al. immediately implies that the DOMINATING SET problem on graphs with n vertices and m edges and of branch-width $\leq \ell$ can be solved in $O(2^{3\ell}n + m^2)$ steps. Notice now that for planar graphs $m = O(n)$. This phase requires $O(2^{3\cdot 3\sqrt{4.5\cdot k}}k+k^2)$ steps. As $3\cdot 3\sqrt{4.5} < 19.1$, we obtain an $O(2^{19.1\sqrt{k}} + n^3)$-step algorithm that finds in planar graph on n vertices a dominating set of size at most k, or reports that no such dominating set exists. However, in the next subsection (Theorem 4.1) we construct a dynamic programming algorithm solving the DOMINATING SET problem on graphs of branch-width $\leq \ell$ in $O(3^{1.5\ell}m)$ steps, where m is the number of edges in a graph. Because $(1.5\cdot\log_2 3)\cdot 3\sqrt{4.5} < 15.13$ and $m = O(k)$, we can reduce the cost of this phase to $O(2^{15.13\sqrt{k}})$ steps and conclude with a time $O(2^{15.13\sqrt{k}} + n^3)$ algorithm.

4.2. Dynamic programming on graphs of bounded branch-width. Let (T', τ) be a branch decomposition of a graph G with m edges and let $\omega' : E(T') \to 2^{V(\mathcal{G})}$ be the order function of (T', τ). We choose an edge $\{x, y\}$ in T', put a new vertex v of degree two on this edge, and make v adjacent to a new vertex r. By choosing r as a root in the new tree $T = T' \cup \{v, r\}$, we turn T into a rooted tree. For every edge of $f \in E(T) \cap E(T')$ we put $\omega(f) = \omega'(f)$. Also we put $\omega(\{x, v\}) = \omega(\{v, y\}) = \omega'(\{x, y\})$ and $\omega(\{r, v\}) = \emptyset$.

For an edge f of T we define E_f (V_f) as the set of edges (vertices) that are "below" f, i.e., the set of all edges (vertices) g such that every path containing g and $\{v, r\}$ in T contains f. With this notation, $E(T) = E_{\{v,r\}}$ and $V(T) = V_{\{v,r\}}$. Every edge f of T that is not incident to a leaf has two children that are the edges of E_f incident to f. We also denote by G_f the subgraph of G formed by edges of G corresponding to the leaves of V_f.

For every edge f of T we color the vertices of $\omega(f)$ in three colors:

black (represented by 1, meaning that the vertex is in the dominating set),

white (represented by 0, meaning that the vertex is dominated at the current step of the algorithm and is not in the dominating set), and

grey (represented by $\hat{0}$, meaning that at the current step of the algorithm we still have not decided to color this vertex white or black).

For every edge f of T we use mapping

$$A_f : \{0, \hat{0}, 1\}^{|\omega(f)|} \to \mathbb{N} \cup \{+\infty\}.$$

For a coloring $c \in \{0, \hat{0}, 1\}^{|\omega(f)|}$, the value $A_f(c)$ stores the minimum cardinality of a set $D_f \subseteq V(G_f)$ such that every nongrey vertex of G_f is dominated by a vertex from D_f and all black vertices are in D_f. More formally, $A_f(c)$ stores the minimum cardinality of a set $D_f(c)$ such that

- every vertex of $V(G_f) \setminus \omega(f)$ is adjacent to a vertex of $D_f(c)$,
- for every vertex $u \in \omega(f)$, $c(u) = 1 \Rightarrow u \in D_f(c)$ and $c(u) = 0 \Rightarrow (u \notin D_f(c)$ and u is adjacent to a vertex from $D_f(c))$.

We put $A_f(c) = +\infty$ if there is no such set $D_f(c)$. Because $\omega(\{r, v\}) = \emptyset$ and $G_{\{r,v\}} = G$, we have that $A_{\{r,v\}}(c)$ is the smallest size of a dominating set in G.

Let f be a nonleaf edge of T and let f_1, f_2 be the children of f. Define $X_1 = \omega(f) - \omega(f_2)$, $X_2 = \omega(f) - \omega(f_1)$, $X_3 = \omega(f) \cap (\omega(f_1) \cap \omega(f_2))$, and $X_4 = (\omega(f_1) \cup \omega(f_2)) - \omega(f)$.

Notice that $X_i \cap X_j \neq \emptyset$, $1 \leq i \neq j \leq 4$, and

$$\omega(f) = X_1 \cup X_2 \cup X_3. \tag{1}$$

Notice now that by the definition of ω it is impossible that a vertex belongs in exactly one of $\omega(f), \omega(f_1), \omega(f_2)$. Therefore, condition $u \in X_4$ implies that $u \in \omega(f_1) \cap \omega(f_2)$.

Hence

$$\omega(f_1) = X_1 \cup X_3 \cup X_4, \tag{2}$$

and

$$\omega(f_2) = X_2 \cup X_3 \cup X_4. \tag{3}$$

We say that a coloring c of $\omega(f)$ is *formed* from coloring c_1 of $\omega(f_1)$ and coloring c_2 of $\omega(f_2)$ if the following hold:

[F1] For every $u \in X_1$, $c(u) = c_1(u)$.

[F2] For every $u \in X_2$, $c(u) = c_2(u)$.

[F3] For every $u \in X_3$, $(c(u) \in \{\hat{0}, 1\} \Rightarrow c(u) = c_1(u) = c_2(u))$ and $(c(u) = 0 \Rightarrow [c_1(u), c_2(u) \in \{\hat{0}, 0\} \wedge (c_1(u) = 0 \vee c_2(u) = 0)])$. (The color 1 ($\hat{0}$) can appear only if both colors in c_1 and c_2 are 1 ($\hat{0}$). The color 0 appears when both colors in c_1, c_2 are not 1 and at least one of them is 0.)

[F4] For every $u \in X_4, (c_1(u) = c_2(u) = 1) \vee (c_1(u) = c_2(u) = 0) \vee (c_1(u) = 0 \wedge c_2(u) = \hat{0}) \vee (c_1(u) = \hat{0} \wedge c_2(u) = 0)$. This property says that every vertex u of $\omega(f_1)$ and $\omega(f_2)$ that does not appear in $\omega(f)$ (and hence does not appear further) should be finally colored either by 1 (if both colors of u in c_1 and c_2 are 1) or 0 (0 can appear if both colors of u in c_1 and c_2 are not 1 and at least one color is 0).

Notice that every coloring of f is formed from some colorings of its children f_1 and f_2. We start computations of values $A_f(c)$ from leaves of T. For every leaf f, $|\omega(f)| \leq 1$, and the number of colorings of $\omega(f)$ is at most three. Thus all possible values of $A_f(c)$ can be computed in $O(m)$ steps.

Then we compute the values of the corresponding functions in bottom-up fashion. The main observation here is that if f_1 and f_2 are the children of f, then the vertex sets $\omega(f_1)$, $\omega(f_2)$ "separate" subgraphs G_1 and G_2; thus the value $A_f(c)$ can be obtained from the information on colorings of $\omega(f_1)$ and $\omega(f_2)$. More precisely, let $\#_1(X_i, c)$, $1 \leq i \leq 4$, be the number of vertices in X_i colored by color 1 in coloring c. For a coloring c we assign

$$A_f(c) = \min\{A_{f_1}(c_1) + A_{f_2}(c_2) - \#_1(X_3, c_1) - \#_1(X_4, c_1) | c_1, c_2 \text{ form } c\}. \tag{4}$$

(Every 1 from X_3 and X_4 is counted in $A_{f_1}(c_1) + A_{f_2}(c_2)$ twice, and $X_3 \cap X_4 = \emptyset$.) The number of steps to compute the minimum in (4) is given by

$$O\left(\sum_c |\{c_1, c_2\} \colon c_1, c_2 \text{ form } c|\right).$$

Let $x_i = |X_i|$, $1 \leq i \leq 4$. For a fixed coloring c of $\omega(f)$, let p be the number of vertices of X_3 colored with 0. By [F3], every 0 of a vertex $u \in X_3$ can be "formed" in three ways, from $\hat{0}$ and 0, or from 0 and 0, or from 0 and $\hat{0}$. By [F4], a color of $u \in X_4$ can be obtained in four ways: 1 can be obtained from 1 and 1; 0 can be obtained either from 0 and 0, or from 0 and $\hat{0}$, or from $\hat{0}$ and 0. Then by [F1]–[F4], the number of colorings that form a fixed coloring c with exactly p vertices of X_3 of color 0 is equal to $3^p 4^{x_4}$. Every vertex of $\omega(f) = X_1 \cup X_2 \cup X_3$ can be colored in one of the three colors. The number of operations needed to estimate (4) for all possible colorings of $\omega(f)$ is

$$\sum_{p=0}^{x_3} 3^{x_1+x_2} \cdot 2^{x_3-p} \cdot 3^p \binom{x_3}{p} 4^{x_4} = 3^{x_1+x_2} 5^{x_3} 4^{x_4}.$$

The obtained bound can be reduced by using the trick due to Alber et al. [1]. The trick is based on the following observation. If for some coloring c of f we replace a color of a vertex u from $\hat{0}$ to 0, then for the new coloring c', $A_f(c) \le A_f(c')$. Thus in (4) we can replace "c_1, c_2 form c" with "c_1 and c_2 satisfies **[F1]**, **[F2]**, **[F3′]**, and **[F4′]**," where **[F3′]** and **[F4′]** are as follows:

[F3′] For every $u \in X_3$, $(c(u) \in \{\hat{0}, 1\} \Rightarrow c(u) = c_1(u) = c_2(u))$ and $(c(u) = 0 \Rightarrow [c_1(u), c_2(u) \in \{\hat{0}, 0\} \wedge (c_1(u) \neq c_2(u))])$.

[F4′] For every $u \in X_4$, $(c_1(u) = c_2(u) = 1) \ \vee [c_1(u), c_2(u) \in \{\hat{0}, 0\} \wedge (c_1(u) \neq c_2(u))]$.

The purpose of properties **[F3′]** and **[F4′]** is to reduce the search space from all coloring forming c to the smaller set of colorings. Thus the number of steps for evaluating $A_f(c)$ is bounded by

$$\sum_{p=0}^{x_3} 3^{x_1+x_2} \cdot 2^{x_3-p} \cdot 2^p \binom{x_3}{p} 3^{x_4} = 3^{x_1+x_2} 4^{x_3} 3^{x_4}.$$

Let ℓ be the branch-width of G. By (1), (2), and (3),

$$\begin{aligned} x_1 + x_2 + x_3 &\le \ell, \\ x_1 + x_3 + x_4 &\le \ell, \\ x_2 + x_3 + x_4 &\le \ell. \end{aligned} \tag{5}$$

The maximum value of the linear function $x_1 + x_2 + x_4 + x_3 \cdot \log_3 4$ subject to constraints (5) is $\frac{3\log_4 3}{2}\ell$. (This is because the value of the corresponding linear program achieves maximum in $x_1 = x_2 = x_4 = 0.5\ell$, $x_3 = 0$.) Thus

$$3^{x_1+x_2} 4^{x_3} 3^{x_4} \le 4^{\frac{3\log_4 3}{2}\ell} = 3^{\frac{3\ell}{2}}.$$

It is easy to check that the number of edges in T is $O(m)$ and the number of steps needed to evaluate $A_{\{r,v\}}(c)$ is $O(3^{\frac{3\ell}{2}} m)$. Summarizing, we get the following theorem.

THEOREM 4.1. *For a graph G on m edges and given a branch decomposition of width $\le \ell$, the dominating set of G can be computed in $O(3^{\frac{3\ell}{2}} m)$ time.*

5. Concluding remarks and open problems. We start this section with a discussion on the optimality of our results. We then give a presentation on several open problems and results that were motivated by this work.

5.1. Can Theorem 3.22 be improved? We have proved that for any planar graph with a dominating set of size $\le k$, $\mathbf{bw}(G) \le 3\sqrt{4.5 \cdot k} < 6.364\sqrt{k}$. The first of the multiplicative factors 3 follows from our results on the structure of planar graphs with a given dominating set in section 3. The second factor $\sqrt{4.5} \approx 2.121$ follows from [28] and is the bound on branch-width of planar graphs (Theorem 2.4). Any improvement to any of these two factors immediately implies an improvement to the time analysis of our fixed-parameter algorithm for a dominating set. However, our approach cannot be strongly improved because the upper bound of Theorem 3.22 is not far from the optimal.

LEMMA 5.1. *There exist planar graphs with a dominating set of size $\le k$ and with branch-width $> 3\sqrt{k}$.*

Proof. Let G be a $(3n+2, 3n+2)$-grid for any $n \ge 1$. Let V' be the vertices of G of degree < 4. Let also V'' be the set of all vertices adjacent to V' in G. We define D as the unique $S \subseteq V(G) - V' - V''$, where $|S| = n^2$ and such that the

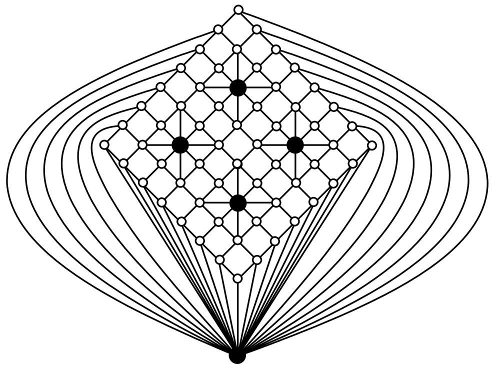

FIG. 13. *An example of the proof of Lemma* 5.1.

distance in G of all pairs $v, u \in D$ in G is a multiple of three. Then for any vertex $v \in D$, and for any possible cycle (square) (v, x, y, z, v) add the edge $\{x, z\}$. The construction is completed by connecting all the vertices in V' with a new vertex v_{new} (see Figure 13). We call the resulting graph J_n. Clearly, $D \cup \{v_{\mathrm{new}}\}$ is a dominating set of J_n of size $k = n^2 + 1 \geq 2$. As the $(3n + 2, 3n + 2)$-grid is a subgraph of J_n we have that $\mathbf{bw}(J_n) \geq 3n + 2 \geq 3\sqrt{k-1} + 2 > 3\sqrt{k}$ (from [36], the (ρ, ρ)-grid has branch-width ρ). ◻

5.2. Open problems and extensions of our results. A (k, r)-*center* in a graph G is a set of at most k vertices, which we call centers, such that any vertex of G is within distance at most r from some center. Extending the results of section 4, [13] gives an algorithm that outputs, if it exists, a (k, r)-center of a planar graph in $r^{O(r\sqrt{k})} + n^{O(1)}$ steps (according to [13], the same result also holds for map graphs). The constants hidden in the first O-notation are based on an extension of Lemma 2.2, bounding the branch-width of any planar graph containing a (k, r)-center by $4(2r + 1)\sqrt{k} + O(r)$. We conjecture that this bound (and subsequently the running time of the algorithm in [13]) can be improved to $(2r + 1)\sqrt{4.5 \cdot k}$. We also suspect that a proof of this conjecture could be based on the same steps as those we used for Theorem 3.22.

An approach similar to the one of section 4 has been applied for a wide number of problems related to the PLANAR DOMINATING SET problem. In this way, our upper bound improves the algorithm complexity analysis for a series of problems when their inputs are restricted to planar graphs. As a sample we mention the following: INDEPENDENT DOMINATING SET, PERFECT DOMINATING SET, PERFECT CODE, WEIGHTED DOMINATING SET, TOTAL DOMINATING SET, EDGE DOMINATING SET, FACE COVER, VERTEX FEEDBACK SET, VERTEX COVER, MINIMUM MAXIMAL MATCHING, CLIQUE TRANSVERSAL SET, DISJOINT CYCLES, and DIGRAPH KERNEL (see [17] for details and extensions to more general graph classes). However, in all of the aforementioned problems, the time analysis is based on algorithms and combinatorial bounds for tree-width. It is an interesting problem whether better speed-up is possible using branch-width instead of tree-width, as we did in this paper. To our knowledge, not much progress has been noted so far on the design of algorithms on graphs of bounded branch-width (see [11, 13, 20]).

It appears that the planarity is not a limit for the existence of bounds like the one in Theorem 3.22. In [27], it was proved that for any D-dominated graph G,

$\mathbf{bw}(G) \leq 3(\sqrt{4.5}+2\sqrt{2 \cdot \mathbf{eg}(G)})\sqrt{|D| + 6 \cdot \mathbf{eg}(G)} = O(\sqrt{|D| \cdot \mathbf{eg}(G)}+\mathbf{eg}(G))$, where $\mathbf{eg}(G)$ is the Euler genus of G. The proof of this bound uses Theorem 3.22 as a basic ingredient. As a consequence of [27], most of the applications mentioned in the previous paragraph also can be extended for graphs of bounded genus. For discussions on the limits of this approach, see [26].

Finally, the idea behind Lemma 2.2 offers a mechanism for proving similar bounds for a wide family of parameters. This is the general family of *bidimensional parameters* introduced in [12] that unified the framework where the algorithmic paradigm of section 4 can be applied. Recent research on bidimensionality extends to several results such as [14, 15, 16, 18].

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