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Exponential Speedup of Fixed-Parameter Algorithms for Classes of Graphs Excluding Single-Crossing Graphs as Minors[1]

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Abstract. We present a fixed-parameter algorithm that constructively solves the k-dominating set problem on any class of graphs excluding a single-crossing graph (a graph that can be drawn in the plane with at most one crossing) as a minor in $O(4^{9.55\sqrt{k}}n^{O(1)})$ time. Examples of such graph classes are the $K_{3,3}$-minor-free graphs and the K_5-minor-free graphs. As a consequence, we extend our results to several other problems such as vertex cover, edge dominating set, independent set, clique-transversal set, kernels in digraphs, feedback vertex set, and a collection of vertex-removal problems. Our work generalizes and extends the recent results of exponential speedup in designing fixed-parameter algorithms on planar graphs due to Alber et al. to other (nonplanar) classes of graphs.

Key Words. Subexponential algorithms, Graph minors, Dominating set.

1. Introduction. According to a 1998 survey book [HHS], there are more than 200 published research papers on solving domination-like problems on graphs. Because this problem is very hard and NP-complete even for special kinds of graphs such as planar graphs, much attention has focused on solving this problem on a more restricted class of graphs. It is well known that this problem can be solved on trees [CGH] or even the generalization of trees, graphs of bounded treewidth [TP1]. The approximability of the dominating set problem has received considerable attention, but it is not known and it is not believed that this problem has constant-factor approximation algorithms on general graphs [ACG$^+$].

Downey and Fellows [DF] introduced a new concept to handle NP-hardness called *fixed-parameter tractability*. Unfortunately, according to this theory, it is very unlikely that the k-dominating set problem has an efficient fixed-parameter algorithm for general graphs. In contrast, this problem is fixed-parameter tractable on planar graphs. Alber et al. [ABF$^+$] demonstrated a solution to the planar k-dominating set in time $O(4^{6\sqrt{34k}}n)$.

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Indeed, this result was the first nontrivial result for the parameterized version of an NP-hard problem where the exponent of the exponential term grows sublinearly in the parameter. Recently, the running time of this algorithm was further improved to $O(2^{27\sqrt{k}}n)$ [KP] and $O(2^{15.13\sqrt{k}}k + n^3 + k^4)$ [FT2]. One of the aims of this paper is to generalize this result to nonplanar classes of graphs.

A graph G is *H-minor-free* if H cannot be obtained from any subgraph of G by contracting edges. A graph is called a *single-crossing graph* if it can be drawn in the plane with at most one crossing. Similar to the approach of Alber et al., we prove that for a single-crossing graph H, the treewidth of any H-minor-free graph G having a k-dominating set is bounded by $O(\sqrt{k})$. We note that planar graphs are both $K_{3,3}$-minor-free and K_5-minor-free, where $K_{3,3}$ and K_5 are both single-crossing graphs. As a result, we generalize current exponential speedup in fixed-parameter algorithms on planar graphs to other kinds of graphs by showing how we can solve the k-dominating set problem on any class of graphs excluding a single-crossing graph as a minor in time $O(4^{9.55\sqrt{k}}n^{O(1)})$. The genesis of our results lies in a result of Hajiaghayi et al. [HNRT], [DHN$^+$] on obtaining the local treewidth of the aforementioned class of graphs.

Using the solution for the k-dominating set problem on planar graphs, Kloks et al. [CKL], [KLL], [GKL] and Alber et al. [ABF$^+$], [AFN] obtained exponential speedup in solving other problems such as vertex cover, independent set, clique-transversal set, kernels in digraph, and feedback vertex set on planar graphs. In this paper we also show how our results can be extended to these problems and many other problems such as variants of dominating set, edge dominating set, and a collection of vertex-removal problems.

Since the results of this paper were announced, several new papers have been developed by using and extending the results and techniques of this paper; see, e.g., [DHN$^+$], [FT2], [DFHT1], [DH1], [DFHT3], [DFHT2] and [DHT2].

This paper is organized as follows. First we introduce the terminology used throughout the paper, and formally define tree decompositions, treewidth, and fixed-parameter tractability in Section 2. In Section 3 we introduce the concept of clique-sum, we prove two general theorems concerning the construction of tree decompositions of width $O(\sqrt{k})$ for these graphs, and finally we consider the design of fast fixed-parameter algorithms for them. In Section 4 we apply our general results to the k-dominating set problem, and in Section 5 we describe how this result can be applied to derive fast fixed-parameter algorithms for many different parameters. In Section 6 we prove some graph-theoretic results that provide a framework for designing fixed-parameter algorithms for a collection of vertex-removal problems. In Section 7 we give some further extensions of our results to graphs with linear local treewidth. We end with some conclusions and open problems in Section 8.

2. Background

2.1. *Preliminaries*. We assume the reader is familiar with general concepts of graph theory such as (un)directed graphs, trees, and planar graphs. The reader is referred to standard references for appropriate background [BM1]. In addition, for exact definitions of various NP-hard graph-theoretic problems in this paper, the reader is referred to Garey and Johnson's book on computers and intractability [GJ].

Our graph terminology is as follows. All graphs are finite, simple, and undirected, unless indicated otherwise. A graph G is represented by $G = (V, E)$, where V (or $V(G)$) is the set of vertices and E (or $E(G)$) is the set of edges. We denote an edge e in a graph G between u and v by $\{u, v\}$. We define n to be the number of vertices of a graph when it is clear from context. We define the *r-neighborhood* of a set $S \subseteq V(G)$, denoted by $N_G^r(S)$, to be the set of vertices at distance at most r from at least one vertex of $S \subseteq V(G)$; if $S = \{v\}$ we simply use the notation $N_G^r(v)$. The *union* of two disjoint graphs G_1 and G_2, $G_1 \cup G_2$, is a graph G such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

For generalizations of algorithms on undirected graphs to directed graphs, we consider underlying graphs of directed graphs. The *underlying graph* of a directed graph H is the undirected graph G in which $V(G) = V(H)$ and $\{u, v\} \in E(G)$ if and only if $(u, v) \in E(H)$ or $(v, u) \in E(H)$.

One way of describing classes of graphs is by using *minors*, introduced below.

DEFINITION 1. *Contracting* an edge $e = \{u, v\}$ is the operation of replacing both u and v by a single vertex w whose neighbors are all vertices that were neighbors of u or v, except u and v themselves. A graph G is a *minor* of a graph H if G can be obtained from a subgraph of H by contracting edges. A graph class $\mathcal{C}$ is a *minor-closed* class if any minor of any graph in $\mathcal{C}$ is also a member of $\mathcal{C}$. A minor-closed graph class $\mathcal{C}$ is *H-minor-free* if $H \notin \mathcal{C}$.

For example, a planar graph is a graph excluding both $K_{3,3}$ and K_5 as minors.

2.2. *Treewidth*. The notion of treewidth was introduced by Robertson and Seymour [RS1] and plays an important role in their fundamental work on graph minors. To define this notion, first we consider the representation of a graph as a tree, which is the basis of our algorithms in this paper.

DEFINITION 2 [RS1]. A *tree decomposition* of a graph $G = (V, E)$, denoted by $TD(G)$, is a pair (χ, T) in which $T = (I, F)$ is a tree and $\chi = \{\chi_i \mid i \in I\}$ is a family of subsets of $V(G)$ such that:

1. $\bigcup_{i \in I} \chi_i = V$;
2. for each edge $e = \{u, v\} \in E$ there exists an $i \in I$ such that both u and v belong to χ_i; and
3. for all $v \in V$, the set of nodes $\{i \in I \mid v \in \chi_i\}$ forms a connected subtree of T.

To distinguish between vertices of the original graph G and vertices of T in $TD(G)$, we call vertices of T *nodes* and their corresponding χ_i's *bags*. The maximum size of a bag in $TD(G)$ minus one is called the *width* of the tree decomposition. The *treewidth* of a graph G, denoted $\mathrm{tw}(G)$, is the minimum width over all possible tree decompositions of G.

Many NP-complete problems have linear-time or polynomial-time algorithms when they are restricted to graphs of bounded treewidth. There are a few techniques for obtaining such algorithms. The main technique is called *computing tables of characterizations of partial solutions*. This technique is a general dynamic programming approach, first introduced by Arnborg and Proskurowski [AP]. Bodlaender [Bo4] gave a better presentation of this technique. Other approaches applicable for solving problems on graphs of bounded treewidth are *graph reduction* [ACPS], [BdF] and *describing the problems in certain types of logic* [ALS], [Co].

2.3. *Fixed-Parameter Tractability.* Developing practical algorithms for NP-hard problems is an important issue. Recently, Downey and Fellows [DF] introduced a new approach to cope with this NP-hardness, namely *fixed-parameter tractability*. For many NP-complete problems, the inherent combinatorial explosion is often due to a certain part of a problem, namely a *parameter*. The parameter is often an integer and small in practice. The running times of simple algorithms may be exponential in the parameter but polynomial in the problem size. For example, it has been shown that k-vertex cover has an algorithm with running time $O(kn + 1.271^k)$ [CKJ] and hence this problem is fixed-parameter tractable.

DEFINITION 3 [DF]. A parameterized problem $L \subset \Sigma^* \times \mathbb{N}$ is *fixed-parameter tractable* (*FPT*) if there is an algorithm that correctly decides, for input $(x, k) \in \Sigma^* \times \mathbb{N}$, whether $(x, k) \in L$ in time $f(k)n^c$, where n is the size of the main part of the input x, $|x| = n$, k is a parameter (usually an integer), c is a constant independent of k, and f is an arbitrary function.

3. General Results on Clique-Sum Graphs. In this section we define the general framework of our results. A basic tool is the graph summation operation, which also plays an important role in the work of Hajiaghayi et al. [HNRT], [Ha] to obtain the local treewidth of *clique-sum* graphs, defined formally below.

DEFINITION 4. Suppose G_1 and G_2 are graphs with disjoint vertex-sets and $k \geq 0$ is an integer. For $i = 1, 2$, let $W_i \subseteq V(G_i)$ form a clique of size k and let G_i' $(i = 1, 2)$ be obtained from G_i by deleting some (possibly no) edges from $G_i[W_i]$ with both endpoints in W_i. Consider a bijection $h : W_1 \to W_2$. We define a *k-sum* G of G_1 and G_2, denoted by $G = G_1 \oplus_k G_2$ or simply by $G = G_1 \oplus G_2$, to be the graph obtained from the union of G_1' and G_2' by identifying w with $h(w)$ for all $w \in W_1$. The images of the vertices of W_1 and W_2 in $G_1 \oplus_k G_2$ form the *join set*.

In the rest of this section, when we refer to a vertex v of G in G_1 or G_2, we mean the corresponding vertex of v in G_1 or G_2 (or both). It is worth mentioning that $\oplus$ is not a well-defined operator and it can have a set of possible results. The reader is referred to Figure 1 to see an example of a 5-sum operation.

The following lemma shows how the treewidth changes when we apply a graph summation operation.

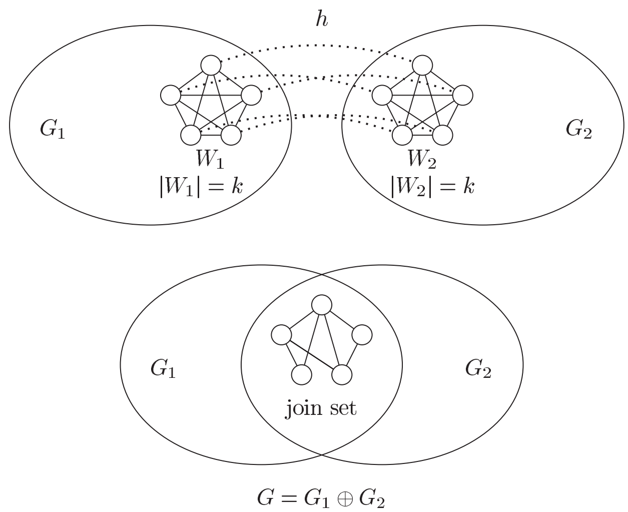

Fig. 1. A k-sum of two graphs G_1 and G_2.

LEMMA 1 [BvLTT]. *For any two graphs G and H,*

$$\mathrm{tw}(G \oplus H) \leq \max\{\mathrm{tw}(G), \mathrm{tw}(H)\}.$$

Let s be an integer where $0 \leq s \leq 3$ and let $\mathcal{C}$ be a finite set of graphs. We say that a graph class $\mathcal{G}$ is a *clique-sum class* if any of its graphs can be constructed by a sequence of i-sums ($i \leq s$) applied to planar graphs and graphs in $\mathcal{C}$. We call a graph *clique-sum* if it is a member of a clique-sum class. We call the pair $(\mathcal{C}, s)$ the *defining pair* of $\mathcal{G}$ and we call the maximum treewidth of graphs in $\mathcal{C}$ the *base* of $\mathcal{G}$ and the *base* of graphs in $\mathcal{G}$. A series of k-sums (not necessarily unique) which generate a clique-sum graph G are called *a decomposition of G into clique-sum operations*.

According to the (nonalgorithmic) result of [RS2], if $\mathcal{G}$ is the class of graphs excluding a single-crossing graph (can be drawn in the plane with at most one crossing) H then $\mathcal{G}$ is a clique-sum class with defining pair $(\mathcal{C}, s)$ where the base of $\mathcal{G}$ is bounded by a constant c_H depending only on H. In particular, if $H = K_{3,3}$, the defining pair is $(\{K_5\}, 2)$ and $c_H = 4$ [Wa] and if $H = K_5$ then the defining pair is $(\{V_8\}, 3)$ and $c_H = 4$ [Wa]. Here by V_8 we mean the graph obtained from a cycle of length eight by joining each pair of diagonally opposite vertices by an edge. For more results on clique-sum classes see [Di1].

From the definition of clique-sum graphs, one can observe that, for any clique-sum graph G which excludes a single-crossing graph H as a minor, any minor G' of G is also a clique-sum graph which excludes the same graph H as a minor.

We call a clique-sum graph class $\mathcal{G}$ *α-recognizable* if there exists an algorithm that for any graph $G \in \mathcal{G}$ outputs in $O(n^\alpha)$ time a sequence of clique-sums of graphs of total size $O(|V(G)|)$ that constructs G. We call a graph *α-recognizable* if it belongs in some α-recognizable clique-sum graph class.

One of the ingredients of our results is the following constructive version of the result in [RS2].

THEOREM 1 [DHT1], [DHN$^+$]. *For any graph G excluding a single-crossing graph H as a minor, we can construct in $O(n^4)$ time a series of clique-sum operations $G = G_1 \oplus G_2 \oplus \cdots \oplus G_m$ where each G_i, $1 \le i \le m$, is a minor of G and is either a planar graph or a graph of treewidth at most c_H. Here each $\oplus$ is a* 0-, 1-, 2- *or* 3-*sum.*

In the remainder of the paper we assume that c_H is the smallest integer for which Theorem 1 holds. Notice that, according to the terminology introduced before, any graph class excluding a single-crossing graph as a minor is a 4-recognizable clique-sum graph class. As particular cases of Theorem 1 we mention that $K_{3,3}$-minor-free graphs are 1-recognizable [As] and K_5-minor-free graphs are 2-recognizable [KM]. For more examples of graph classes that can be characterized by clique-sum decompositions, see the work of Diestel [Di1], [Di2].

A *parameterized graph class* (or just *graph parameter*) is a family $\mathcal{F}$ of classes $\{\mathcal{F}_i, i \ge 0\}$ where $\bigcup_{i\ge 0} \mathcal{F}_i$ is the set of all graphs and for any $i \ge 0$, $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$. Given two parameterized graph classes $\mathcal{F}^1$ and $\mathcal{F}^2$ and a natural number $\gamma \ge 1$ we say that $\mathcal{F}^1 \preccurlyeq_\gamma \mathcal{F}^2$ if for any $i \ge 0$, $\mathcal{F}^1_i \subseteq \mathcal{F}^2_{\gamma \cdot i}$.

In the rest of this paper, we identify a parameterized problem with the *parameterized graph class* corresponding to its "yes" instances.

THEOREM 2. *Let $\mathcal{G}$ be an α_1-recognizable clique-sum graph class with base c and let $\mathcal{F}$ be a parameterized graph class. In addition, we assume that each graph in $\mathcal{G}$ can be constructed using i-sums where $i \le s \le 3$. Suppose also that there exist two positive real numbers β_1, β_2 such that*:

(1) *For any $k \ge 0$, planar graphs in $\mathcal{F}_k$ have treewidth at most $\beta_1\sqrt{k} + \beta_2$ and such a tree decomposition can be found in $O(n^{\alpha_2})$ time.*
(2) *For any $k \ge 0$ and any $i \le s$, if $G_1 \oplus_i G_2 \in \mathcal{F}_k$ then $G_1, G_2 \in \mathcal{F}_k$.*

Then, for any $k \ge 0$, the graphs in $\mathcal{G} \cap \mathcal{F}_k$ all have treewidth $\le \max\{\beta_1\sqrt{k} + \beta_2, c\}$ and such a tree decomposition can be constructed in $O(n^{\max\{\alpha_1,\alpha_2\}} + (\sqrt{k})^s \cdot n)$ time.

PROOF. Let $G \in \mathcal{G} \cap \mathcal{F}_k$ and assume that $G = G_1 \oplus G_2 \oplus \cdots \oplus G_m$ where each G_i, $1 \le i \le m$, is either a planar graph or a graph of treewidth at most c. We use induction on m, the number of G_i's. For $m = 1$, $G = G_1$ is either a planar graph that from (1) has treewidth at most $\beta_1\sqrt{k} + \beta_2$ or a graph of treewidth at most c. Thus the basis of the induction is true for both cases. We assume the induction hypothesis is true for $m = h$, and we prove the hypothesis for $m = h + 1$. Let $G' = G_1 \oplus G_2 \oplus \cdots \oplus G_h$ and $G'' = G_{h+1}$. Thus $G = G' \oplus G''$. By (2), both G' and G'' belong in $\mathcal{F}_k$. By the induction hypothesis, $\mathrm{tw}(G') \le \max\{\beta_1\sqrt{k} + \beta_2, c\}$ and from (1) $\mathrm{tw}(G'') \le \max\{\beta_1\sqrt{k} + \beta_2, c\}$. The proof, for $m = h + 1$, follows from this fact and Lemma 1.

To construct a tree decomposition of the aforementioned width, first we construct a tree decomposition of width at most $\beta_1\sqrt{k} + \beta_2$ for each planar graph in $O(n^{\alpha_2})$ time. We also note that using Bodlaender's algorithm [Bo3], we can obtain a tree decomposition of width c for any graph of treewidth at most c in linear time (the hidden constant only depends on c). Then having tree decompositions of G_i's, $1 \le i \le m$, in the rest of the algorithm, we glue together the tree decompositions of G_i's using the construction given in the proof of Lemma 1. To this end, we introduce an array *Nodes* indexed by all subsets of $V(G)$ of size at most s. In this array, for each subset whose elements form a clique, we specify a node of the tree decomposition which contains this subset. We note that for each clique C in G_i, there exists a node z of $TD(G)$ such that all vertices of C appear in the bag of z [BM2]. This array is initialized as part of a preprocessing stage of the algorithm. Now, for the $\oplus$ operation between $G_1 \oplus \cdots \oplus G_h$ and G_{h+1} over the join set W, using array *Nodes*, we find a node α in the tree decomposition of $G_1 \oplus \cdots \oplus G_h$ whose bag contains W. Because we have the tree decomposition of G_{h+1}, we can find the node α' of the tree decomposition whose bag contains W by brute force over all subsets of size at most s of bags. Simultaneously, we update array *Nodes* by subsets of $V(G)$ which form a clique and appear in bags of the tree decomposition of G_{h+1}. Then we add an edge between α and α'. As the number of nodes in a tree decomposition of G_{h+1} is in $O(|V(G_{h+1})|)$ and each bag has size at most $O(\sqrt{k})$ (and thus there are at most $O((\sqrt{k})^s)$ choices for a subset of size at most s), this operation takes $O((\sqrt{k})^s|V(G_{h+1})|)$ time for G_{h+1}.

The claimed running time follows from the time required to determine a set of clique-sum operations, the time required to construct tree decompositions, the time needed for gluing tree decompositions together, and the fact that $\sum_{i=1}^{m}|V(G_i)| = O(|V(G)|)$. $\square$

Notice that Condition (2) of Theorem 2 is not necessary when $\mathcal{G}$ excludes a single-crossing graph and $\mathcal{F}$ is closed under taking of minors. Indeed, from Theorem 1, we have that in the sequence of operations $G = G_1 \oplus G_2 \oplus \cdots \oplus G_m$, each G_i is a minor of G and therefore, if $G \in F_k$ then each G_i is also a member of F_k. We resume this observation to the following.

THEOREM 3. *Let $\mathcal{G}$ be the class of graphs excluding some single-crossing graph H as a minor and let $\mathcal{F}$ be any minor-closed parameterized graph class. Suppose that there exist real numbers $\beta_0 \ge 4$, β_1 such that any planar graph in $\mathcal{F}_k$ has treewidth at most* $\max\{\beta_1\sqrt{k} + \beta_0, c_H\}$ *and such a tree decomposition can be found in $O(n^{\alpha})$. Then graphs in $\mathcal{G} \cap \mathcal{F}_k$ all have treewidth $\le \beta_1\sqrt{k} + \beta_0$ and such a tree decomposition can be constructed in $O(n^{\max\{\alpha,4\}})$ time.*

THEOREM 4. *Let $\mathcal{G}$ be a graph class and let $\mathcal{F}$ be some parameterized graph class. Suppose also for some positive real numbers $c, \alpha_1, \alpha_2, \beta_1, \beta_2, \delta$ the following hold*:

(1) *For any $k \ge 0$, the graphs in $\mathcal{G} \cap \mathcal{F}_k$ all have treewidth* $\le \max\{c, \beta_1\sqrt{k} + \beta_2\}$ *and such a tree decomposition can be decided and constructed* (*if it exists*) *in $O(n^{\alpha_2})$ time. We also assume testing membership in $\mathcal{G}$ takes $O(n^{\alpha_1})$ time.*
(2) *Given a tree decomposition of width at most w of a graph, there exists an algorithm deciding whether the graph belongs in $\mathcal{F}_k$ in $O(\delta^w n)$ time.*

Then there exists an algorithm deciding in $O(\delta^{\max\{c,\beta_1\sqrt{k}+\beta_2\}} + n^{\max\{\alpha_1,\alpha_2\}})$ time whether an input graph G belongs in $\mathcal{G} \cap \mathcal{F}_k$.

PROOF. First, we can test membership in $\mathcal{G}$ in $O(n^{\alpha_1})$ time. Then we can apply the algorithm from (1) and (assuming success) supply the resulting tree decomposition to the algorithm from (2). □

4. Fixed-Parameter Algorithms for Dominating Set. In this section we describe some of the consequences of Theorems 2 and 4 on the design of efficient fixed-parameter algorithms for a collection of parameterized problems where their inputs are clique-sum graphs.

A *dominating set* of a graph G is a set of vertices of G such that each of the rest of vertices has at least one neighbor in the set. We represent the *k-dominating set* problem with the parameterized graph class $\mathcal{DS}$ where $\mathcal{DS}_k$ contains graphs which have a dominating set of size $\leq k$. Our target is to show how we can solve the k-dominating set problem on clique-sum graphs, where H is a single-crossing graph, in time $O(c^{\sqrt{k}} n^{O(1)})$ instead of the current algorithms which run in time $O(c^k n^{O(1)})$ for some constant c. By this result, we extend the current exponential speedup in designing algorithms for planar graphs [AFN] to a more generalized class of graphs. In fact, planar graphs are both $K_{3,3}$-minor-free and K_5-minor-free graphs, where both $K_{3,3}$ and K_5 are single-crossing graphs.

According to the result of [KP] Condition (1) of Theorem 2 is satisfied for $\beta_1 = 15.6$, $\beta_2 = 50$, and $\alpha_2 = 1$. Moreover, from [FT2], Condition (1) is also satisfied for $\beta_1 = 9.55$, $\beta_2 = 0$, and $\alpha_2 = 4$.

The next lemma shows that Condition (2) of Theorem 2 also holds.

LEMMA 2. *If $G = G_1 \oplus_m G_2$ has a k-dominating set, then both G_1 and G_2 have dominating sets of size at most k.*

PROOF. Let the k-dominating set of G be S and let W be the join set of $G_1 \oplus_k G_2$. Without loss of generality we show that G_1 has a dominating set of size k. If $S_1 = S \cap V(G_1)$ is a dominating set for G_1 then the result immediately follows, otherwise there exists vertex $w \in V(G_1)$ which is dominated by a vertex $v \in V(G_2) - V(G_1)$. One can observe that all such vertices w are in W. Because $v \in S$, but $v \notin S_1$, set $S_1' = S_1 + \{w\}$ has at most k vertices and because W is a clique in G_1, S_1' is a dominating set of size at most k in G_1. □

Let $\mathcal{G}$ be any α-recognizable clique-sum class. Now by applying Theorem 2 for $\beta_1 = 9.55$, $\beta_2 = 0$, $\alpha_1 = \alpha$, and $\alpha_2 = 4$ we have the following.

THEOREM 5. *If $\mathcal{G}$ is an α-recognizable clique-sum class of base c, then any member G of $\mathcal{G}$ with a dominating set of size at most $\leq k$ has treewidth at most* $\max\{c, 9.55\sqrt{k}\}$ *and the corresponding tree decomposition of G can be constructed in $O(n^{\max\{\alpha,4\}})$ time.*

From Theorem 5, we get that Condition (1) of Theorem 4 is satisfied for $\beta_1 = 9.55$, $\beta_2 = 0$, $\alpha_2 = \max\{\alpha, 4\}$, and $\alpha_2 = 4$. The main result in [AN] shows that for the graph parameter $\mathcal{DS}$ Condition (2) of Theorem 4 is also satisfied for $\delta = 4$. We conclude with the following.

THEOREM 6. *There is an algorithm that in $O(4^{9.55\sqrt{k}}n + n^{\max\{\alpha,4\}})$ time solves the k-dominating set problem for any α-recognizable clique-sum graph of base c.*[4]

COROLLARY 1. *There is an algorithm that solves the k-dominating set problem for any graph class excluding some single-crossing graph as a minor in $O(4^{9.55\sqrt{k}}n + n^4)$ time.*

For the special cases of K_5-minor-free graphs and $K_{3,3}$-minor-free graphs, we may apply Theorem 2 for $\beta_1 = 15.6$, $\beta_2 = 50$, and $\alpha_2 = 1$ and derive the following.

COROLLARY 2. *There is an algorithm that solves the k-dominating set problem for any K_5-minor-free graph in $O(4^{15.6\sqrt{k}+50}n + n^2)$ time and for any $K_{3,3}$-minor-free graph in $O(4^{15.6\sqrt{k}+50}n)$ time.*

5. Algorithms for Parameters Bounded by the Dominating-Set Number. We provide a general methodology for deriving fast fixed-parameter algorithms in this section. First, we consider the following theorem which is an immediate consequence of Theorem 4.

THEOREM 7. *Let $\mathcal{G}$ be a graph class and let $\mathcal{F}^1, \mathcal{F}^2$ be two parameterized graph classes where $\mathcal{F}^1 \preccurlyeq_\gamma \mathcal{F}^2$ for some natural number $\gamma \geq 1$. Suppose also that there exist positive real numbers $\alpha_1, \alpha_2, \beta_1, \beta_2, \delta$ such that*:

(1) *For any $k \geq 0$, the graphs in $\mathcal{G} \cap \mathcal{F}_k^2$ all have treewidth $\leq \beta_1\sqrt{k} + \beta_2$ and such a tree decomposition can be decided and constructed (if it exists) in $O(n^{\alpha_2})$ time. We also assume testing membership in $\mathcal{G}$ takes $O(n^{\alpha_1})$ time.*
(2) *There exists an algorithm deciding whether a graph of treewidth $\leq w$ belongs in $\mathcal{F}_k^1$ in $O(\delta^w n)$ time.*

Then:

(1) *For any $k \geq 0$, the graphs in $\mathcal{G} \cap \mathcal{F}_k^1$ all have treewidth at most $\beta_1\sqrt{\gamma k} + \beta_2$ and such a tree decomposition can be constructed in $O(n^{\alpha_2})$ time.*
(2) *There exists an algorithm deciding in $O(\delta^{\beta_1\sqrt{\gamma k}+\beta_2} + n^{\max\{\alpha_1,\alpha_2\}})$ time whether an input graph G belongs in $\mathcal{G} \cap \mathcal{F}_k^1$.*

PROOF. Consequence (1) follows immediately from the definition of $\preccurlyeq_\gamma$. Consequence (2) follows from Theorem 4. □

[4] In the rest of this paper we assume that constants, e.g., c, are small and they do not appear in the powers, because they are absorbed into the O notation.

The idea of our general technique is given by the following theorem that is a direct consequence of Theorems 5 and 7.

THEOREM 8. *Let $\mathcal{F}$ be a parameterized graph class satisfying the following two properties*:

(1) *It is possible to check membership in $\mathcal{F}_k$ of a graph G of treewidth at most w in $O(\delta^w n)$ time for some positive real number δ.*
(2) $\mathcal{F} \preccurlyeq_\gamma \mathcal{DS}$.

Then:

(1) *Any clique-sum graph G of base c in $\mathcal{F}_k$ has treewidth at most* $\max\{9.55\sqrt{\gamma k}+8, c\}$.
(2) *We can check whether an input graph G is in $\mathcal{F}_k$ in $O(\delta^{9.55\sqrt{\gamma k}}n + n^{\max\{\alpha,4\}})$ on an α-recognizable clique-sum graph of base c.*

In what follows we explain how Theorem 8 applies for a series of graph parameters. In particular, we explain why Conditions (1) and (2) are satisfied for each problem.

5.1. *Variants of the Dominating Set Problem*. A *k-dominating set with property Π* on an undirected graph G is a k-dominating set D of G which has the additional property Π and the *k-dominating set with property Π problem* is the task to decide, given a graph $G = (V, E)$, a property Π, and a positive integer k, whether or not there is a k-dominating set with property Π. Some examples of this type of problems, which are mentioned in [ABF$^+$], [TP1] and [TP2], are the *k-independent dominating set* problem, the *k-total dominating set* problem, the *k-perfect dominating set* problem, the *k-perfect independent dominating set* problem, also known as *k-perfect code*, and the *k-total perfect dominating set* problem. For each Π, we denote the corresponding dominating set problem by $\mathcal{DS}^\Pi$.

Another variant is *the weighted dominating set problem* in which we have a graph $G = (V, E)$ together with an integer weight function $w\colon V \to \mathbb{N}$ with $w(v) > 0$ for all $v \in V$. The weight of a vertex set $D \subseteq V$ is defined as $w(D) = \sum_{v\in D} w(v)$. A k-weighted dominating set D of an undirected graph G is a dominating set D of G with $w(D) \le k$. The *k-weighted dominating set* problem is the task of deciding whether or not there exits a k-weighted dominating set. We use the parameterized class $\mathcal{WDS}$ to denote the k-weighted dominating set problem.

Condition (1) of Theorem 8 holds for $\delta = 4$ because of the following.

THEOREM 9 [ABF$^+$]. *If a tree decomposition of width w of a graph is known*, *then a solution to $\mathcal{DS}^\Pi$ or to $\mathcal{WDS}$ can be determined in at most $O(4^w \cdot n)$ time.*

Clearly, $\mathcal{DS}^\Pi \preccurlyeq_1 \mathcal{DS}$ and $\mathcal{WDS} \preccurlyeq_1 \mathcal{DS}$ and Condition (2) also holds. Therefore Theorem 8 holds for $\gamma = 1$ and $\delta = 4$ for $\mathcal{DS}^\Pi$ and $\mathcal{WDS}$.

Another related problem is the Y-domination problem ($\mathcal{DS}^Y$) introduced in [BBHS].

DEFINITION 5. Let Y be a finite set of integers. A *Y-domination* is an assignment $f\colon V \to Y$ such that for each vertex x, $f(N[x]) = \sum_{v\in N[x]} f(x) \ge 1$ where $N[x]$ stands for the neighborhood of x including x itself. An *efficient Y-domination* is an assignment f with $f(N[x]) = 1$ for all vertices $x \in V$. The *value* of a Y-domination f is $|\{x \mid f(x) > 0\}|$. The weight of a Y-domination is $\sum_{x \in V} f(x)$. Two Y-dominations are *equivalent* if they have the same closed neighborhood sum at every vertex. The *Y-domination problem* asks whether the input graph G has an efficient Y-domination of value at most k.

Using the generalized dynamic programming approach, Kloks and Cai [KC] presents an algorithm which runs in time $O(|Y|^w n)$ to decide whether a graph G of treewidth at most w has an efficient Y-domination of value at most k. It is worth mentioning that, according to Bange et al. [BBHS], a graph G has an efficient Y-domination if and only if all equivalent Y-dominations have the same weight, and thus there is no need to worry about the actual weight of an efficient Y-domination. Therefore, we have that Condition (1) of Theorem 8 holds for $\delta = |Y|$.

One can easily see that for Y-domination f of a graph $G = (V, E)$, $D = \{x | f(x) > 0\}$ is a dominating set, because each vertex x has at least one vertex with a positive number assigned to it in $N[x]$. Thus if f is a Y-domination of G with value at most k, then G also has a dominating set of size k. Therefore, $\mathcal{DS}^Y \preccurlyeq_1 \mathcal{DS}$ and Condition (2) holds as well. Theorem 8 applies for $\gamma = 1$ and $\delta = |Y|$.

5.2. *Vertex Cover.* The k-vertex cover problem ($\mathcal{VC}$) asks whether there exists a subset C of at most k vertices such that every edge of G has at least one endpoint in C. This problem is one of the most popular problems in combinatorial optimization.

A great number of researchers believe that there is no polynomial-time approximation algorithm achieving an approximation factor strictly smaller than $2 - \varepsilon$, for a positive constant ε, unless $P = NP$. Currently, the best known lower bound for this factor is 1.36 [DS] and the best upper bound is 2 which can be obtained easily. The best current fixed-parameter tractable algorithm has time $O(1.271^k + k|V|)$ [CKJ]. In this section we present an exponentially faster algorithm for this problem on clique-sum graphs.

Without loss of generality, we can restrict our attention to graphs with no vertex of degree 0. One can observe that if a graph G has a vertex cover of size k, then it has also a k-dominating set. Therefore $\mathcal{VC} \preccurlyeq_1 \mathcal{DS}$ and Condition (1) of Theorem 8 holds. Moreover, Condition (2) holds because we can solve the vertex cover problem in time $O(2^w)$ if we know the tree decomposition of width w of a graph G [AFN]. Therefore, Theorem 8 applies for $\gamma = 1$ and $\delta = 2$ for the k-vertex problem.

A simple standard reduction to the problem kernel due to Buss and Goldsmith [BG] is as follows: Each vertex of degree greater than k must be in the vertex cover of size k, because otherwise, not all edges can be covered. Thus we can obtain a subgraph G' of G which has at most k^2 edges and at most $k^2 + k$ vertices and k' is obtained from k reduced by the number of vertices of degree more than k. Chen et al. [CKJ] showed that in Buss and Goldsmith's approach one can even obtain a problem kernel with at most $2k$ vertices in $O(nk + k^3)$ time. Thus, using this result with the consequence (2) of Theorem 8 for $\mathcal{VC}$, we obtain the following result.

THEOREM 10. *There exists an algorithm which decides the k-vertex cover problem in $O(2^{9.55\sqrt{k}}k + kn + k^3 + n^{\max\{\alpha,4\}})$ time on an α-recognizable clique-sum graph.*

5.3. *Edge Dominating Set*. Another related problem is the edge dominating set problem $\mathcal{EDS}$ that given a graph G asks for a set $E' \subseteq E$ of k or fewer edges such that every edge in E shares at least one endpoint with some edge in E'. Again without loss of generality we can assume that graph G has no vertex of degree 0.

One can observe that if a graph G has a k-edge dominating set E', we can obtain a vertex cover of size $2k$ by including both endpoints of each edge $e \in E'$. This means that $\mathcal{EDS} \preccurlyeq_2 \mathcal{VC}$. In the previous section we showed that $\mathcal{VC} \preccurlyeq_1 \mathcal{DS}$ therefore, Condition (2) of Theorem 8 holds for $\mathcal{EDS}$ when $\gamma = 2$. Condition (1) holds because the edge dominating set problem can be solved in $c_{\mathsf{eds}}^{w} n$ [Bo1, Bak] (where c_{eds} is a small constant) on a tree decomposition of width w for a graph G. We conclude that Theorem 8 applies for $\gamma = 2$ and $\delta = c_{\mathsf{eds}}$.

THEOREM 11. *We can find a k-edge dominating set in $O(c_{\mathsf{eds}}^{9.55\sqrt{2k}} n + n^{\max\{\alpha,4\}})$ time on an α-recognizable clique-sum graph.*

5.4. *Clique-Transversal Set*. A *clique-transversal set* of a connected graph G is a subset of vertices intersecting all the maximal cliques of G [BNR], [CCCY], [AST], [GR]. Because the vertex cover problem is NP-complete even restricted to triangle-free planar graphs [CKL], [Ue], the clique-transversal problem remains NP-complete on clique-sum graphs. The *k-clique-transversal* problem $\mathcal{CT}$ asks whether the input graph has a clique-transversal set of size $\leq k$.

If a graph G has a k-clique-transversal, then it has a dominating set of size at most k, because every vertex of G is contained in at least one maximal clique. This implies that $\mathcal{CT} \preccurlyeq_1 \mathcal{DS}$ and Condition (2) of Theorem 8 holds for $\gamma = 1$. Using the general dynamic programming technique, we can solve the k-clique-transversal problem on a graph G of treewidth at most w in $O(c_{\mathsf{ct}}^{w} n)$ for some constant c_{ct}. (The approach is very similar to that by Chang et al. [CKL].) Therefore, Theorem 8 applies for $\gamma = 1$ and $\delta = c_{\mathsf{ct}}$.

THEOREM 12. *We can find a k-clique-transversal set in $O(c_{\mathsf{ct}}^{9.55\sqrt{k}} n + n^{\max\{\alpha,4\}})$ time on an α-recognizable clique-sum graph.*

5.5. *Maximal Matching*. A *matching* in a graph G is a set E' of edges without common endpoints. A matching in G is *maximal* if there is no other matching in G containing it. The k-maximal matching problem $\mathcal{MM}$ asks whether an input graph G has a maximal matching of size $\leq k$.

Let E' be the edges of a maximal matching of G. Notice that the set of endpoints of the edges in E' is a dominating set of G. Therefore $\mathcal{MM} \preccurlyeq_2 \mathcal{DS}$ and Condition (2) of Theorem 8 holds. Condition (1) holds because the problem can be solved in $c_{\mathsf{mm}}^{w} n$ [Bo1] on a tree decomposition of width w for a graph G. Hence Theorem 8 gives the following result.

THEOREM 13.

(1) *Any clique-sum graph of base c with a minimum maximal marching of size k has treewidth $\leq 9.55\sqrt{2k} + \max\{8, c\}$.*
(2) *One can decide whether an α-recognizable clique-sum graph G has a minimum maximal matching of size at most k in $O(c_{\mathsf{mm}}^{9.55\sqrt{2k}} n + n^{\max\{\alpha,4\}})$ time.*

5.6. *Kernels in Digraphs*. A set S of vertices in a digraph $D = (V, A)$ is a kernel if S is independent and every vertex in $V - S$ has an out-neighbor in S. It has been shown that the problem of deciding whether a digraph has a kernel is NP-complete [GJ]. Fraenkel [Fr] showed that the kernel problem remains NP-complete even for planar digraphs D with indegree and outdegree at most 2 and total degree at most 3. The k-kernel problem $\mathcal{KER}$ asks whether a graph has a kernel of size k. Moreover, we define the co-$\mathcal{KER}$ problem as the one asking whether an n-vertex graph has a kernel of size $n - k$.

Here, we again observe that if a digraph D has a kernel of size at most k, then its underlying graph G has a dominating set of cardinality at most k. Also for a connected digraph $D = (V, A)$ and kernel K, $V - K$ is a dominating set in the underlying graph of D. Resuming these two facts we have $\mathcal{KER} \preccurlyeq_1 \mathcal{DS}$ and co-$\mathcal{KER} \preccurlyeq_1 \mathcal{DS}$ and Condition (2) of Theorem 8 holds for both problems. We note that a slight variation of Condition (1) also holds because Gutin et al. [GKL] give an $O(3^w kn)$-time algorithm solving the k-kernel problem on graphs of treewidth at most w using the general dynamic programming approach. The straightforward adaptation of Theorem 8 to this variation of Condition (1) gives the following.

THEOREM 14.

(1) *Any clique-sum graph of base c that has a kernel of size k or $n - k$ has treewidth* $\leq 9.55\sqrt{k} + \max\{8, c\}$.
(2) *One can decide whether an α-recognizable clique-sum graph G of base c has a kernel of size k in $O(3^{9.55\sqrt{k}}nk + n^{\max\{\alpha,4\}})$ time.*
(3) *One can decide whether an α-recognizable clique-sum graph G of base c has a kernel of size $n - k$ in $O(3^{9.55\sqrt{k}}n(n - k) + n^{\max\{\alpha,4\}})$ time.*

6. Fixed-Parameter Algorithms for Vertex-Removal Problems. In this section we present general results allowing the construction of $O(c^{\sqrt{k}}n)$-time algorithms for a collection of vertex-removal problems. To this end, we start with some definitions. For any graph class $\mathcal{G}$ and any nonnegative integer k the graph class k-almost($\mathcal{G}$) contains any graph $G = (V, E)$ where there exists a subset $S \subseteq V(G)$ of size at most k such that $G[V - S] \in \mathcal{G}$. We note that using this notation if $\mathcal{G}$ contains all the edgeless graphs or forests then k-almost($\mathcal{G}$) is the class of graphs with vertex cover $\leq k$ or feedback vertex set $\leq k$.

We define $\mathcal{T}_r$ to be the class of graphs with treewidth $\leq r$. It is known that, for $1 \leq i \leq 2$, $\mathcal{T}_i$ is exactly the class of K_{i+2}-minor-free graphs (see, e.g., [Bo5]). We now present a series of consequences of Theorem 3 for solving a collection of vertex-removal problems on classes of graphs excluding a single-crossing graph as a minor. First, we need the following combinatorial lemma.

LEMMA 3. *Planar graphs in k-almost($\mathcal{T}_2$) have treewidth $\leq 9.55\sqrt{k}$. Moreover, such a tree decomposition can be found in $O(n^4)$ time.*

PROOF. Our target is to prove that planar graphs in k-almost($\mathcal{T}_2$) are subgraphs of planar graphs in $\mathcal{DS}_k$ and the result will follow from the fact that from [FT2], Condition (1) of Theorem 2 is also satisfied for $\beta_1 = 9.55$, $\beta_2 = 0$, and $\alpha_2 = 4$.

Let G be a planar graph and let S be a set of $\leq k$ vertices in G where $G[V - S]$ is K_4-minor-free. Using Lemma 1, we can assume that G is a biconnected graph. In addition, because k-almost($\mathcal{T}_2$) is a minor closed class, we can assume that G does not have a 2-cut (a cut of size 2). In fact, if G has a 2-cut $\{u, v\}$, each of the connected components of $G - \{u, v\}$ plus edge $\{u, v\}$ is a minor of G and thus by induction, we can assume that they satisfy the conditions of the theorem. Then using Lemma 1, we can glue the corresponding tree-decompositions together and obtain the desired result for G. All these operations take at most $O(n^3)$ time.

Therefore, in the rest of the proof we assume that G does not have 1- or 2-cuts. A consequence of this is that all the vertices of G have degree at least 3. Another consequence is that two faces of G can have in common either a vertex or an edge (otherwise, a 2-cut appears). Consider any planar embedding of G. We call a face of this embedding *exterior* if it contains a vertex of S, otherwise we call it interior. For each exterior face choose a vertex in S and connect it with the rest of its vertices. We call the resulting graph H and we note that (a) G is a subgraph of H, (b) $H[V - S] = G[V - S]$, and (c) all the vertices of the exterior faces of H are dominated by some vertex in S. We claim that S is a dominating set of H. Suppose, towards a contradiction, that there is a vertex v that is not dominated by S. From (c) we can assume that all of the faces containing v are interior. Let H' be the graph induced by the vertices of these faces. As they are all interior, H' should be a subgraph of $H[V - S]$. Let $(x_1, \ldots, x_q, x_1)$ be a cyclic order of the neighbors of v and notice that $q \geq 3$. Let also F_i be the face of H containing the vertices $x_i, v, x_{\mathsf{next}(i)}$, $1 \leq i \leq q$, where $\mathsf{next}(i) = (i + 1) \bmod q + 1$. We note that all these faces are pairwise distinct otherwise v will be a 1-cut for H and G. Let P_i be the path connecting x_i and $x_{\mathsf{next}(i)}$ in H' avoiding v and containing only vertices of F_i. Recall now that two faces of H have either v or an edge containing v in common. Therefore, it is impossible for two paths $P_i, P_j, i \neq j$, to share an internal vertex. This implies that the contraction of all the edges but one of each of these paths transforms H' to a wheel W_q that, as $q \geq 3$, can be further contracted to a K_4 (a wheel W_q is the graph constructed taking a cycle of length q and connecting all its vertices with a new vertex v). As H' is a subgraph of the graph $H[V - S]$ (b) implies that $G[V - S]$ contains a K_4, and this is a contradiction. As S is now a dominating set for H, the treewidth of H is at most $9.55\sqrt{k}$. From (a) we have that G is a subgraph of a planar graph in $\mathcal{DS}_k$ and this completes the proof of the theorem. $\square$

As mentioned before, k-almost($\mathcal{T}_2$) is a minor closed graph class. Moreover, if $\mathcal{O} \subseteq \mathcal{T}_2$, then for any k, k-almost($\mathcal{O}$) $\subseteq$ k-almost($\mathcal{T}_2$). Using Theorem 3 we now conclude the following general result.

THEOREM 15. *Let $\mathcal{O}$ be any class of graphs with treewidth ≤ 2 and let $\mathcal{G}$ be the class of graphs excluding some single-crossing graph H as a minor. Then the following hold*:

(1) *For any $k \geq 0$, all graphs in k-*almost($\mathcal{O}$) *that also belong to $\mathcal{G}$ have treewidth $\leq \max\{9.55\sqrt{k}, c_H\}$. Moreover, the corresponding tree decomposition can be found in $O(n^4)$ time.*
(2) *Suppose also that there exists an $O(\delta^w n)$ algorithm that decides whether a given graph belongs in k-*almost($\mathcal{O}$) *for graphs of treewidth at most w. Then one can decide whether a graph in $\mathcal{G}$ belongs in k-*almost($\mathcal{O}$) *in $O(\delta^{9.55\sqrt{k}} n + n^4)$ time.*

If $\{O_1, \dots, O_r\}$ is a finite set of graphs, we denote by minor-excl$(O_1, \dots, O_r)$ the class of graphs that are O_i-minor-free for all $i = 1, \dots, r$.

As examples of problems for which Theorem 15 can be applied, we mention the problems of checking whether a graph, after removing k vertices, is *edgeless* ($\mathcal{G} = \mathcal{T}_0$), or has *maximum degree* ≤ 2 ($\mathcal{G} =$ minor-excl$(K_{1,3})$), or becomes a *a star forest* ($\mathcal{G} =$ minor-excl(K_3, P_3)), or a *caterpillar* ($\mathcal{G} =$ minor-excl$(K_3$, subdivision of $K_{1,3})$), or a *forest* ($\mathcal{G} = \mathcal{T}_1$), or *outerplanar* ($\mathcal{G} =$ minor-excl$(K_4, K_{2,3})$), or *series-parallel*, or has treewidth ≤ 2 ($\mathcal{G} = \mathcal{T}_2$).

We consider the cases where $\mathcal{G} = \mathcal{T}_0$ and $\mathcal{G} = \mathcal{T}_1$ in the next two subsections.

6.1. *Feedback Vertex Set*. A *feedback vertex set* (*FVS*) of a graph G is a set U of vertices such that every cycle of G passes through at least one vertex of U. The previous known fixed-parameter algorithms for solving the k-feedback vertex set problem had running time $O((2k+1)^k n^2)$ [DF] and alternatively time $O((917k^4)!\,(n+m))$ [Bo2] (m is the number of edges.) Also there exists a randomized algorithm which needs $O(c4^k kn)$ time with probability at least $1 - (1 - 1/4^k)^{c4^k}$ [BBYG]. The *k-feedback vertex set* problem ($\mathcal{FVS}$) asks whether an input graph has a feedback vertex set of size $\leq k$.

Kloks et al. [KLL] proved that the feedback vertex set problem on planar graphs of treewidth at most w can be solved in $O(c_{\mathsf{fvs}}^w n)$ time for some constant c_{fbs}. The complexity of their algorithm is based on the fact that the number of edges of a planar graph is bounded by a simple linear function of its vertices (i.e., $3n - 6$). As we have a similar bound $3n - 5$ for $K_{3,3}(K_5)$-minor-free graphs [As], [KM], one can easily observe that the algorithm of [KLL] works also for the more general case. Therefore, Theorem 15 can be applied for $\mathcal{G} = \mathcal{T}_1$ and $\delta = c_{\mathsf{fvs}}$ and we have the following.

THEOREM 16. *If $\mathcal{G}$ is a graph class excluding some single-crossing graph H as a minor, then*:

(1) *If G has a feedback vertex set of size at most k then G has treewidth at most* $\max\{9.55\sqrt{k}, c_H\}$.
(2) *We can check whether some n-vertex graph in $\mathcal{G}$ has a feedback vertex set of size $\leq k$ in $O(c_{\mathsf{fvs}}^{9.55\sqrt{k}} n + n^4)$ time.*

Theorem 16 generalizes the results of [KLL] to any class of graphs excluding some single-crossing graph H as a minor.

6.2. *Improving Bounds for Vertex Cover.* Alber et al. [AFN] proved that planar graphs in $\mathcal{VC}_k$ have treewidth at most $4\sqrt{3}\sqrt{k} + 5 < 6.93\sqrt{k} + 5$. An easy improvement of this result is the following:

LEMMA 4. *If a planar graph has a vertex cover of size $\leq k$ then its treewidth is bounded by* $5.52\sqrt{k}$

PROOF. Again using Lemma 1, we may assume that G is a biconnected graph. Let S be a vertex cover in G where $|S| \leq k$. Consider a planar embedding of G.

Construct a triangulation H of G as follows: for any face F we add edges connecting only vertices of $F \cap S$. This operation constructs a triangulation as there is no pair of vertices in $F - S$ that are consecutive in F. Moreover, as all the added edges have endpoints in S, S is a vertex cover of H. We will prove that $\mathrm{tw}(H) \leq 5.52\sqrt{k}$.

We may assume that H is a triangulation without double edges. To see this, consider two edges e_1 and e_2 connecting vertices x and y and apply Lemma 1 on the graphs G_{in} and G_{ex} induced by the vertices included in each of the closed disks bounded by the cycle where the two edges of this cycle are identified.

Notice now that for each vertex $v \in V(H) - S$, all its neighbors are members of S. This means that $|V(H) - S| \leq r$ where r is the number of faces of $J = H[S]$. As H has no double edges, neither does J and therefore $|E(J)| \leq 3|V(J)| - 6$. It is known that $r \leq |E(J)| - |V(J)| + 2$ and we get that $r \leq 2|V(J)| - 4$. We conclude that $|V(H)| \leq |V(J)| + 2|V(J)| - 4 = 3|S| - 4 = 3k - 4$. From [FT1], we know that any n-vertex planar graph has treewidth at most $(9/2\sqrt{2})\sqrt{n}$. This means that $\mathrm{tw}(H) \leq (9/2\sqrt{2})\sqrt{3k-4} \leq (9/2\sqrt{2})\sqrt{3}\sqrt{k}$. As G is a subgraph of H and $(9/2\sqrt{2})\sqrt{3} < 5.52$, the result follows. $\square$

Applying Theorem 15, we have that Condition (1) of Theorem 4 holds if $\mathcal{F}$ is the class of graphs with vertex cover $\leq k$ and $\mathcal{G}$ is any graph class excluding some single-crossing graph H as a minor when $c = c_H$, $\alpha_1 = 4$, $\alpha_2 = 4$, $\beta_1 = 5.52$, and $\beta_2 = 0$. Also, as we mentioned in Section 5.2 it is possible to decide in $O(2^w n)$ time if a graph has a vertex cover of size at most k. Therefore, Condition (2) holds for $\delta = 2$. Concluding, we have the following improvement of the results of Section 5.2 for any graph class excluding some single-crossing graph H as a minor.

THEOREM 17. *If $\mathcal{G}$ is some graph class excluding some single-crossing graph H as a minor then the following hold*:

(1) *If $G \in \mathcal{G}$ has a vertex cover of size at most k then G has treewidth at most* $\max\{5.52\sqrt{k}, c_H\}$.
(2) *There is an algorithm which checks whether some graph $G \in \mathcal{G}$ has a vertex cover of size $\leq k$ in $O(2^{5.52\sqrt{k}}k + kn + k^3 + n^4)$ time.*

Because $\mathcal{EDS} \preccurlyeq_2 \mathcal{VC}$, we can also obtain an $O({c_{\mathsf{eds}}}^{5.52\sqrt{2k}}n + n^4)$-time algorithm for the edge dominating set problem on graphs excluding some single-crossing graph as a minor.

7. Further Extensions. In this section we obtain fixed-parameter algorithms with exponential speedup for k-vertex cover and k-edge dominating set on classes of graphs that are not necessarily classifiable as single-crossing minor-free graphs. Our approach, similar to the Alber et al.'s approach [AFN], is a general one that can be applied to other problems.

Baker [Bak] developed several approximation algorithms to solve NP-complete problems for planar graphs. To extend these algorithms to other graph families, Eppstein [Ep] introduced the notion of bounded local treewidth, defined formally below, which is a generalization of the notion of treewidth. Intuitively, a graph has bounded local treewidth (or locally bounded treewidth) if the treewidth of an r-neighborhood of each vertex $v \in V(G)$ is a function of r, $r \in \mathbb{N}$, and not $|V(G)|$.

DEFINITION 6. The *local treewidth* of a graph G is the function $\mathrm{ltw}^G\colon \mathbb{N} \to \mathbb{N}$ that associates with every $r \in \mathbb{N}$ the maximum treewidth of an r-neighborhood in G. We set $\mathrm{ltw}^G(r) = \max_{v \in V(G)}\{\mathrm{tw}(G[N_G^r(v)])\}$, and we say that a graph class $\mathcal{C}$ has *bounded local treewidth* (*or locally bounded treewidth*) when there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $G \in \mathcal{C}$ and $r \in \mathbb{N}$, $\mathrm{ltw}^G(r) \leq f(r)$.

A graph is called an *apex graph* if deleting one vertex produces a planar graph. Eppstein [Ep] showed that a minor-closed graph class $\mathcal{E}$ has bounded local treewidth if and only if $\mathcal{E}$ is H-minor-free for some apex graph H.

So far, the only graph classes studied with small local treewidth are the class of planar graphs and more generally bounded genus graphs [Ep], the class of almost-embeddable graphs [Gr], and finally the class of clique-sum graphs [HNRT], [DHN$^+$]. All such graph classes have linear local treewidth with small hidden constants. For example, for any planar graph G, $\mathrm{ltw}^G(k) \leq 3k - 1$ [Ep], and for any $K_{3,3}$-minor-free or K_5-minor-free graph G, $\mathrm{ltw}^G(k) \leq 3k + 4$ [HNRT], [DHN$^+$]. For these classes of graphs, there are efficient algorithms for constructing tree decompositions.

Eppstein [Ep] showed how the concept of the kth outer face in planar graphs can be replaced by the concept of the kth layer (or level) in graphs of locally bounded treewidth. The kth layer (L_k) of a graph G consists of all vertices at distance k from an arbitrary fixed vertex v of $V(G)$. We denote *consecutive layers from* i *to* j by $L[i, j] = \bigcup_{i \leq k \leq j} L_k$.

Here we generalize the concept of *layerwise separation*, introduced by Alber et al. [AFN] for planar graphs, to general graphs.

DEFINITION 7. Let G be a graph layered from a vertex v, and let r be the number of layers. A *layerwise separation of width* w *and* size s for G is a sequence $(S_1, S_2, \ldots, S_r)$ of subsets of V, with the property that $S_i \subseteq \bigcup_{j=i}^{i+(w-1)} L_j$; S_i separates layers L_{i-1} and L_{i+w}; and $\sum_{j=1}^{r} |S_j| \leq s$. Here we let $S_i = \emptyset$ for all $i < 1$ and $i > r$.

Now we relate the concept of layerwise separation to parameterized problems.

DEFINITION 8. A parameterized problem P has the *Layerwise Separation Property* (*LSP*) of width w and size-factor d, if for each instance (G, k) of the problem P, graph G admits a layerwise separation of width w and size dk.

For example, we can obtain constants $w = 2$ and $d = 2$ for the vertex cover problem. In fact, consider a k-vertex cover C on a graph G and set $S_i = (L_i \cup L_{i+1}) \cap C$. The S_i's form a layerwise separation. Similarly, we can get constants $w = 2$ and $d = 4$ for the edge dominating set problem (see [AFN] for further examples).

LEMMA 5. *Let P be a parameterized problem on instance (G, k) that admits a problem kernel of size dk. Then the parameterized problem P on the problem kernel has the LSP of width* 1 *and size-factor d.*

PROOF. Consider the problem kernel (G', k') for an instance (G, k) and obtain layering L' for G' from arbitrary vertex v. Then clearly the sequence $S_i = L'_i$ for $i = 1, \dots, r'$ (r' is the number of layers) is a layerwise separation of width 1 and size $k' \leq dk$ for G'. □

In fact, using Lemma 5 and the problem kernel of size $2k$ (see Section 5.2) for the vertex cover problem, this problem has the LSP of width 1 and size-factor 2.

Now we are ready to present the main theorem of this section.

THEOREM 18. *Suppose for a graph G from a minor-closed class of graphs*, $\operatorname{ltw}(G) \leq cr + c'$ *and a tree decomposition of width $ch + c'$ can be constructed in time $f(n, h)$ for any h consecutive layers. Also assume G admits a layerwise separation of width w and size dk. Then we have* $\operatorname{tw}(G) \leq 2\sqrt{2cdk} + cw + c'$. *Such a tree decomposition can be computed in time $O(kf(n, \sqrt{k}))$.*

PROOF. The proof is very similar to the proof of Theorem 15 of Alber et al.'s work [AFN] and for the sake of brevity we only mention the differences and omit the lengthy details. In the proof the concept of the kth outer face in planar graphs will be replaced by the concept of the kth layer (or level) in graphs of locally bounded treewidth. More precisely, Alber et al. [AFN] use the fact that the treewidth of an h-outerplanar graph is $3h - 1$, but in our proof we use the fact that, for any graph G, the treewidth of any h consecutive layers is at most $ch + c'$ [Gr], [DHN$^+$]. In addition, as mentioned before, Eppstein [Ep] showed that a minor-closed graph class $\mathcal{E}$ has bounded local treewidth if and only if $\mathcal{E}$ is H-minor-free for some apex graph H. (A simpler proof of this theorem can be found in [DH2].) From Thomason [Th], we know that any graph G excluding an r-clique as a minor cannot have more than $(0.319 + o(1))(r\sqrt{\log r})|V(G)|$ edges. This implies that for graph G mentioned in the statement of the theorem, $|E(G)| = O(|V(G)|)$, similar to the corresponding relation for planar graphs. This fact is used for analyzing the running time. □

COROLLARY 3. *For any H-minor-free graph G, where H is a single-crossing graph, that admits a layerwise separation of width w and size dk, we have* $\operatorname{tw}(G) \leq 2\sqrt{6dk} + 3w + c_H$. *Such a tree decomposition can be computed in time $O(k^{5/2}n + kn^4)$. Furthermore, for any $K_{3,3}(K_5)$-minor-free graph G that admits a layerwise separation of width w and size dk, we have* $\operatorname{tw}(G) \leq 2\sqrt{6dk} + 3w + 4$. *Such a tree decomposition can be computed in time $O(k^{5/2}n)$ $(O(k^{5/2}n + kn^2))$.*

PROOF. The proof follows directly from Theorem 18 and the fact that for any single-crossing-minor-free graph G, we can construct a tree decomposition of width $3h + c_h$ for any h consecutive layers in $O(h^3 \cdot n + n^4)$ time; for a $K_{3,3}$-minor-free or K_5-minor-free graph G, the running time can be reduced to $O(h^3 n)$ or $O(h^3 n + n^2)$, respectively [Ha], [DHN$^+$] ($c_H = 4$ for these graphs.) □

Finally, we have this general theorem.

THEOREM 19. *Suppose for a graph G from a minor-closed class of graphs*, $\mathrm{ltw}(G) \leq cr + c'$. *Let P be a parameterized problem on G such that P has the LSP of width w and size-factor d and there exists an $O(\delta^w n)$-time algorithm*, *given a tree decomposition of width w for G*, *which decides whether problem P has a solution of size k on G. Then there exists an algorithm which decides whether P has a solution of size k on G in time* $O(2^{(11/3)(2\sqrt{2cdk}+cw+c')}n^{3.01} + \delta^{3.698(2\sqrt{2cdk}+cw+c')}n)$.

PROOF. The proof follows from Theorem 18, the fact that for graph G, the treewidth of any h consecutive layers is at most $ch + c'$ [Gr], [DHN$^+$], and finally the result of Amir [Am], which says for any graph G of treewidth w, we can construct a tree decomposition of width at most $(11/3)w$ in time $O(2^{3.698w}n^{3.01})$. □

For example, Theorem 19 gives an exponential speed up, i.e., an algorithm with running time $O(2^{O(\sqrt{gk})}k^{3.01} + kn + k^3 + n^4)$ (because $c = O(g)$ [Ep]), for solving vertex cover on graphs of bounded genus.

Recently, it was established that all minor-closed classes of graphs with bounded local treewidth, i.e., all minor-closed graph classes excluding an apex graph, in fact have linear local treewidth [DH1]. Therefore Theorem 19 applies generally to any such class of graphs.

8. Conclusions and Future Work. In this paper we considered H-minor-free graphs, where H is a single-crossing graph, and proved that if these graphs have a k-dominating set then their treewidth is at most $c\sqrt{k}$ for a small constant c. As a consequence, we obtained exponential speedup in designing FPT algorithms for several NP-hard problems on these graphs, especially $K_{3,3}$-minor-free or K_5-minor-free graphs. In fact, our approach is a general one that can be applied to several problems which can be reduced to the dominating set problem as discussed in Section 5 or to problems that themselves can be solved exponentially faster on planar graphs [AFN]. Here, we present several open problems that are possible extensions of this paper.

One topic of interest is finding other problems to which the technique of this paper can be applied. Moreover, it would be interesting to find other classes of graphs than H-minor-free graphs, where H is a single-crossing graph, on which the problems can be solved exponentially faster for parameter k. A partial answer to this question is the class of map graphs [DFHT1].

For several problems in this paper, Kloks et al. [CKL], [KLL], [GKL], [KC] introduced a reduction to the problem kernel on planar graphs. Because graphs excluding a single-crossing graph are similar to planar graphs, in the sense of having a linear number of edges and not having a clique of more than a constant size, we believe that one might obtain similar results for these graphs.

As mentioned before, Theorem 15 holds for any class of graphs with treewidth ≤ 2. It is an open problem whether it is possible to generalize it to apply to any class of graphs of treewidth $\leq h$ for arbitrary fixed h. Moreover, there exists no general method for designing $O(\delta^w n)$-time algorithms for vertex-removal problems in graphs with treewidth

$\leq w$. If this becomes possible, then Theorem 17 will have considerable algorithmic applications.

Finally, as a matter of practical importance, it would be interesting to obtain a constant coefficient better than 9.55 for the treewidth of planar graphs having a k-dominating set, which would lead to a direct improvement on our results.

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