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Constructive Linear Time Algorithms For Branchwidth*

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Abstract

We prove that, for any fixed k, one can construct a linear time algorithm that checks if a graph has branchwidth$\leq k$ and, if so, outputs a branch decomposition of minimum width.

1 Introduction

This paper considers the problem of finding branch decompositions of graphs with small branchwidth. The notion of branchwidth has a close relationship to the more well-known notion of treewidth, a notion that has come to play a large role in many recent investigations in algorithmic graph theory. (See Section 2 for definitions of treewidth and branchwidth.) One reason for the interest in this notion is that many graph problems can be solved by linear time algorithms, when the inputs are restricted to graphs with some uniform upper bound on their treewidth. Most of these algorithms first try to find a tree decomposition of small width, and then utilize the advantages of the tree structure of the decomposition (see [1], [4]).

The branchwidth of a graph differs from its treewidth by at most a multiplicative constant factor (see [18]). As branchwidth is also reflecting some optimal tree structure arrangement, it is possible to have algorithmic applications analogous to those of treewidth. Hence, instead of using tree decompositions, one also

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can use branch decompositions as starting point for the linear time algorithms for problems restricted to graphs with bounded treewidth (and hence also bounded branchwidth). In fact, in some cases, it appears that branchwidth is more convenient to use, and seems to give better constant factors in the implementation of the algorithms; for instance, Cook used branch decompositions as an important ingredient in a practical approximation algorithm for the Traveling Salesman Problem [10], and remarked that branchwidth was the more natural notion (instead of treewidth) to use for that problem [9]: where tree decompositions primarily are concerned with vertices, branch decompositions deal more with edges (in a loose sense). We also mention that the branchwidth of planar graphs can be computed in polynomial time (see [20]). As both treewidth and branchwidth are NP-complete parameters (see [2, 20]), it appears an interesting task to find algorithms solving the following problems (k is assumed to be a fixed constant).

- $\Pi_k^d(B)$ ($\Pi_k^d(T)$): Check if an input graph has branchwidth (treewidth) at most k.
- $\Pi_k^c(B)$ ($\Pi_k^c(T)$): Given a graph with branchwidth (treewidth) at most k, output a minimum width branch (tree) decomposition.

According to the results of Robertson and Seymour, for any minor closed class of graphs there exist a finite set of graphs, its *obstruction set*, such that a graph G belongs in the class iff no element of the obstruction set is a minor of G (see e.g. [17]). It is also known that for, any k, the class of graphs where treewidth (or branchwidth) is bounded by a fixed k is minor closed. An immediate consequence of this fact (using results from Robertson and Seymour and the algorithm from [5]) is the existence of a linear time algorithm solving $\Pi_k^d(B)$ or $\Pi_k^d(T)$. Unfortunately, in this way, we only get a non-constructive proof (see [11, 12]) of the existence of such an algorithm, but in order to construct the algorithm, we must know the corresponding obstruction set. Additionally, we would like to have an algorithm that non only decides on branchwidth, but also constructs corresponding branch decompositions.

Much research has been done towards the construction of linear time algorithms solving $\Pi_k^d(T)$ and $\Pi_k^c(T)$. In [5], a linear (on the size of the input) time algorithm for treewidth was constructed. For further results concerning on related graph theoretic parameters see [3, 7, 8, 13, 16, 14, 15, 19, 21, 19, 22, 23, 24].

In this paper, we find analogous results to those of [5] for the parameter of branchwidth. Namely, we prove that, for any fixed k, one can construct a linear time algorithm that solves $\Pi_k^d(B)$ and $\Pi_k^c(B)$. An immediate consequence of this result is that, for any fixed k, one can construct (i) a sentence in monadic second order logic expressing whether a graph has branchwidth at most k or not and (ii) the obstruction set of the class of graphs of branchwidth at most k.

2 Definitions and Preliminary Results

Given a graph $G = (V, E)$ we denote its vertex set V and edge set E with $V(G)$ and $E(G)$ respectively. Given two graphs $G_i, i = 1, 2$ we define $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. For any vertex set $S \in V(G)$, we define as $G[S]$ the subgraph of G induced by S. We also denote as $N_G(S)$ the set of vertices in $V(G)$ adjacent with vertices of S in G. If $v \in V(G)$, we set $N_G(v) = N_G(\{v\})$. The degree $d_G(v)$ (or simply $d(v)$ when there is no doubt about G) of a vertex in $V(G)$ is the cardinality of $N_G(v)$. We denote as K_r the complete graph with r vertices and as $K_{q,r}$ the complete bipartite graph with parts consisting of q and r vertices each. We call a vertex in $V(G)$ *pendant* (*isolated*) if $d_G(v) = 1$ ($d_G(v) = 0$). We call the pendant vertices of a tree *leaves*. We also call an edge in G *pendant* if it contains at least one pendant vertex. We denote as $A(G)$ ($I(G)$) the set of all the pendant (isolated) vertices of a graph G. Given a tree T and a vertex t we define $\mathcal{C}(T, t) = \{T[V(T_1) \cup \{t\}], \ldots, T[V(T_r) \cup \{t\}]\}$ where $T_1, \ldots, T_r$ are the connected components of $T[V(T) - \{t\}]$. Given now an edge $\{t_1, t_2\} \in E(T)$ we define $\mathcal{C}(T, t_1, t_2) = (T[V(T_1') \cup \{t_2\}], T[V(T_1') \cup \{t_2\}])$ where for $i = 1, 2$ T_i' is the graph in $\mathcal{C}(T, t_{3-i})$ that contains t_i as a vertex. Given two vertices $x_1, x_2 \in V(T)$ then we define $T(x_1, x_2) = T[V(T_1'') \cap T(T_2'')]$ where, for $i = 1, 2$, T_i'' is the graph in $\mathcal{C}(T, x_i)$ that contains x_{3-i} as a vertex then. Finally given a tree T and a set of leaves $A \subseteq A(T)$ we define the *subtree of T spanned by A* as the subtree of T that contains A as leaves and the minimum number of edges. We call a tree *caterpilar* if it contains a path whose neighborhood includes its vertex set. We call the edges of this path *ridge edges*.

Let T_1, T_2 be trees. We call the set of isomorphisms between T_1 and T_2 as $\mathcal{I}(T_1, T_2)$.

Let $f : A \to B$ be a function and $A' \subseteq A$. We denote its domain as $\mathcal{D}(f)$. If $S \in B$ then we set $f - S = \{(a, f(a) - S) \mid a \in A\}$. We denote as $f|_{A'}$ the restriction of f to the pairs whose first elements belongs to A' i.e. $f|_{A'} = \{(a, b) \in f \mid a \in A'\}$. If $B' \subseteq B$ we define $f \bar{\cap} B' = \{(x, \phi(x) \cap B') \mid x \in A\}$ We also define $\max(f) = \max\{f(x) \mid x \in B\}$.

If S is a collection of objects for which operation "$\cup$" is defined then we define $\cup S = \cup_{s \in S} s$.

Let $f_i : A_i \to B, i = 1, 2$ be two functions and let $\sigma : A_1 \to A_2$ be a bijection between A_1 and A_2. Let also A be a set and $\rho : A \to A_1$ a bijection between A and A_1. Then we define $f_1 \cup_\sigma f_2 : A \to B$ such that $\forall_{x \in A}(f_1 \cup_\sigma f_2)(x) = f_1(\rho(x)) \cup f_2(\sigma(\rho(x)))$. We define $f_1 \cap_\sigma f_2 : A \to B$ by replacing $\cup_\sigma$ with $\cap_\sigma$ in the aforementioned definition. If "$\star$" is a relation between the elements of B then we define $f_1 \star_\sigma f_2$ iff $\forall_{x \in A_1} f_1(x) \star f_2(\sigma(x))$.

Given two graphs G, H we say that H is a *minor* of G (denoted by $H \leq G$) if H can be obtained by a series of the following operations: vertex deletions, edge deletions, and edge contractions (a contraction of an edge $\{u, v\}$ in G is the operation that replaces u and v by a new vertex whose neighbors are the vertices that where adjacent to u and/or v). Let $\mathcal{G}$ be a class of graphs. We say that $\mathcal{G}$ is *closed under taking of minors* when all minors of any graph in $\mathcal{G}$ belong also in $\mathcal{G}$. Robertson and Seymour proved (see e.g. [17]) that any class of graphs $\mathcal{G}$ contains a finite set of minor minimal elements. We call such a set the *obstruction set* of $\mathcal{G}$.

It follows that if $\mathcal{G}$ is closed under taking of minors, then, for any graph H, $G \in \mathcal{G}$ iff there is no graph in the obstruction set of $\mathcal{G}$ such that $H \leq G$.

2.1 Treewidth

We give now the formal definitions of treewidth and branchwidth.

A *tree decomposition* of a graph G is a pair $(X, U) = (\{X_i \mid i \in V(U)\}, U)$, where $\{X_i \mid i \in I\}$ is a collection of subsets of V and U is a tree, such that

- $\bigcup_{i \in I} X_i = V(G)$,
- for each edge $\{v, w\} \in E(G)$, there is an $i \in I$ such that $v, w \in X_i$, and
- for each $v \in V$ the set of nodes $\{i \mid v \in X_i\}$ forms a subtree of U.

The *width* of a tree decomposition $(\{X_i \mid i \in I\}, U = (I, F))$ equals $\max_{i \in I}\{|X_i| - 1\}$. The *treewidth* of a graph G is the minimum width over all tree decompositions of G.

A *rooted* tree decomposition is a tree decomposition $D = (X, U)$ in which U is a rooted tree. Let $D = (X, U)$ be a rooted tree decomposition of a graph G For each node i of T, let U_i be the subtree of U, rooted at node i. We set $V_i = \cup_{v \in V(U_i)} X_v$ and let $G_i = G[V_i]$. Notice that if r is the root of U, then $G_r = G$. We call G_i the subgraph of G *rooted* at i. We finally set, for any $i \in V(U)$, $D_i = (X^i, U_i)$

where $X^i = \{X_v \mid v \in V(U_i)\}$. Observe that for each node $i \in V(U)$, D_i is a tree decomposition of G_i.

Let $D = (X, U)$ be a tree decomposition of a graph G where $X = \{X_i \mid i \in V(U)\}$. D is called a *nice* tree decomposition if the following are satisfied:

1. Every node of U has at most two children.

2. If a node i has two children j, h then $X_i = X_j = X_k$.

3. If a node i has one child, then either $|X_i| = |X_j| + 1$ and $X_j \subset X_i$ or $|X_i| = |X_j| - 1$ and $X_i \subset X_j$.

Lemma 1 *For any constant $k \geq 1$, given a tree decomposition of a graph G of width $\leq k$ and $O(|V(G)|$ nodes, there exist an algorithm that, in $O(|V(G)|)$ time, finds a nice tree decomposition of G of width $\leq k$ and with at most $4|V(G)|$ nodes.*

We now observe that a nice tree decomposition $(\{X_i \mid i \in V(U)\}, U)$ contains nodes of the following four possible types. A node $i \in V(U)$ is called

"**start**" if $i \in A(U)$,

"**join**" if i has two children,

"**forget**" if i has one child j and $|X_i| < |X_j|$,

"**introduce**" if i has one child j and $|X_i| > |X_j|$,

We may also assume that if i is a **start** node then $|X_i| = 1$: the effect of **start** nodes with $|X_i| > 1$ can be obtained by using a **start** node with an one vertex set, and then $|X_i| - 1$ **introduce** nodes, which add all the other vertices.

2.2 branchwidth

A *branch decomposition* B of a graph G is a pair (T, θ), where T is a ternary tree (a tree with vertices of degree 1 or 3) and θ is a bijection from the set of leaves of T to $E(G)$. We define the description $\mathrm{Des}(B) = (T, \alpha, \beta, \gamma)$ of B as a quadruple where

- $\alpha : E(T) \to 2^{V(G)-A(G)-I(G)}$ is a function such that $\alpha(e)$ is the set of vertices $v \in V(G)$ for which there are leaves t_1, t_2 in T in different components of $T(V(T), E(T) - \{e\})$ with $\theta(t_1)$ and $\theta(t_2)$ both containing v.

- $\beta : A(T) \to 2^{A(G)}$ is a function such that $\forall_{v \in A(T)} \beta(v) = \theta(v) \cap A(G)$.

- $\gamma : E(T) \to \mathcal{S}$ such that each edge $e \in E(T)$ is mapped through γ to the sequence of integers $(|\alpha(e)|)$ (notice that $\forall_{e \in E(T)}\ \gamma(e)$ is sequence consisting of only one number – this somehow overloaded definition will be justified by considerations to be made later in this paper).

The *width* of (T, θ) is equal to $\max(\alpha)$. edges of T. The *branchwidth* of G is the minimum width over all the branch decompositions of G (in case where $|E(G)| \leq 1$, then we define the branchwidth to be 0; if $|E(G)| = 0$, then G has no branch decomposition; if $|E(G)| = 1$, then G has a branch decomposition consisting of a tree with one vertex – the width of this branch decomposition is considered to be 0).

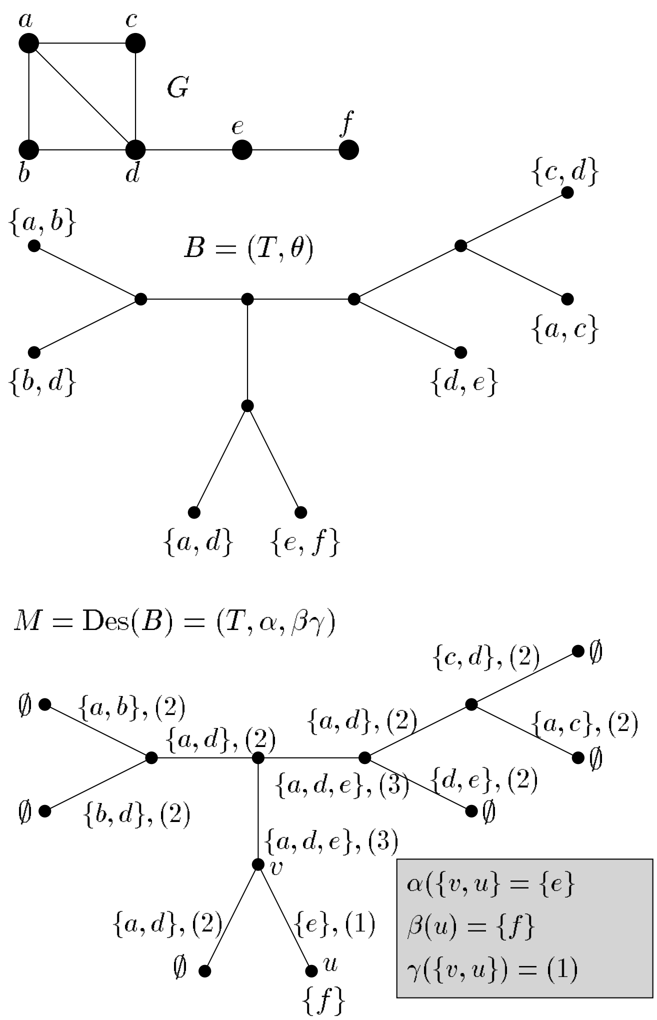


Figure 1: A branch decomposition $B = (T, \theta)$ of a graph G and its description $M = \mathrm{Des}(B) = (T, \alpha, \beta, \gamma)$.

We modify the definition of θ above so that it is a function $\theta : A \to E(G) \cup \{\emptyset\}$, mapping only some leaves $A \subseteq A(G)$ to either an edge or to the empty set and having the property that for every edge e in G there is a unique leaf $t \in A(T)$ that is mapped to e through θ. Then we call (T, θ) *extended branch decomposition*. One can easily observe that for any ternary subtree T' of T the pair (T', θ') where $\theta' = \theta|_{A(T') \cap A(T)}$ is an extended branch decomposition of $(V(G), \{\theta(t) : t \in A(T') \cap A(T)\})$ (clearly, the leaves of T' that are not leaves of T are not mapped to any edge through θ'). It is easy to see that any extended branch decomposition (T, θ) of a graph G can be transformed to an branch decomposition in $O(|V(T)|)$ time. This gives us the right, from now on, whenever we refer to branch decompositions we will assume that they are extended. Moreover, given any branch decomposition (T, θ) we will denote as $\tilde{A}(T)$ the leaves of T that are mapped through θ to some edge of G or to the empty set. We call the leaves in $\tilde{A}(T)$ *external leaves* of T and the leaves in $A(T) - \tilde{A}(T)$ *internal leaves* of T. For reasons of consistency with this modification, we revise the definition of $\mathrm{Des}(B)$ above by putting $\tilde{A}(G)$ instead of $A(G)$ so that $\theta(v)$ is well defined.

2.3 Sequences of integers

We denote as $\mathcal{S}$ the set of all the sequences of positive integers. If $A = (a_1, \ldots, a_{|A|}) \in \mathcal{S}$ and $1 \leq k \leq l \leq |A|$, we define $A_{k,l} = (a_k, \ldots, a_l)$, $\max(A) = \max_{1 \leq i \leq |A|}\{a_i\}$ and for any positive integer t we set $A + t = (a_1 + t, \ldots, a_{|A|} + t)$. The *typical sequence* $\tau(A)$ of a sequence of integers A is the sequence obtained after iterating the following operations, until none is possible any more.

(i) If for some $i, 1 \leq i \leq |A| - 1$ $a_i = a_{i+1}$, then set $A = (a_1, \ldots, a_{|A|}) \leftarrow (a_1, \ldots, a_i, a_{i+2}, \ldots, a_{|A|})$.
(ii) If the sequence contains two elements a_i and a_j such that $j - i \geq 2$ and $\forall_{i<k<j}\ a_i \leq a_k \leq a_j$ or $\forall_{i<k<j}\ a_i \geq a_k \geq a_j$, then set $A = (a_1, \ldots, a_{|A|}) \leftarrow (a_1, a_i, a_j, \ldots, a_{|A|})$.

As an example we mention that if $A = (5, 5, 6, 7, 7, 7, 4, 4, 3, 5, 4, 6, 8, 2, 9, 3, 4, 6, 7, 2, 7, 5, 4, 4, 6, 4)$, then $\tau(A) = (5, 7, 3, 8, 2, 9, 2, 7, 4)$. We call a sequence A *typical* if $\tau(A) = A$ i.e. it is not possible to apply (i) or (ii) on A. We denote the set of all the typical sequnces as $\hat{\mathcal{S}}$.

Let $A, B \in \mathcal{S}$. We say that $A \sqsubset B$ when A can be obtained from B after a series of operations (i) and (ii) above.

Let $A, B \in \mathcal{S}$ where $A = (a_1, \ldots, a_{|A|})$ and $B = (b_1, \ldots, b_{|B|})$. If $|A| = |B|$ then we say that $A \leq B$ if $\forall_{1 \leq i \leq |A|}\ a_i \leq b_i$. We define the set of extensions of A as

$$e(A) = \{A^* = (a_1^*, \ldots, a_{A^*}^*) \mid \exists_{1=t_1<\ldots<t_{|A|+1}}\ \forall_{1\leq i \leq |A|}\ \forall_{t_i \leq k < t_{i+1}}\ a_k^* = a_i\}.$$

We say that $A \prec B$ if there exist extensions $A^* \in e(A), B^* \in e(B)$ such that $|A^*| = |B^*|$ and $A^* \leq B^*$. For example if $A = (5, 7, 4, 8)$ and $B = (1, 7, 2, 6, 4)$ then $B \prec A$ because $A^* = (5, 7, 7, 7, 4, 8, 8, 8, 8)$ is an extension of A, $B^* = (1, 7, 2, 6, 4, 4, 4, 4, 4)$ is an extension of A, and $B^* \leq A^*$. We also say that $A \equiv B$ when $\tau(A) = \tau(B)$.

Given $A = (a_1, \ldots, a_{|A|})$ and $B = (b_1, \ldots, b_{|B|})$ we set $A \oplus B = (a_1, \ldots, a_{|A|}, b_1, \ldots, b_{|B|})$. We also say that $A \prec\!\!\sqsubset B$ if there exist a $C \in \mathcal{S}$ such that $A \prec C$ and $C \sqsubset B$. The proof of the following Lemma can be found in [6] (Lemma 3.19).

Lemma 2 *Let $A, A', B, B' \in \hat{\mathcal{S}}$ such that $A \prec A'$ and $B \prec B'$. Then $\tau(A \oplus B) \prec \tau(A' \oplus B')$.*

We also present, without proof, the two following easy Lemmata.

Lemma 3 *Let $A, B_i \in \hat{\mathcal{S}}, i = 1, \ldots, q$ such that $A \sqsubset B_1 \oplus \cdots \oplus B_q$ ($A \prec\!\!\sqsubset B_1 \oplus \cdots \oplus B_q$). Then $\forall_{1\leq m\leq |A|} \exists_{1\leq r \leq q}\ A_{1,m} \sqsubset B_1 \oplus \cdots \oplus B_r \wedge A_{m,|A|} \sqsubset B_r \oplus \cdots \oplus B_q$ ($\forall_{1\leq m\leq |A|} \exists_{1\leq r \leq q}\ A_{1,m} \prec\!\!\sqsubset B_1 \oplus \cdots \oplus B_r \wedge A_{m,|A|} \prec\!\!\sqsubset B_r \oplus \cdots \oplus B_q$).*

Lemma 4 *Let $A, B, C, C' \in \hat{\mathcal{S}}$ four typical sequences such that $C = \tau(A \oplus B)$ and $C \prec C'$. Then one can construct two sequences $A', B' \in \hat{\mathcal{S}}$ such that $C' = \tau(A' \oplus B')$, $A \prec A'$ and $B \prec B'$.*

Lemma 5 *Let $A_1, \ldots, A_r, B \in \hat{\mathcal{S}}$ such that $A_1 \oplus \cdots \oplus A_r \prec B$. Then there exist a sequence $B_1, \ldots, B_r$ of typical sequences such that $B_1 \oplus \cdots \oplus B_r \equiv B$ and $\forall_{1\leq i \leq r}\ A_i \prec B_i$.*

Let $A = (a_1, \ldots, a_{|\mathbf{A}|})$ and $B = (b_1, \ldots, b_{|\mathbf{B}|})$ be two sequences in $\mathcal{S}$ where $|A| = |B|$. We say that $A \sim B$ iff $\forall_{1\leq i\leq |A|}\ a_i \neq a_{i+1} \Leftrightarrow b_i = b_{i+1}$ (and, therefore, $b_i \neq b_{i+1} \Leftrightarrow a_i = a_{i+1}$).

Let now $A, B \in \hat{\mathcal{S}}$. The *interleaving* $A \otimes B$ of A and B is a set of sequences in $\mathcal{S}$ defined as follows

$$A \otimes B = \{A^* + B^* \mid A^* \in E(A) \text{ and } B^* \in E(B) \text{ and } |A^*| \sim |B^*|\}.$$

Observe that all the sequences in $A \otimes B$ have length $|A| + |B| - 1$.

Let $A, B \in \mathcal{S}$ where $A = (a_1, \dots, a_{|A|}), B = (b_1, \dots, b_{|B|})$ and such that $|A| = |B| = r$. We define $A + B = (a_1 + b_1, \dots, a_r + b_r)$. We will need the following lemmata.

Lemma 6 *Let A, B, C be sequences such that $|B| = |C|$ and $A = B + C$. Then there exist a sequence $A' \in \tau(B) \otimes \tau(C)$ such that $\tau(A') \prec \tau(A)$.*

Lemma 7 *Let A, B be two typical sequence and C a sequence such that $C \in A \otimes B$. Suppose also that A', B' be two typical sequence such that $A \prec A'$ and $B \prec B'$. Then there exist a sequence $C' \in A' \otimes B'$ such that $C \prec C'$.*

2.4 Branch representations and branch Models

Let G be a graph and $R \subseteq V(G)$. Let also $M = (T, \alpha, \beta, \gamma)$ be a quadruple where T is a ternary tree, and α, β, γ be functions where α maps each edge of T to a subset of $R - A(G) - I(G)$, β maps each vertex of a set $A \subseteq A(T)$ to a subset of $R \cap A(G)$ and with the property that $\forall_{t_1, t_2 \in A, t_1 \neq t_2}\ \beta(t_1) \cap \beta(t_2) = \emptyset$, and γ maps each edge of T to a sequence in $\mathcal{S}$. We call $M = (T, \alpha, \beta, \gamma)$ *branch representation of G rooted on R* and the tree T *underlining tree* of the branch representation M. Given a branch representation M, we denote its underlining tree as $T(M)$. In accordance to the definition of β in subsection 2.2, we denote the subset A used in the definition of β above as $\tilde{A}(T)$ and we call its vertices *external leaves* of T (if a leaf of T is not external, we call it *internal*). We define $V(M) = (\cup_{e \in E(T)}\ \alpha(e)) \bigcup (\cup_{v \in \tilde{A}(T)}\ \beta(v))$ and we observe that $V(M) \subseteq R - I(G)$ (we call $V(M)$ *vertex set* of M). Suppose now that T' is a ternary subtree of T. We define $M|_{T'} = (T', \alpha|_{E(T')}, \beta|_{A(T') \cap \tilde{A}(T)}, \gamma|_{E(T')})$. Clearly, $M|_{T'}$ is a branch representation of G rooted on R where $\tilde{A}(T') = A(T') \cap \tilde{A}(T)$. Given a set $R' \subseteq R$ we define $M\|_{R'} = (T, \alpha \bar{\cap} R', \beta \bar{\cap} R', \gamma)$. Notice that $M\|_{R'}$ is a branch representation of G rooted on R'. Finally, given a vertex $x \in R$ and a ternary subtree T' of T we call T' *x-free* if $x \notin V(M|_{T'})$.

For any $v \in R$ we set $V_x(M) = V_x^{\alpha}(M) \cup V_x^{\beta}(M)$ where $V_x^{\beta}(M) = \{t \in \tilde{A}(T) \mid x \in \beta(t)\}$ and $V_x^{\alpha}(M) = V(\{e \in E(T) \mid x \in \alpha(e)\})$. We call a branch representation $M = (T, \alpha, \beta, \gamma)$ *branch model of G rooted on R* if $\forall_{x \in R - I(G)}\ T[V_v(M)]$ is a subtree (connected subforest) of T whose leaves form a subset of $\tilde{A}(T)$. It is not hard to verify that if $\forall_{x \in R - I(G)}\ V_x(M) \neq \emptyset$ then $V(M) = R - I(G)$ and, in such a case, we call the branch representation M *complete*, otherwise, we call it *incomplete*. As we will see in Lemma 14, given a branch decomposition $B = (T, \theta)$ of G we have that $\mathrm{Des}(B)$ is a complete branch model of G

rooted on $V(G)$. We call a branch representation $M = (T, \alpha, \beta, \gamma)$ *typical* when $\forall_{e \in E(T)} \gamma(e)$ is a typical sequence. Let $M = (T, \alpha, \beta, \gamma)$ be a branch represen-

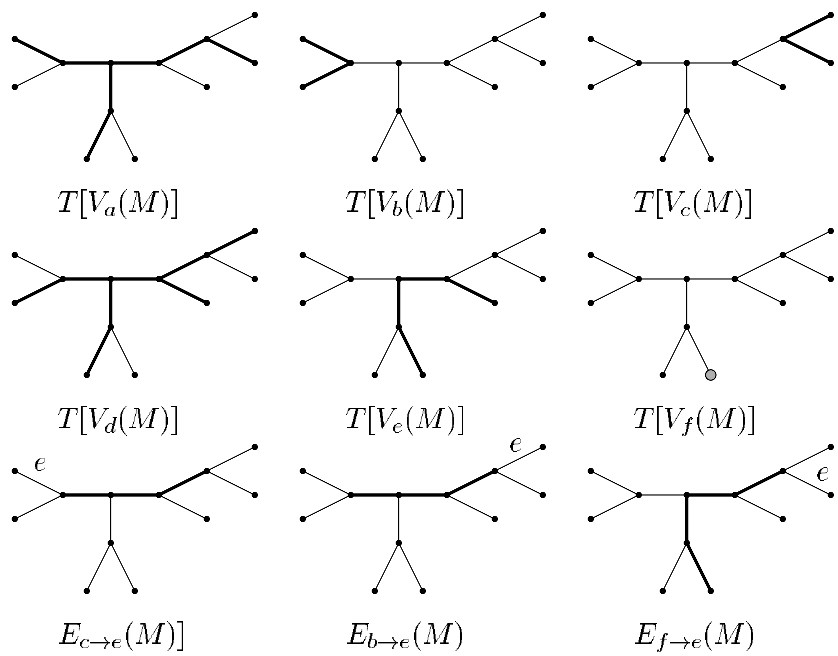


Figure 2: Values of $V_x(M)$ and $E_{x\to e}(M)$ for the description M of the branch decomposition B of Figure 1.

tation rooted on some vertex set $R \subseteq V(G)$. A path $P = (v_1, \ldots, v_r), r \geq 3$ in T is a *trunk* of M if $\forall_{2\leq i\leq r-1}\ N_T(v_i)$ contains an external leaf v_i' such that $\beta(v_i') = \emptyset$. If additionally $\forall_{2\leq i\leq r-1}\ \alpha(\{v_{i-1}, v_i\}) = \alpha(\{v_i, v_{i+1}\})$ then the trunk P is a *spine*. A subgraph $(\{v, y, x, w\}, \{\{v, y\}, \{v, w\}, \{v, x\}\})$ of T (isomorphic to $K_{1,3}$) is called *fork* if $w, x \in \tilde{A}(T), \beta(w) = \beta(x) = \emptyset, \alpha(\{v, w\}) \subseteq \alpha(\{v, y\})$, and $\alpha(\{v, x\}) \subseteq \alpha(\{v, y\})$. Finally, given a vertex $x \in R - I(G)$ (and an edge $e \in E(T)$) we define $E_x(M)$ ($E_{x\to e}(M)$) as the minimum size edge set that should be added in $T[V_x(M)]$ ($T[V_x(M) \cup e]$) in order to make it connected (it is easy to see that this edge set is uniquely defined). Observe that $E_x(M) = \emptyset \Leftrightarrow T(V_x(M))$ is connected. Notice that if M is a branch model then $\forall_{x\in R}\ E_x(M) = \emptyset$ and $\forall_{x\in R}\ E_{x\to e}(M)$ either induces a path where e is the first (or the last) edge or is empty (in case $x \in \alpha(e)$).

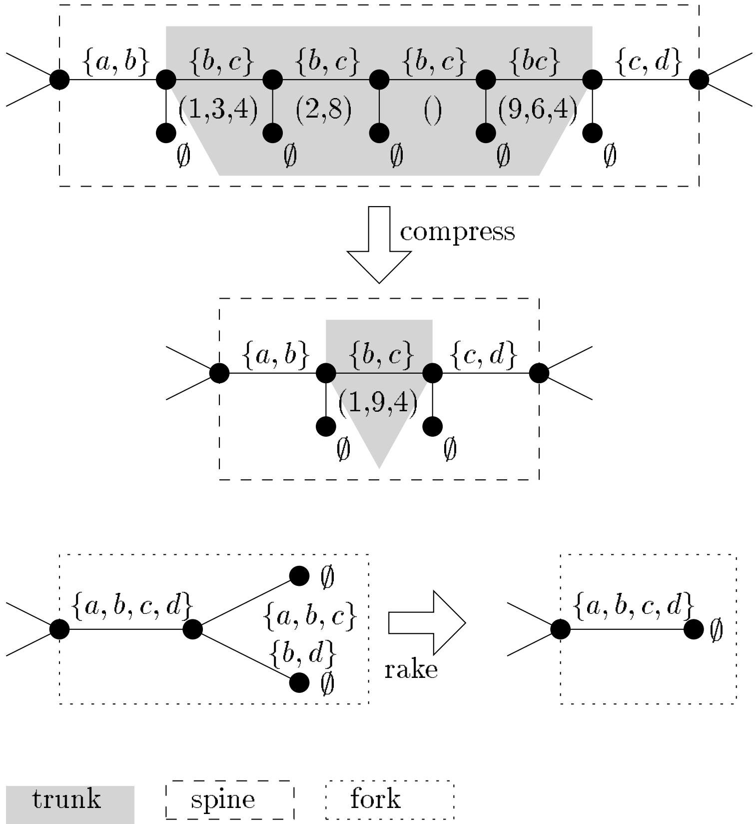


Figure 3: An example of the operations compress and rake.

Let G be a graph and $R \subseteq V(G)$. Let also $M = (T, \alpha, \beta, \gamma)$ be a branch representation of G rooted on R. The *characteristic* $C(M)$ of M is defined to be the output of the following procedure.

PROCEDURE $\mathsf{Com}(T, \alpha, \beta, \gamma)$

Input: A branch representation $M = (T, \alpha, \beta, \gamma)$ rooted on R

Output: A branch representation $M = (T, \alpha, \beta, \gamma)$ rooted on R.

1: Apply one of the following operations until this is no longer possible.

- (*Compress Operation*) If $P = (v_1, \ldots, v_r)$ is a spine in M then

  set $\alpha \leftarrow \alpha|_{E(T)-E(P)} \cup (\{v_1, v_r\}, \alpha(\{v_1, v_2\}))$,

  set $\beta \leftarrow \beta|_{\tilde{A}(T)-N_T(\{v_2,\ldots,v_{r-1}\})}$,

  set $\gamma \leftarrow \gamma|_{E(T)-E(P)} \cup (\{v_1, v_r\}, \tau(\gamma(\{v_1, v_2\}) \oplus \cdots \oplus \gamma(\{v_{r-1}, v_r\})))$,

  replace path P in T with edge $\{v_1, v_r\}$.

- (*Rake operation*) If $(\{v, y, x, w\}, \{\{v, y\}, \{v, w\}, \{v, x\}\})$ is a fork in M having w, x as leaves then

  set $\alpha \leftarrow \alpha|_{E(T)-\{\{v,w\},\{v,x\}\}}$,

  set $\beta \leftarrow \beta|_{\tilde{A}(T)-\{w,x\}} \cup (v, \emptyset)$,

  set $\gamma \leftarrow \gamma|_{E(T)-\{\{v,w\},\{v,x\}\}}$,

  remove vertices x and w and edges $\{v, w\}$ and $\{v, x\}$ from T.

2: end.

Let $M_i = (T_i, \alpha_i, \beta_i, \gamma_i), i = 1, 2$ be two branch representations. We say that $M_1 \overset{\phi}{=} M_2$ if there exist an isomorphism $\phi : T_1 \to T_2$ such that $\forall_{\{t_1,t_2\}\in E(T_1)}$ $\alpha_1(\{t_1, t_2\}) = \alpha_2(\{\phi(t_1), \phi(t_2)\}), \gamma_1(\{t_1, t_2\}) = \gamma_2(\{\phi(t_1), \phi(t_2)\})$, and $\forall_{t\in V(T_1)} \beta_1(t) = \beta_2(\phi(t))$. We say that $M_1 = M_2$ when there exist an isomorphism $\phi : T_1 \to T_2$ such that $M_1 \overset{\phi}{=} M_2$. If in the definition of "$=$" we replace $\gamma_1(\{t_1, t_2\}) = \gamma_2(\{\phi(t_1), \phi(t_2)\})$ with $\gamma_1(\{t_1, t_2\}) \prec \gamma_2(\{\phi(t_1), \phi(t_2)\})$ we define relation "$\prec$" for branch representations. We set $M_1 \equiv M_2 \overset{\text{def}}{\Longleftrightarrow} C(M_1) = C(M_2)$. Suppose now that M is a branch representation and M' is the result of the application of a series of rake or compress operations on M. Then we call M' (M) *descendant* (*ancestor*) of M (M') and we denote it as $M' \sqsubset M$ $(M' \sqsubset M)$.

Finally, we say that $M_1 \prec\!\sqsubset M_2$ when there exist a branch representation M' such that $M_1 \prec M'$ and $M' \sqsubset M_2$.

Notice that if $M_1 \sqsubset M_2$ or $M_1 \prec\!\sqsubset M_2$ then $V(M_1) = V(M_2)$.

It is easy to see that relations "$\prec$","$\equiv$", "$=$", "$\sqsubset$", and "$\prec\!\sqsubset$" are transitive. We call a branch representation M *dense* if $M = C(M)$, i.e. none of the two operations of procedure Com can be applied on M.

It is easy to observe that if a branch representation M is dense and $\tilde{A}(T(M)) = A(T(M))$ then $T(M)$ has at most $V(M)$ leaves.

Given a ternary tree T we denote as $\mathcal{T}(T)$ the set of all its ternary subtrees. Notice that each edge $e \in E(T)$ corresponds to a ternary subtree $T_e = (e, \{e\})$ of T. We denote as $\mathcal{T}_e(T)$ the set of the ternary subtrees of T that are corresponding to edges in $E(T)$.

Lemma 8 *Let $\hat{M} = (\hat{T}, \hat{\alpha}, \hat{\beta}, \hat{\gamma})$ and $M = (T, \alpha, \beta, \gamma)$, be two branch representations such that $\hat{M} \sqsubset M$ ($\hat{M} \prec\!\!\sqsubset M$). Then, there exist two functions $\psi : \mathcal{T}_e(\hat{T}) \to \mathcal{T}(T)$ and $\omega : V(\hat{T}) \to V(T)$ such that*

1. $\forall_{\hat{e}_1,\hat{e}_2 \in E(\hat{T}), \hat{e}_1 \neq \hat{e}_2}\ V(\psi(\hat{T}_{\hat{e}_1})) \cap V(\psi(\hat{T}_{\hat{e}_2})) = \{\omega(\hat{v}) \mid \hat{v} \in \hat{e}_1 \cap \hat{e}_2\}$,
2. $\cup\{T^{\diamond} \in \mathcal{T}(T) : \exists_{\hat{e} \in E(\hat{T})} \psi(\hat{T}_{\hat{e}}) = T^{\diamond}\} = T$,
3. $\forall_{\hat{e} \in E(\hat{T})}\ \hat{M}|_{\hat{T}_{\hat{e}}} \sqsubset M|_{\psi(\hat{T}_{\hat{e}})}$ ($\forall_{\hat{e} \in E(\hat{T})}\ \hat{M}|_{\hat{T}_{\hat{e}}} \prec\!\!\sqsubset M|_{\psi(\hat{T}_{\hat{e}})}$),
4. $\forall_{\hat{t}_1,\hat{t}_2 \in V(\hat{T})}\ T(\omega(\hat{t}_1), \omega(\hat{t}_2)) = \cup_{\hat{e} \in E(\hat{T}(\hat{t}_1,\hat{t}_2))}\ \psi(\hat{T}(\hat{e}))$.

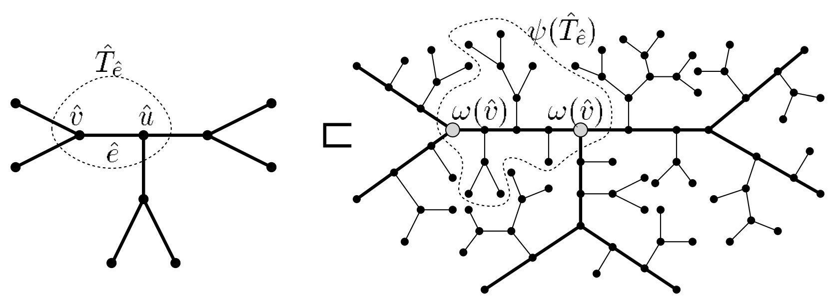


Figure 4: A representation of functions ω, and ψ in Lemma 8.

Let $\hat{M}, M$ be branch representations where $\hat{M} \sqsubset M$. Let also $\hat{T}, T$ be the underlining trees of $\hat{M}$ and M respectively. We denote as $\psi_{\hat{M},M}$, $\omega_{\hat{M},M}$ the functions defined according to Lemma 8. The images of $\psi_{\hat{M},M}$ define a set of ternary subtrees of T without common edges and whose union is T. We call this set $\mathcal{T}(\psi_{\hat{M},M})$.

Let $v \in V(T) - A(T)$ and let e_1, e_2, e_3 the three edges of T that contain v. We call vertex v x-critical when exactly one or two trees in $\{T_{e_1}, T_{e_2}, T_{e_3}\}$ are x-free.

It is easy to verify that if $\hat{M}, M$ are branch representations where $\hat{M} \sqsubset M$ then if $\hat{v}$ is x-critical then also $\omega_{\hat{M},M}(\hat{v})$ is x-critical.

It is easy to verify that $M \sqsubset M' \Rightarrow C(M) = C(M')$ and $M \prec\!\sqsubset M' \Rightarrow C(M) \prec C(M')$.

The following Lemma follows easily from the definitions and Lemma 2.

We will also need the following three easy lemmata.

$$\begin{tikzcd}
\hat{M}_1 \arrow[r, phantom, "\sqsubset"] \arrow[d, "\mathsf{Com}"] & M \arrow[d, dash, "\sqcup"] \\
\hat{M} & M_2 \arrow[l, "\mathsf{Com}"']
\end{tikzcd}$$

Figure 5: A scheme for Lemma 9.

Lemma 9 *Let M_1, M_2 be two branch models of a graph G rooted on some set $R \subseteq V(G)$. Then, if $M_1 \equiv M_2$ then there exist a branch model M such that $M_i \sqsubset M, i = 1, 2$ i.e. M is a common predecessor of $M_i, i = 1, 2$.*

$$\begin{tikzcd}
M^{+*} \arrow[d, dash, "\sqcup"] & M^* \arrow[l, "||_{X_R}"'] \arrow[d, dash, "\sqcup"] \\
M^+ & M \arrow[l, "||_{X_R}"']
\end{tikzcd}$$

Figure 6: A scheme for Lemma 10.

Lemma 10 *Let M^+, M, M^{+*} be three branch models of a graph G such that $V(M) = V(G)$, $V(M^+) = V(M^{+*}) = R$, $M^+ \sqsubset M^{+*}$, and $M^+ = M\|_R$ where $R \subseteq V(G)$. Then there exist a branch model M^* such that $M \sqsubset M^*$ and $M^{+*} = M^*\|_R$*

Let $M_i = (T_i, \alpha_i, \beta_i, \gamma_i), i = 1, 2, 3$ be three branch representations of a graph G all rooted on $R \subseteq V(G)$. Let also $V(T_1) \cap V(T_2) \cap V(T_3) = \{x\} \notin \tilde{A}(T_1) \cup \tilde{A}(T_2) \cup \tilde{A}(T_3)$ (we call vertex x *common leaf* of M_1, M_2, M_3 and we say that they are *touching*). Then we define $[M_1 \oplus M_2 \oplus M_3] = (T_1 \cup T_2 \cup T_3, \alpha_1 \cup \alpha_2 \cup \alpha_3, \beta_1 \cup \beta_2 \cup \beta_3, \gamma_1 \cup \gamma_2 \cup \gamma_3)$

Lemma 11 *Let $M_1, i = 1, \ldots, 6$ be six branch representations of a graph G rooted on $R \subseteq V(G)$. Suppose also that $\forall_{j=1,2,3}\ M_j \sqsubset M_{j+3}$ and $\forall_{j=0,1}\ M_{1+3j}, M_{2+3j}, M_{3+3j}$ are touching. Then $[M_1 \oplus M_2 \oplus M_3] \sqsubset [M_4 \oplus M_5 \oplus M_6]$. Moreover, the lemma holds if we replace "$\sqsubset$" with "$\prec\!\sqsubset$".*

Let M be a complete branch representation that is not a branch model. We define A so that it contains all the vertices $x \in V(M) = R - I(G)$ such that $T[V_x(M)]$ is not connected, i.e. $E_x(M) \neq \emptyset$. We will now give a procedure that can transform such a branch representation to a branch model if it is applied successively for each $x \in A$.

PROCEDURE $\mathsf{Norm}(M, x)$
Input: A branch representation $M = (T, \alpha, \beta, \gamma)$ rooted on R, and a vertex $x \in V(M)$.
Output: A branch representation $M = (T, \alpha, \beta, \gamma)$ rooted on R and such that $E_x(M) = \emptyset$.

1: If $E_x(M) \neq \emptyset$ then apply the following steps.
• $\beta \leftarrow \beta - \{x\}$.
• For any $e \in E_x(M)$ set $\alpha(e) \leftarrow \alpha(e) \cup \{x\}$ and $\gamma(e) \leftarrow \gamma(e) + 1$.
2: output M.
3: end.

The following Lemma can be easily proved using Lemma 8 and Lemma 11.

Lemma 12 *Let $\hat{M}, M$ be two branch representations such that $\hat{M} \sqsubset M$ ($\hat{M} \prec\!\sqsubset M$) and let $\hat{e} = \{\hat{t}_1, \hat{t}_2\}$ be an edge of $\hat{T} = T(\hat{M})$. Let also $\hat{T}_1, \hat{T}_2$ (resp. $\hat{T}_4, \hat{T}_5$) be the two of the three trees in $\mathcal{C}(\hat{T}, \hat{t}_1)$ (resp. $\mathcal{C}(\hat{T}, \hat{t}_2)$) that do not contain $\hat{t}_2$ (resp. $\hat{t}_1$) as a vertex. We set $\hat{T}_3 = (\{\hat{t}_1, \hat{t}_2\}, \{\{\hat{t}_1, \hat{t}_2\}\})$ and $\hat{M}_i = M|_{\hat{T}_i}$ for $i = 1, \ldots, 5$. Then, there exist 5 subtrees T_1, T_2, T_3, T_4, T_5 of $T = T(M)$ such that $\cup_{i=1,\ldots,5} T_i = T, \cap_{i=1,2,3} V(T_i) = \{\omega(\hat{t}_1)\}$, $\cap_{i=3,4,5} V(T_i) = \{\omega(t_2)\}$, and if $M_i = M|_{T_i}, i = 1, \ldots, 5$ then $\forall_{i=1,\ldots,5}\ \hat{M}_i \sqsubset M_i$ ($\hat{M}_i \prec\!\sqsubset M_i$).*

Lemma 13 *Suppose that $\hat{M}$ and M are two representation models where $\hat{M} \sqsubset M$. Let also $x \in V(\hat{M}) = V(M)$. Then $\mathsf{Norm}(\hat{M}, x) \sqsubset \mathsf{Norm}(M, x)$. Moreover, the lemma holds if we replace "$\sqsubset$" with "$\prec$" or "$\prec\!\sqsubset$".*

Proof W.l.o.g. we will assume that $\hat{V}_x(\hat{M})$ induces only two connected components $\hat{C}_1$, $\hat{C}_2$ in $\hat{T}$ (if they are one, then the proof is trivial and if they are more is a straightforward generalization of the present one). For simplicity in the notation we set $\psi \leftarrow \psi_{\hat{M},M}$ and $\omega \leftarrow \omega_{\hat{M},M}$. Using Lemma 8 (Relation iii.), one can easily see that $V_x(M)$ induces in T two connected components, namely $C_i = \cup_{\hat{e} \in E(\hat{C}_i)} \psi(\hat{e}), i = 1, 2$. Let $\hat{P} = (\hat{v}_1, \ldots, \hat{v}_{\hat{r}})$ be the shortest path connecting $V(\hat{C}_1)$ with $V(\hat{C}_2)$ in $\hat{T}$. Clearly, the edges induced by this path are the edges in $E_x(M)$. We set $v_1 = \omega(\hat{v}_1)$ and $v_r = \omega(\hat{v}_{\hat{r}})$. Notice that $\hat{T}(\hat{v}_1, \hat{v}_{\hat{r}})$ is x-free. From Lemma 8 (Relation iv.) have that $T(v_1, v_r) = \cup_{\hat{e} \in E(\hat{T}(\hat{v}_1, \hat{v}_{\hat{r}}))} \psi(T_{\hat{e}})$ and we conclude that $T(v_1, v_r)$ is x-free as well. Observe now that, as $\hat{v}_1$ and $\hat{v}_{\hat{r}}$ are x-critical, then also v_1 and v_2 are x critical and now it is now easy to see that the shortest path connecting $V(C_1)$ with $V(C_2)$ is the subpath $P = (v_1, \ldots, v_r)$ of $T(v_1, v_r)$ connecting v_1 and v_r in T. Suppose that the outputs of $\mathsf{Norm}(\hat{T}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, x)$ and $\mathsf{Norm}(T, \alpha, \beta, \gamma, x)$ are $\hat{M}'$ and M' respectively. We will prove that $\hat{M}' \sqsubset M'$.

Let $\hat{e}_1 = \{\hat{v}_1, \hat{v}_2\}$ be the first edge of $\hat{P}$ and let $\hat{U}^1_1, \hat{U}^1_2$ (resp. $\hat{U}^1_4, \hat{U}^1_5$) be the two of the three trees in $\mathcal{C}(\hat{T}, \hat{v}_1)$ (resp. $\mathcal{C}(\hat{T}, \hat{v}_2)$) that do not contain $\hat{v}_2$ (resp. $\hat{v}_1$) as a vertex (we exclude the case where $\hat{e}_1$ is a pendant edge of $\hat{T}$ as it is similar and easier). We also set $\hat{U}^1_3 = \hat{T}(\hat{v}_1, \hat{v}_2) = \hat{T}_{\hat{e}_1}$. Using Lemma 12 we have that there exist 5 subtrees $U^1_1, U^1_2, U^1_3, U^1_4, U^1_5$ of T such that $\cup_{i=1,\ldots,5} U^i_i = T, \cap_{i=1,2,3} U^1_i = \{\omega(\hat{v}_1)\} = \{v_1\}, \cap_{i=3,4,5} U^1_i = \{\omega(\hat{v}_2)\} = \{v_2\}$, and

$$\forall_{i=1,\ldots,5} \ \hat{M}|_{\hat{U}^1_i} \ \sqsubset \ M_1|_{U^1_i}. \tag{1}$$

Observe that $\hat{v}_{\hat{r}}$ is a vertex of U^1_4 or U^1_5. W.l.o.g. we assume that $\hat{v}_{\hat{r}} \in V(U^1_4)$ and as $\omega(\hat{v}_2) = V(U^1_3) \cap V(U^1_4)$ then $\omega(\hat{v}_2)$ is a vertex of P. Clearly, the vertices that are between $v_1 = \omega(\hat{v}_1)$ and $\omega(\hat{v}_2)$ in P are the vertices of the path connecting $v_1 = \omega(\hat{v}_1)$ and $\omega(\hat{v}_2)$ in U^1_3. We denote as E_1 the edges of this path. Suppose that $M|_{U^1_3} = (U^1_3, \alpha^1_3, \beta^1_3, \gamma^1_3)$ and that $\hat{M}|_{\hat{U}^1_3} = (\hat{U}^1_3, \hat{\alpha}^1_3, \hat{\beta}^1_3, \hat{\gamma}^1_3)$. We define $M^{1\prime}_3 = (\hat{U}^1_3, \alpha^{1\prime}_3, \beta^{1\prime}_3, \gamma^{1\prime}_3)$ where $\forall_{e \in E_1} \alpha^{1\prime}_3(e) = \alpha^1_3(e) \cup \{x\}$, $\forall_{e \in E(U^1_3) - E_1} \alpha^{1\prime}_3(e) = \alpha^1_3(e)$, $\beta^{1\prime}_3 = \beta^1_3$, $\forall_{e \in E_1} \gamma^{1\prime}_3(e) = \gamma_3(e) + 1$, and $\forall_{e \in E(U^1_3) - E_1} \gamma^{1\prime}_3(e) = \gamma^1_3(e)$. We also define $\hat{M}^{1\prime}_3 = (\hat{T}_{\hat{e}_1}, \hat{\alpha}^{1\prime}_3, \hat{\beta}^{1\prime}_3, \hat{\gamma}^{1\prime}_3)$ where $\hat{\alpha}^{1\prime}_3 = \{(\hat{e}, \hat{\alpha}^1_3(\hat{e}) \cup \{x\})\}$, $\hat{\beta}^{1\prime}_3 = \hat{\beta}^1_3$, and $\hat{\gamma}^{1\prime}_3 = \hat{\gamma}^1_3 + 1$. Recall that $(\hat{\alpha}^1_3, \hat{\beta}^1_3, \hat{\gamma}^1_3) \sqsubset (\alpha^1_3, \beta^1_3, \gamma^1_3)$ and using the definitions of $\hat{M}^{1\prime}_3$ and $M^{1\prime}_3$ one can

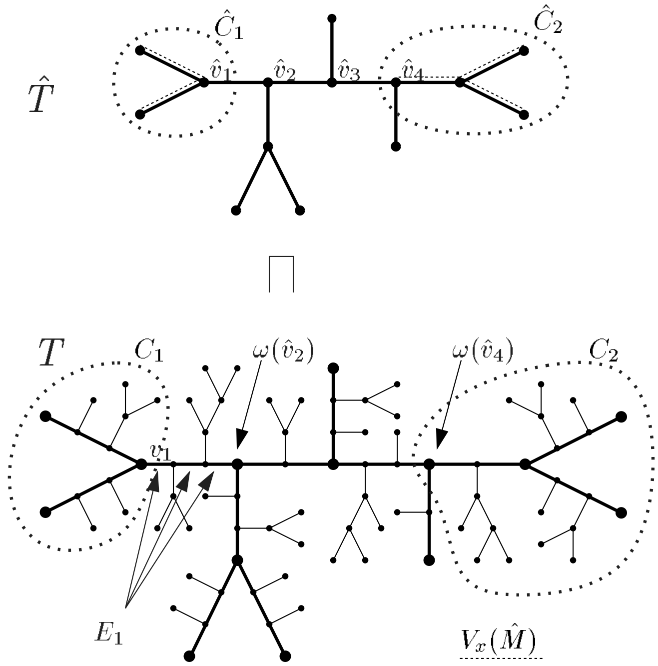


Figure 7: A scheme for the proof of Lemma 13.

easily conclude that

$$\hat{M}_3^{1\prime} \quad \sqsubset \quad M_3^{1\prime}. \tag{2}$$

Notice now that $M_1' = [[M|_{U_1} \oplus M|_{U_2} \oplus M_3^{1\prime}] \oplus M|_{U_4} \oplus M|_{U_5}]$ and $\hat{M}_1' = [[\hat{M}|_{\hat{U}_1} \oplus \hat{M}|_{\hat{U}_2} \oplus \hat{M}_3^{1\prime}] \oplus \hat{M}|_{\hat{U}_4} \oplus \hat{M}|_{\hat{U}_5}]$ are well defined and using (1), (2), and Lemma 11 we have that $\hat{M}_1' \sqsubset M_1'$. Notice that $\hat{M}_1$ is the result of the insertion of x for the first edge of $\hat{P}$ and M_1 is the result of the insertion of x for the edges in E_1 i.e. the edges of the subpath of P that connect $v_1 = \omega(\hat{v}_1)$ and $\omega(\hat{v}_2)$ in T.

Let now $\hat{e}_2 = \{\hat{v}_2, \hat{v}_3\}$ be the second edge of $\hat{P}$. Applying the same arguments for M_1' $\hat{M}_1'$ and $\hat{e}_2 = \{\hat{v}_2, \hat{v}_3\}$ as we did before for M, $\hat{M}$, and $\hat{e}_1 = \{\hat{v}_1, \hat{v}_2\}$, we first define the subtrees $U_1^2, U_2^2, U_3^2, U_4^2, U_5^2$ of T, then we construct the branch models $\hat{M}_3^{2\prime}, M_3^{2\prime}$ and we will finally end up with two equivalent branch representations $\hat{M}_2'$ and M_2' where $\hat{M}_2'$ is the result of the insertion of x for the second edge of $\hat{P}$ and M_2' is the result of the insertion of x for the edges in E_2 i.e. the edges of the subpath of P that connect $\omega(\hat{v}_2)$ and $\omega(\hat{v}_3)$ in T. Going on in this way we have that for $i = 1, \ldots, |\hat{P}| - 1$ $\hat{M}_i' \sqsubset M_i'$ and, as $\hat{M}' = \hat{M}'_{|\hat{P}|-1}$ and $M' = M'_{|\hat{P}|-1}$, we conclude that $\hat{M}' \sqsubset M'$ and we are done. The proof of the "$\prec\!\sqsubset$" version of the lemma is simply obtained if in the above proof we replace "$\sqsubset$" with "$\prec\!\sqsubset$" and use the "$\prec\!\sqsubset$" versions of Lemmata 8, 12 and 11. $\square$

3 The algorithm

3.1 Introducing a new edge in a graph

Lemma 14 *Let $B = (T, \theta)$ be a branch decomposition of a graph G. Then* $\mathrm{Des}(B) = (T, \alpha, \beta, \gamma)$ *is a complete branch model of G rooted on $V(G)$.*

Proof It is trivial to check that $(T, \alpha, \beta, \gamma)$ is a complete branch representation rooted on $V(G)$. It remains to prove that $\forall_{x \in V(G)-I(G)}\ T[V_x(M)]$ is a subtree (i.e. connected subforest) of T whoose leaves is a subset of $\tilde{A}(G)$. Let $E_x = \{e_1, \ldots, e_r\}, r = d_G(x)$ be the edges of G containing x and let $L_x = \{t \in \tilde{A} \mid \theta(t) \in E_x\}$. Let T' subtree of T spanned by L_x. It is enough to prove that $\forall_{e \in E(T)} x \in \alpha(e) \Leftrightarrow e \in E(T')$. Let e be an edge of T'. It is easy to see that $T - e$ consists of two connected components and each one of them has a leaf of L_x mapped through θ to an edge of E_x. Certainly, this means that $x \in \alpha(e)$. Suppose now that e is an edge where $x \in \alpha(e)$. This means that there exist two edges $e_1, e_2 \in E_x$ where $t_i = \theta^{-1}(e_i), i = 1, 2$ belong to different connected components of $T - e$. As e belongs to the unique path connecting t_1 and t_2 in T and $t_i \in V(T'), i = 1, 2$ we conclude that $e \in E(T')$ and this completes the proof of the lemma. $\square$

Suppose that $M = (T, \alpha, \beta, \gamma)$ is a complete branch model of a graph G rooted on $V(G)$. Suppose also that M is the description of a branch decomposition $B = (T, \theta)$ of G. Then, it is easy to observe that the following hold.

$$\forall_{e \in E(T)}\ |\gamma(e)| = 1, \tilde{A}(T) = A(T),\ \forall_{x,y \in A(T)} \alpha(e_x) = \alpha(e_y) \Rightarrow$$

$$\alpha(e_x) = \emptyset,\ \text{and}\ E(G) = \{\alpha(e_x) \cup \beta(x) \mid x \in A(T)\}.$$

Where, if x is a pendant vertex, e_x is the unique pendant edge that contains it. We call a branch model with the above property *entire*. It is clear that if a branch model is entire then it is also complete. Actually, one can prove the following.

Lemma 15 *Let M be an entire branch model $M = (T, \alpha, \beta, \gamma)$ of a graph G rooted on $V(G)$ and such that* $\mathrm{width}(M) \leq k$. *Then there exist a branch decomposition B of G with width $\leq k$ and such that $B = \mathrm{Des}(M)$.*

Proof Notice that $\forall_{x \in A(T)}$ the set $\alpha(e_x) \cup \beta(x)$ is either the empty set or an edge of G. Let $\theta : \tilde{A}(T) = A(T) \to E(G)$ be a function mapping each external leave of T to the set $\alpha(e_x) \cup \beta(x)$. It is now easy to verify that $B = (T, \theta)$ is a branch decomposition of G rooted on $V(G)$ and such that $\mathrm{Des}(B) = M$. $\square$

Lemma 16 *Let $B = (T, \theta)$ be a branch decomposition of G and $M = \mathrm{Des}(B) = (T, \alpha, \beta, \gamma)$. Let also e_{new} be an edge not in $E(G)$ and $e_{\text{ins}} = \{t_1, t_2\}$ be an edge of T. Suppose that $B' = (T', \theta')$ be the branch decomposition of $G' = (V(G), E(G) \cup \{e_{\text{new}}\})$ where $T' = (V(T) \cup \{t_{\text{mid}}, t_{\text{leaf}}\}, (E(T) - \{e_{\text{ins}}\}) \cup \{\{t_1, t_{\text{mid}}\}, \{t_{\text{mid}}, t_{\text{leaf}}\}, \{t_{\text{mid}}, t_2\}\})$ and $\theta' = \theta \cup \{(t_{\text{leaf}}, e_{\text{new}})\}$. Then if $\mathrm{Des}(B') = (T', \alpha', \beta', \gamma')$ then*

(i) $\forall_{e \in E(T') - \{\{t_1, t_{\text{mid}}\}, \{t_{\text{mid}}, t_{\text{leaf}}\}, \{t_{\text{mid}}, t_2\}\}}\ \alpha'(e) = \alpha(e) \cup \{v \in e_{\text{new}} \mid e \in E_{v \to e_{\text{ins}}}(M)\}$,

$\forall_{i=1,2}\ \alpha'(\{t_{\text{mid}}, t_i\}) = \alpha(e_{\text{ins}}) \cup \{v \in e_{\text{new}} \mid t_i \in V(E_{v \to e_{\text{ins}}}(M) - e_{\text{ins}})\}$,

$\alpha'(t_{\text{mid}}, t_{\text{leaf}}) = e_{\text{new}} - A(G')$.

(ii) $\forall_{v \in \tilde{A}(T') - \{t_{\text{leaf}}\}}\ \beta'(v) = \beta(v) - e_{\text{new}}$, *where* $\tilde{A}(T') = \tilde{A}(T) \cup \{t_{\text{leaf}}\}$,

$\beta'(t_{\text{leaf}}) = e_{\text{new}} \cap A(G')$,

(iii) $\forall_{e \in E(T') - \{\{t_1, t_{\text{mid}}\}, \{t_{\text{mid}}, t_{\text{leaf}}\}, \{t_{\text{mid}}, t_2\}\}}\ \gamma'(e) = \gamma(e) + |\{v \in e_{\text{new}} \mid e \in E_{v \to e_{\text{ins}}}(M)\}|$,

$\forall_{i=1,2}\ \alpha'(\{t_{\text{mid}}, t_i\}) = \alpha(e_{\text{ins}}) + |\{v \in e_{\text{new}} \mid t_i \in V(E_{v \to e_{\text{ins}}}(M) - e_{\text{ins}})\}|$,

$\gamma'(t_{\text{mid}}, t_{\text{leaf}}) = (|e_{\text{new}} - A(G')|)$.

Proof In order to prove (i), we notice first that $\forall_{e \in T - \{e_{\text{ins}}\}}\ \alpha(e) \subseteq \alpha'(e)$ and $\forall_{i=1,2}\ \alpha'(\{t_i, t_{\text{mid}}\}) \subseteq \alpha(e_{\text{ins}})$. Moreover, as only vertices from e_{new} are introduced, we have that $\forall_{e \in T - \{e_{\text{ins}}\}}\ \alpha'(e) - \alpha(e) \subseteq e_{\text{new}}$ and $\forall_{i=1,2}\ \alpha'(\{t_i, t_{\text{mid}}\}) - \alpha(e_{\text{ins}}) \subseteq e_{\text{new}}$.

Let $x \in e_{\text{new}}$. We assume that x is not an isolated vertex in G (if x is an isolated vertex in G, then $V_x(M) = \emptyset$ and, as no edge in T' should be mapped through α' to a set containing x, (i) is directly justified). Clearly, $V_x(M) \neq \emptyset$ and if $x \notin A(G)$ then $V_x^{\alpha} \neq \emptyset$, otherwise $V_x^{\beta} \neq \emptyset$. Using the definition of branchwidth one can observe that, towards constructing α', x must be introduced to the value of α for any edge e in a shortest path of T connecting $\{t_1, t_2\}$ with either the vertices in V_x^{α} (in case $x \notin A(G)$) or the unique vertex in V_x^{β} (in case $x \in A(G)$). If $e_{\text{ins}} \subseteq V_x^{\alpha}$ then this path has no edges and x is not necessary to be introduced, otherwise it is uniquely defined and contains one of t_1, t_2. Notice also that if this path contains $t_i, i = 1$ or 2 as an endpoint, then x must be introduced in $\alpha'(\{t_i, t_{\text{mid}}\})$ (and not in $\alpha'(\{t_{3-i}, t_{\text{mid}}\})$). Notice also that x is not a pendant vertex in G' and therefore

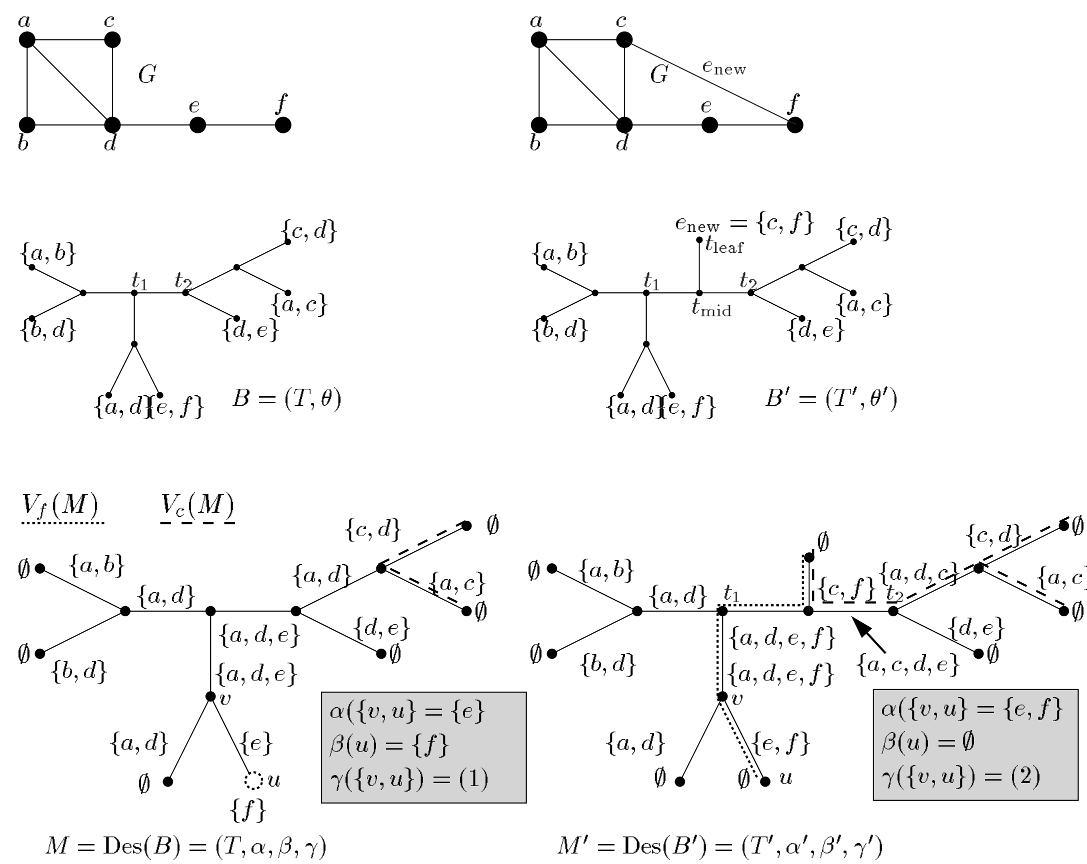


Figure 8: An example of Lemma 16

$x \in \alpha'(\{t_{\text{mid}}, t_{\text{leaf}}\})$. It is now easy to see that the above implications justify (i) when x is not an isolated vertex in G. This completes the proof of (i). (i) follows immediately from the definition of functions β and θ. Finally, we omit (iii) as it is almost the same with (i). $\square$

3.2 Characteristic of a branch decomposition

Let $B = (T, \theta)$ be an branch decomposition of the graph G rooted on R. Let also $M = \mathrm{Des}(B)$. We define $\mathrm{Des}_R(B) = M\|_R$. Using Lemma 14 we can easily verify that $\mathrm{Des}_R(B)$ is a branch model of G rooted on R and we call it *description of B with respect to R*. We set $C_R(B) = C(\mathrm{Des}_R(B))$ and we call $C_R(B)$ *characteristic* of B *with respect to R*. Clearly, $C_R(B)$ is dense and typical and is an ancestor of $\mathrm{Des}_R(B)$.

Very similarly to [6] and [7] one can prove the following useful lemmata.

Lemma 17 *There exists a function $\delta(k)$ such that for any graph G and any dense branch model M be of G rooted on some set $R \subseteq V(G)$ with $|R| \leq k$, we have that $|E(T(M))| \leq \delta(k)$.*

Lemma 18 *Let $D = (X, U)$ be a tree decomposition of G with width $\leq l$ and $i \in V(U)$ be some node of U. The number of different characteristics with respect to X_i of all possible branch decompositions of G_i with branchwidth at most k, is bounded from a function depending only on k and l, i.e. is a constant not depending on $|V(G)|$.*

A set $FS(i)$ of characteristics of branch decompositions of a graph G_i (i is a node of tree decomposition) with width at most k is called a *full set of characteristics* at i if for each branch decomposition B of G_i with branchwidth at most k, there is a branch decomposition B' such that $C_{X_i}(B') \prec C_{X_i}(B)$ and $C_{X_i}(B') \in FS(i)$, i.e. the characteristic of B' is in $FS(i)$. The following lemma can be derived directly from the definitions.

Lemma 19 *A full set of characteristics at i is non-empty if and only if the branchwidth of G_i is at most k. If some full set of characteristics at i is non-empty, then every full set of characteristics at this node is non-empty.*

An important consequence of Lemma 19 is that the branchwidth of G is at most k, if and only if any full set of characteristics of $G_r = G$ is non-empty (r is the root node of the tree decomposition). In what follows, we will show how to compute a full set of characteristics at a node i in $O(1)$ time, when the full sets of characteristics of the children of i in U are given.

3.3 Introducing a new edge in a branch model

PROCEDURE $\mathsf{Int}(M, e_{\text{ins}}, m, S, W)$
Input: A branch model $M = (T, \alpha, \beta, \gamma)$ rooted on R, an edge $e_{\text{ins}} = \{t_1, t_2\}$ of T, an integer $m, 1 \leq m \leq |\gamma(e_{\text{ins}})|$ and two vertex sets S, W where $W \subseteq S \subseteq R$.
Output: A branch model $M = (T, \alpha, \beta, \gamma)$ rooted on R.

1: (*Splitting step*) Construct the branch model $M' = (T', \alpha', \beta', \gamma')$ by applying the following four steps.

• Set $T' = (V(T) \cup \{t_{\text{mid}}, t_{\text{leaf}}\}, (E(T) - \{e_{\text{ins}}\}) \cup \{\{t_1, t_{\text{mid}}\}, \{t_{\text{mid}}, t_{\text{leaf}}\}, \{t_{\text{mid}}, t_2\}\})$,

• Set α' so that

$$\forall_{e \in E(T') - \{\{t_1, t_{\text{mid}}\}, \{t_{\text{mid}}, t_{\text{leaf}}\}, \{t_{\text{mid}}, t_2\}\}} \ \alpha'(e) = \alpha(e),$$
$$\alpha'(\{t_{\text{mid}}, t_2\}) = \alpha'(\{t_{\text{mid}}, t_1\}) = \alpha(e_{\text{ins}}),$$
$$\alpha'(\{t_{\text{mid}}, t_{\text{leaf}}\}) = S - W.$$

• Set $\beta' \leftarrow \beta \cup \{(t_{\text{leaf}}), W\}$ (and therefore $\tilde{A}(T') = \tilde{A}(T) \cup \{t_{\text{leaf}}\}$).

• Set γ' so that

$$\forall_{e \in E(T') - \{\{t_1, t_{\text{mid}}\}, \{t_{\text{mid}}, t_{\text{leaf}}\}, \{t_{\text{mid}}, t_2\}\}} \ \gamma'(e) = \gamma(e),$$
$$\gamma'(\{t_{\text{mid}}, t_2\}) = \gamma(e_{\text{ins}})_{1,m},$$
$$\gamma'(\{t_{\text{mid}}, t_1\}) = \gamma(e_{\text{ins}})_{m, |\gamma(e_{\text{ins}})|},$$
$$\gamma'(\{t_{\text{mid}}, t_{\text{leaf}}\}) = (|S - W|).$$

2: (*Normalizing step*) For any $x \in S - W$ set $M' \leftarrow \mathsf{Norm}(M', x)$.

3: output $(T', \alpha', \beta', \gamma')$.

4: end.

The next lemma follows easily from Lemma 16, Procedure Int, and the definitions of α, β, and γ.

Lemma 20 *Let G be a graph rooted on $R \subseteq V(G)$ and G' be the graph obtained by G after introducing an edge e_{new} with both endpoints in R. Suppose that $B = (T, \theta)$ is a branch decomposition of G, $e_{\text{ins}} = \{t_1, t_2\}$ is some edge of T, and $B' = (T', \theta')$ is a branch decomposition of G' where $T' = (V(T) \cup \{t_{\text{mid}}, t_{\text{leaf}}\}, (E(T) - \{e_{\text{ins}}\}) \cup \{\{t_1, t_{\text{mid}}\}, \{t_{\text{mid}}, t_{\text{leaf}}\}, \{t_{\text{mid}}, t_2\}\})$ and $\theta' = \theta \cup \{(t_{\text{leaf}}, e_{\text{new}})\}$. Let also $M = \text{Des}_R(B)$. Then $\text{Des}_R(B') = \mathsf{Int}(M, e_{\text{ins}}, 1, e_{\text{new}}, W)$ where $W = e_{\text{new}} - A(G') =$*

$e_{\rm new} - I(G)$.

Lemma 21 *Let* $M = (T,\alpha,\beta,\gamma), \hat{M} = (\hat{T},\hat{\alpha},\hat{\beta},\hat{\gamma})$ *be two branch models of a graph* G *rooted on* $R \subseteq V(G)$ *where* $\hat{M} \sqsubset M$ ($\hat{M} \prec\!\sqsubset M$) *and such that* $\hat{T} = (\{\hat{t}_1,\hat{t}_2\},\{\{\hat{t}_1,\hat{t}_2\}\})$ *and* $\tilde{A}(\hat{T}) = \emptyset$ *(i.e.* $\hat{T}$ *consists of only one edge whose endpoints are internal leaves). Let also* $s = |\hat{\gamma}(\{\hat{t}_1,\hat{t}_2\})|$*. Then for any* $m, 1 \leq m \leq s$ *there exist an edge* $\{t_r,t_{r+1}\} \in E(T)$ *such that if* $\mathcal{C}(T,t_r,t_{r+1}) = (T^r,T^{r+1})$, $M^r = M|_{T^r}$, $M^{r+1} = M|_{T^{r+1}}$, $\hat{M}^1 = (\hat{T},\hat{\alpha},\hat{\beta},\{(\{\hat{t}_1,\hat{t}_2\},\hat{\gamma}(\{\hat{t}_1,\hat{t}_2\})_{1,m})\}$*, and* $\hat{M}^2 = (\hat{T},\hat{\alpha},\hat{\beta},\{(\{\hat{t}_1,\hat{t}_2\},\hat{\gamma}(\{\hat{t}_1,\hat{t}_2\})_{m,s})\}$*, then* $\hat{M}^1 \sqsubset M^2$ *and* $\hat{M}^r \sqsubset M^{r+1}$ ($\hat{M}^1 \prec\!\sqsubset M^2$ *and* $\hat{M}^r \prec\!\sqsubset M^{r+1}$)*.*

proof Let $\omega \leftarrow \omega_{\hat{M},M}$ and let $P = (t_1,\dots,t_q)$ be the path connecting $\omega(\hat{t}_1)$ and $\omega(\hat{t}_2)$ in T (we assume that $t_1 = \omega(\hat{t}_1)$ and $t_q = \omega(\hat{t}_2)$. Clearly,

$$\hat{\gamma}(\{\hat{t}_1,\hat{t}_2\}) \quad \sqsubset \quad \gamma(\{t_1,t_2\}) \oplus \cdots \oplus \gamma(\{t_{q-1},t_q\}). \tag{3}$$

From relation (3) and Lemma 3we have that there exist some $r, 1 \leq r < q$ such that

$$\hat{\gamma}(\{\hat{t}_1,\hat{t}_2\})_{1,m} \quad \sqsubset \quad \gamma(\{t_1,t_2\}) \oplus \cdots \oplus \gamma(\{t_r,t_{r+1}\}) \text{ and} \tag{4}$$

$$\hat{\gamma}(\{\hat{t}_1,\hat{t}_2\})_{m,s} \quad \sqsubset \quad \gamma(\{t_r,t_{r+1}\}) \oplus \cdots \oplus \gamma(\{t_{q-1},t_q\}) \tag{5}$$

For $i = 0,1$ we define as T^{r+i} the tree of $\mathcal{C}(T,t_r,t_{r+1})$ that do not contain t_{r+1-i} as a vertex. We now set $M^{r+i} = M|_{T^{r+i}}, i = 0,1$ and, from Relations (4) and (5), we easily have the required. The proof of the "$\prec\!\sqsubset$" version of the lemma is obtained by the current one if we replace "$\sqsubset$" with "$\prec\!\sqsubset$" in Relations (3),(4), and (5). $\square$

Lemma 22 *Let* $B = (T,\theta)$ *be a branch decomposition of a graph* G *rooted on* $R \subseteq V(G)$*. We assume that* $M = \mathrm{Des}_R(B)$ *and* $\hat{M} = C_R(B) = C(M)$ *Then for any vertex sets* $S \subseteq R$*, and* $W \subseteq S, W \subseteq I(G)$ *the following hold.*

(i) For any $\hat{e}_{\rm ins} \in E(\hat{T})$ *and any* $m, 1 \leq m \leq |\hat{\gamma}(\hat{e}_{\rm ins})|$ *there exists an edge* $e_{\rm ins} \in T$ *such that* $\mathsf{Com}(\mathsf{Int}(\hat{M},\hat{e}_{\rm ins},m,S,W)) = \mathsf{Com}(\mathsf{Int}(M,e_{\rm ins},1,S,W))$*.*

(ii) For any $e_{\rm ins} \in E(T)$ *there exists an edge* $\hat{e}_{\rm ins} \in \hat{T}$ *and an* $m, 1 \leq m \leq |\hat{\gamma}(\hat{e}_{\rm ins})|$ *such that* $\mathsf{Com}(\mathsf{Int}(\hat{M},\hat{e}_{\rm ins},m,S,w)) \prec \mathsf{Com}(\mathsf{Int}(M,e_{\rm ins},1,S,W))$*.*

Proof (i) Set $M = (T,\alpha,\beta,\gamma)$, $\hat{M} = (\hat{T},\hat{\alpha},\hat{\beta},\hat{\gamma})$, $\hat{e}_{\rm ins} = \{\hat{t}_1,\hat{t}_2\}$ and $s = |\hat{\gamma}(\hat{e}_{\rm ins})|$. Notice that, as $\hat{M} = C(M)$, $\hat{M}$ is a dense and typical branch model. We apply Lemma 12 for edge $\hat{e}_{\rm ins}$ and define $T_i, \hat{T}_i, M_i, \hat{M}_i,\ i = 1,\dots,5$, accordingly (we exclude the case where $\hat{e}_{\mathrm{ins}}$ is a pendant edge of $\hat{T}$ as it is similar and easier). We have that

$$\forall_{1\leq i\leq 5}\ \hat{M}_i \quad = \quad C(M_i) \tag{6}$$

We set $\mathcal{C}(T_3, t_1, t_2) = (T_3^1, T_3^2)$, $M_3^1 = M|_{T_3^1}$, $M_3^2 = M|_{T_3^2}$, $\hat{M}_3^1 = (\hat{T}_3, \hat{\alpha}|_{\{\hat{e}_{\mathrm{ins}}\}}, \{\}, \{(\hat{e}_{\mathrm{ins}}, \hat{\gamma}(\hat{e}_{\mathrm{ins}})_{1,m})\})$, and $\hat{M}_3^2 = (\hat{T}_3, \hat{\alpha}|_{\{\hat{e}_{\mathrm{ins}}\}}, \{\}, \{(\hat{e}_{\mathrm{ins}}, \hat{\gamma}(\hat{e}_{\mathrm{ins}})_{m,s})\})$. From the fact that $\hat{M}_3 = C(M_3)$ and Lemma 21 we have that there exist an edge $e_{\mathrm{ins}} = \{t_1, t_2\} \in E(T_3)$ such that

$$\hat{M}_3^1 = C(M_3^1), \quad \text{and} \quad \hat{M}_3^2 = C(M_3^2) \tag{7}$$

Clearly, in order to prove the required, it is sufficient to prove that $\mathsf{Int}(\hat{M}, \hat{e}_{\mathrm{ins}}, m, S, W) \sqsubset \mathsf{Int}(M, , e_{\mathrm{ins}}, 1, S, W)$. In what follows, we will proceed applying in parallel the two steps of the procedures $\mathsf{Int}(\hat{M}, \hat{e}_{\mathrm{ins}}, m, S, W)$ and $\mathsf{Int}(M, e_{\mathrm{ins}}, 1, S, W)$. We will show that the corresponding branch models constructed after each step are relevant. Clearly, before the splitting step, we have that $\hat{M} \sqsubset M$.

We will first prove that the same holds after the splitting step. Observe that the branch models $M_1' = [M_1 \oplus M_2 \oplus M_3^1]$, $M_2' = [M_4 \oplus M_5 \oplus M_3^2]$, $\hat{M}_1' = [\hat{M}_1 \oplus \hat{M}_2 \oplus \hat{M}_3^1]$, and $\hat{M}_2' = [\hat{M}_4 \oplus \hat{M}_5 \oplus \hat{M}_3^2]$, are well defined and from (6),(7), and Lemma 11 we have that

$$\hat{M}_1' \sqsubset M_1' \quad \text{and} \quad \hat{M}_2' \sqsubset M_2'. \tag{8}$$

Set $M_3' = (\{(\{t_{\mathrm{mid}}, t_{\mathrm{leaf}}\}, S - W)\}, \{t_{\mathrm{leaf}}, W\}, \{(\{t_{\mathrm{mid}}, t_{\mathrm{leaf}}\}, (|S - W|))\})$ and $\hat{M}_3' = (\{(\{\hat{t}_{\mathrm{mid}}, \hat{t}_{\mathrm{leaf}}\}, S - W)\}, \{t_{\mathrm{leaf}}, W\}, \{(\{\hat{t}_{\mathrm{mid}}, \hat{t}_{\mathrm{leaf}}\}, (|S - W|)\})$ Clearly,

$$\hat{M}_3' \quad \sqsubset \quad M_3'. \tag{9}$$

We now replace in M_1' (M_2') vertex t_2 (t_1) with t_{mid} and in $\hat{M}_1'$ ($\hat{M}_2'$) we replace $\hat{t}_2$ ($\hat{t}_1$) with $\hat{t}_{\mathrm{mid}}$. Notice that $M' = (T', \alpha', \beta', \gamma') = [M_1' \oplus M_2' \oplus M_3']$ and $\hat{M}' = (\hat{T}', \hat{\alpha}', \hat{\beta}', \hat{\gamma}') = [\hat{M}_1' \oplus \hat{M}_2' \oplus \hat{M}_3']$ are well defined and are the two branch representations constructed after the application in parallel of the splitting step are M_1', M_2', M_3' and $\hat{M}_1', \hat{M}_2', \hat{M}_3'$ are touching the common vertices are t_{mid} and $\hat{t}_{\mathrm{mid}}$ respectively. From (8),(9), and Lemma 11 we have that

$$\hat{M}' \quad \sqsubset \quad M'. \tag{10}$$

Notice now that $\hat{M}'$ and M' are representation models but not necessarily branch models. The *normalizing step* transforms both of them to branch models by introducing and/or removing vertices of $V(M')$ ($V(\hat{M}')$) from M' ($\hat{M}'$). The results of

this step satisfy Relation (10) because of Lemma 13 and this completes the proof of (i).
(ii) We define $\hat{M}$ and M as in the proof of (i). Let $\hat{e}_{\rm ins} = \{\hat{t}_1, \hat{t}_2\}$ be the unique edge of $\hat{T}$ such that $e_{\rm ins} \in E(\psi_{\hat{M},M}(\hat{T}_{\hat{e}_{\rm ins}}))$. We also set $e_{\rm ins} = \{t_1, t_2\}$ and $s = |\hat{\gamma}(\hat{e}_{\rm ins})|$. As we did in the proof of (i), we apply Lemma 12 for edge $\hat{e}_{\rm ins}$ and define $T_i, \hat{T}_i, M_i, \hat{M}_i$, $i = 1, \ldots, 5$, accordingly (we exclude the case where $\hat{e}_{\rm ins}$ is a pendant edge of $\hat{T}$ as it is similar and easier). We observe that (6) holds as well. Let $P = (e_1, \ldots, e_r)$ be the path in T_3 connecting $\omega(\hat{t}_1)$ with $\omega(\hat{t}_2)$ (clearly, $\omega(\hat{t}_1) \in e_1$ and $\omega(\hat{t}_2) \in e_r$). We distinguish two cases.
Case I. $e_{\rm ins}$ is an edge in P. We assume that $e_{\rm ins} = e_j, 1 \leq j \leq r$. We observe that $\hat{\gamma}(\hat{e}_{\rm ins}) \sqsubset \gamma(e_1) \oplus \cdots \oplus \gamma(e_r)$. Suppose first that there exist an integer m such that

$$\hat{\gamma}(\hat{e}_{\rm ins})_{1,m} \quad \sqsubset \quad \gamma(e_1) \oplus \cdots \oplus \gamma(e_j) \text{ and} \tag{11}$$

$$\hat{\gamma}(\hat{e}_{\rm ins})_{m,s} \quad \sqsubset \quad \gamma(e_j) \oplus \cdots \oplus \gamma(e_r). \tag{12}$$

Notice that using (11) and (12) we can define $M_3^1, M_3^2, \hat{M}_3^1, \hat{M}_3^2$ as we did in the proof of (i). Clearly, (7) is satisfied as well and then it is sufficient to follow the steps of the proof of (i) in order to prove that $\mathsf{Com}(\mathsf{Int}(M, e_{\rm ins}, 1, S, W)) = \mathsf{Com}(\mathsf{Int}(\hat{M}, \hat{e}_{\rm ins}, m, S, W))$ which is a stronger version of the required. We now examine the case where there exist no m such that (11) and (12) are satisfied. We observe instead that there exist some $m', 2 \leq m' \leq r-1$ and two integers $\lambda, \mu, 1 \leq \lambda < j < \mu \leq r$ such that

$$\hat{\gamma}(\hat{e}_{\rm ins})_{1,m'} \quad \sqsubset \quad \gamma(e_1) \oplus \cdots \oplus \gamma(e_\lambda) \text{ and} \tag{13}$$

$$\hat{\gamma}(\hat{e}_{\rm ins})_{m'+1,s} \quad \sqsubset \quad \gamma(e_\mu) \oplus \cdots \oplus \gamma(e_r). \tag{14}$$

Clearly $\gamma(e_\lambda)$ and $\gamma(e_\mu)$ are sequences consisting of only one integer. We denote these integers as q_λ and q_μ respectively. It is easy to notice that $q_\lambda \neq q_\mu$. If $q_\lambda < q_\mu$ then we set $m \leftarrow m'$, otherwise we set $m \leftarrow m'+1$. We will examine the case where $q_\lambda < q_\mu$ (the other case is similar). We observe that

$$\forall_{\lambda+1 \leq h \leq \mu-1} \hat{\gamma}(\hat{e_{\rm ins}})_{m,m} \quad \prec \quad \gamma(e_h) \tag{15}$$

Combining now (13),(14), and (15) we conclude to the following.

$$\hat{\gamma}(\hat{e}_{\rm ins})_{1,m} \quad \prec\!\!\sqsubset \quad \gamma(e_1) \oplus \cdots \oplus \gamma(e_j) \text{ and} \tag{16}$$

$$\hat{\gamma}(\hat{e}_{\rm ins})_{m,s} \quad \prec\!\!\sqsubset \quad \gamma(e_j) \oplus \cdots \oplus \gamma(e_r). \tag{17}$$

Let $\mathcal{C}(T_3, t_1, t_2) = (T_3^1, T_3^2)$. We define $M_3^1 = M|_{T_3^1}$, $M_3^2 = M|_{T_3^2}$, $\hat{M}_3^1 = (\hat{T}_3, \hat{\alpha}|_{\{\hat{e}_{\rm ins}\}}, \{\}, \{(\hat{e}_{\rm ins}, \hat{\gamma}(\hat{e}_{\rm ins})_{1,m})\})$, and $\hat{M}_3^2 = (\hat{T}_3, \hat{\alpha}|_{\{\hat{e}_{\rm ins}\}}, \{\}, \{(\hat{e}_{\rm ins}, \hat{\gamma}(\hat{e}_{\rm ins})_{m,s})\})$. Using (16) and (17) one can verify the following.

$$\hat{M}_3^1 \prec\!\sqsubset M_3^1 \quad (18)$$
$$\hat{M}_3^2 \prec\!\sqsubset M_3^2 \quad (19)$$

Following now the methodology of the proof of (i), we will proceed applying in parallel the steps of procedures $\mathsf{Int}(M, e_{\text{ins}}, 1, S, W)$ and $\mathsf{Int}(\hat{M}, \hat{e}_{\text{ins}}, m, S, W)$. We define the branch models $\hat{M}_3'$ and M_3' as in the proof of (i). Clearly, the "$\prec\!\sqsubset$" version of (9) holds. We now replace in M_3^1 (M_3^2) vertex t_2 (t_1) with t_{mid} and in $\hat{M}_3^1$ ($\hat{M}_3^2$) we replace $\hat{t}_2$ ($\hat{t}_1$) with $\hat{t}_{\text{mid}}$. Similarly to the proof of (i) we have that $M' = [[M_1 \oplus M_2 \oplus M_3^1] \oplus M_3' \oplus [M_3^1 \oplus M_4 \oplus M_5]]$ and $\hat{M}' = [[\hat{M}_1 \oplus \hat{M}_2 \oplus \hat{M}_3^1] \oplus \hat{M}_3' \oplus [\hat{M}_3^1 \oplus \hat{M}_4 \oplus \hat{M}_5]]$ are the results of the application of the splitting step. Using now (6),(18),(19),(9), and the "$\prec\!\sqsubset$" version of Lemma 11 we have that

$$\hat{M}' \prec\!\sqsubset M' \quad (20)$$

We need now to prove that (20) holds for the branch models occurring after the normalization step and this follows immediately from the "$\prec\!\sqsubset$" version of Lemma 13. *Case II.* e_{ins} does not belong to P. Let e_j, e_{j+1} be the two neighboring edges of P that are closer to e_{ins} in T_3. We denote the common endpoint of e_j, e_{j+1} as v. Let also U_3 be the tree of $\mathcal{C}(T, v)$ that contains e_{ins} as an edge. Recall that $\tau(\gamma(e_1) \oplus \cdots \oplus \gamma(e_r)) = \hat{\gamma}(\hat{e}_{\text{ins}})$. Clearly, $r \geq 2$. Observe that we can choose an integer $m, 1 \leq m \leq s$ such that either relations (11) and (12) hold. or there exist two integers $\lambda, \mu, 1 \leq \lambda < j < \mu \leq r$ such that relations (13) and (14) hold. Each of the above cases is faced by a case analysis very similar to the one of *Case I.* The only difference is that now in the definition of M_3' we should set $M_3' = (T^\diamond, \alpha^\diamond, \beta^\diamond, \gamma^\diamond)$ where

$T^\diamond = (V(U_3) \cup \{t_{\text{mid}}, t_{\text{leaf}}\}, E(U_3) - \{t_1, t_2\} \cup \{\{t_1, t_{\text{mid}}\}, \{t_{\text{mid}}, t_2\}, \{t_{\text{mid}}, t_{\text{leaf}}\}\})$,
$\alpha^\diamond = \alpha|_{E(U_3)-\{\{t_1,t_2\}\}} \cup \{(\{t_1, t_{\text{mid}}\}, \alpha(e_{\text{ins}})), (\{t_{\text{mid}}, t_2\}, \alpha(e_{\text{ins}})), (\{t_{\text{mid}}, t_{\text{leaf}}\}, \emptyset)\}$,
$\beta^\diamond = \beta \cup \{(t_{\text{leaf}}, \emptyset)\}$,
$\gamma^\diamond = \gamma|_{E(U_3)-\{\{t_1,t_2\}\}} \cup \{(\{t_1, t_{\text{mid}}\}, \gamma(e_{\text{ins}})), (\{t_{\text{mid}}, t_2\}, \gamma(e_{\text{ins}})), (\{t_{\text{mid}}, t_{\text{leaf}}\}, ())\})$

and observe that the "$\prec\!\sqsubset$" version of relation 9 holds as well (i.e. $\hat{M}_3' \prec\!\sqsubset M_3'$). □

The proof of the following lemma is similar (and easier) to the one of Lemma 22.

Lemma 23 *Let $M^i = (T^i, \alpha^i, \beta^i, \gamma^i), i = 1, 2$ be two branch models of G rooted on R such that $M^1 \prec M^2$. Then for any $S \subseteq R$, $W \subseteq S, W \subseteq I(G)$, $e_{\text{ins}}^1 \in E(T^1)$, and $m_1, 1 \leq m_1 \leq |\gamma^1(e_{\text{ins}}^1)|$ there exist $e_{\text{ins}}^2 \in T^2$ and $m_2, 1 \leq m_2 \leq |\gamma^2(e_{\text{ins}}^2)|$ such that $\mathsf{Com}(\mathsf{Int}(T^1, \alpha^1, \beta^1, \gamma^1, e_{\text{ins}}^1, m_1, S, W)) \prec \mathsf{Com}(\mathsf{Int}(T^2, \alpha^2, \beta^2, \gamma^2, e_{\text{ins}}^2, m_2, S, W))$.*

3.4 A full set for an introduce node

We will now consider the case where i is an **introduce** node. Let j be the child of i.

Clearly $V_i = V_j \cup \{x\}$ where $x \notin V_j$. Suppose that $E_x = \{e_1, \ldots, e_r\}, 0 \leq r \leq |X_j| \leq l$ is the set of edges containing x in G_i (notice that, $N_{G_j}(x) \cup_{e \in E_x} e \subseteq X_j$). If $E_x = \emptyset$, then, we simply set $FS(i) = FS(j)$. What remains is to examine the case where $|E_x| \geq 1$.

We define $G_i^p = (V(G_i), E(G_j) \cup \{e_1, \ldots, e_p\}), 0 \leq p \leq r$. Clearly, $FS(j)$ is a full set of characteristics for $G_i^0 = G_j$. Notice also that $G_i = G_i^p$. Suppose that we have a full set of characteristics $FS(i, p-1)$ for $G_i^{p-1}, 1 \leq p \leq r$ (which is the case when $p = 1$). It is sufficient to give a $O(1)$ time algorithm constructing a full set of characteristics $FS(i, p)$ for G_i^p.

ALGORITHM $\mathsf{Introduce\text{-}edge}$

Input: A full set of characteristics $FS(i, p-1)$ for G_i^{p-1}.
Output: A full set of characteristics $FS(i, p)$ for G_i^p.

1: Initialize $FS(i, p) = \emptyset$.
2: For each characteristic $M \in FS(i, p-1)$, each edge $e_{\text{ins}} \in E(T)$, and any integer $m, 1 \leq m \leq |\gamma(e_{\text{ins}})|$, set $M' = \mathsf{Com}(\mathsf{Int}(M, e_{\text{ins}}, m, e_p, e_p \cap A(G_i^p)))$ and if $\max(\gamma') \leq k$, then set $FS(i,p) \leftarrow FS(i,p) \cup \{M'\}$.
3: end.

Lemma 24 *The set $FS(i, p)$ constructed by the above algorithm is a full set of characteristics.*

Proof. We will prove first that $FS(i, p)$ is a set of characteristics. Let $\hat{M}' \in FS(i, p)$. We will show that there exists a branch decomposition of G_i^p with $\hat{M}'$ as a characteristic. We set $W = e_p \cap A(G_i^p) = e_p \cap I(G_i^{p-1})$. Clearly, as $\hat{M}'$ is constructed by the algorithm above, there must be a characteristic $\hat{M} = (\hat{T}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}) \in FS(i, p-1)$, an edge $\hat{e}_{\text{ins}} \in E(\hat{T})$, and an integer $m, 1 \leq m \leq |\hat{\gamma}(\hat{e}_{\text{ins}})|$ such that

$$\mathsf{Com}(\mathsf{Int}(\hat{M}, \hat{e}_{\text{ins}}, m, e_p, W)) \;\; = \;\; \hat{M}'. \qquad (21)$$

As $\hat{M} \in FS(i, p-1)$ we have that there exist a branch decomposition $B = (T, \theta)$ of G_i^{p-1} such that $\hat{M} = C_{X_i}(B)$, i.e B that has $\hat{M}$ as characteristic with respect to X_i. Let $\mathrm{Des}_{X_i}(B) = M$. Clearly, $\hat{M} = C(M)$. From Lemma 22.i we have that there exist an edge $e_{\mathrm{ins}} \in E(T)$ such that

$$\mathsf{Com}(\mathsf{Int}(\hat{M}, \hat{e}_{\mathrm{ins}}, m, e_p, W)) = \mathsf{Com}(\mathsf{Int}(M, e_{\mathrm{ins}}, 1, e_p, W)). \tag{22}$$

Let $B' = (T', \theta')$ be a branch decomposition defined from B as in Lemmata 16 or 20. Clearly, $B' = (T', \theta')$ is a branch decomposition of G_i^p. We claim that $C_{X_i}(B') = \hat{M}'$. Indeed, from Lemma 20, we have that

$$C_{X_i}(B') = \mathsf{Com}(\mathsf{Int}(M, e_{\mathrm{ins}}, 1, e_p, W)). \tag{23}$$

and, now, $C_{X_i}(B') = \hat{M}'$ follows directly from (21), (22) and (23).

It remains now to prove that $FS(i,p)$ is a full set of characteristics. Let $B' = (T', \theta')$ be a branch decomposition of G_i^p. We will show that there exists a branch decomposition $B'^{\sharp} = (T'^{\sharp}, \theta'^{\sharp})$ of G_i^p such that $C_{X_i}(B'^{\sharp}) \prec C_{X_i}(B')$ and $C_{X_i}(B'^{\sharp}) \in FS(i,p)$. Let $e_{\mathrm{ins}} = \{t_{\mathrm{mid}}, t_{\mathrm{leaf}}\} \in E(T')$ be the edge of T' such that $t_{\mathrm{leaf}} \in \tilde{A}(T')$ and $\theta'(t_{\mathrm{leaf}}) = e_p$. Let also $t_i, i = 1,2$ be the vertices of $N_{T'}(t_{\mathrm{mid}}) - \{t_{\mathrm{leaf}}\}$. We set $B = (T, \theta)$ where $T = (V(T) - \{t_{\mathrm{leaf}}, t_{\mathrm{mid}}\}, E(T) - \{\{t_1, t_{\mathrm{mid}}\}, \{t_{\mathrm{mid}}, t_{\mathrm{leaf}}\}, \{t_{\mathrm{mid}}, t_2\}\} \cup \{\{t_1, t_2\}\})$ and $\theta = \theta'|_{\tilde{A}(T) - \{t_{\mathrm{leaf}}\}}$. Let $C_{X_i}(B) = \hat{M} = (\hat{T}, \hat{\alpha}, \hat{\beta}, \hat{\gamma})$ and $\mathrm{Des}_{X_i}(B) = M$. From Lemma 20 we have that

$$\mathsf{Com}(\mathsf{Int}(M, e_{\mathrm{ins}}, 1, e_p, W)) = C_{X_i}(B'). \tag{24}$$

From Lemma 22.(ii) we have that there exist and edge $\hat{e}_{\mathrm{ins}} \in E(\hat{T})$ and an integer $m, 1 \le m \le |\hat{\gamma}(\hat{e}_{\mathrm{ins}})|$ such that

$$\mathsf{Com}(\mathsf{Int}(\hat{M}, \hat{e}_{\mathrm{ins}}, m, e_p, W)) \prec \mathsf{Com}(\mathsf{Int}(M, e_{\mathrm{ins}}, 1, e_p, W)). \tag{25}$$

As $FS(i, p-1)$ is a full set of characteristics, we have that there exists a branch decomposition $B^{\sharp} = (T^{\sharp}, \theta^{\sharp})$ of G_i^{p-1} such that $C_{X_i}(B^{\sharp}) \prec C_{X_i}(B)$ and $C_{X_i}(B^{\sharp}) \in FS(i, p-1)$. Let $C_{X_i}(B^{\sharp}) = \hat{M}^{\sharp} = (\hat{T}^{\sharp}, \hat{\alpha}^{\sharp}, \hat{\beta}^{\sharp}, \hat{\gamma}^{\sharp})$. From Lemma 23 we have that there exists an edge $\hat{e}^{\sharp}_{\mathrm{ins}}$ and an integer $m^{\sharp}, 1 \le m^{\sharp} \le \hat{\gamma}^{\sharp}(\hat{e}^{\sharp}_{\mathrm{ins}})$ such that

$$\mathsf{Com}(\mathsf{Int}(\hat{M}^{\sharp}, \hat{e}^{\sharp}_{\mathrm{ins}}, m^{\sharp}, e_p, W)) \prec \mathsf{Com}(\mathsf{Int}(\hat{M}, \hat{e}_{\mathrm{ins}}, m, e_p, W)). \tag{26}$$

We set $B'^{\sharp} = (T'^{\sharp}, \theta'^{\sharp})$ where $T'^{\sharp}$ and $\theta'^{\sharp}$ are defined as in Lemmata 16 or 20 (notice that $|\tilde{A}(T)| = |\tilde{A}(T^{\sharp})| = |E(G_i^{p-1})|$). Let $\mathrm{Des}_{X_i}(B^{\sharp}) = M^{\sharp}$. From Lemma 22.i there exist an edge $e^{\sharp}_{\mathrm{ins}} \in E(T(M^{\sharp}))$ such that

$$\mathsf{Com}(\mathsf{Int}(\hat{M}^{\sharp}, \hat{e}^{\sharp}_{\mathrm{ins}}, m^{\sharp}, e_p, W)) = \mathsf{Com}(\mathsf{Int}(M^{\sharp}, e^{\sharp}_{\mathrm{ins}}, 1, e_p, W)). \tag{27}$$

$$
\begin{tikzcd}[column sep=large]
B', M' = \mathrm{Des}_{X_i}(B') \arrow[r, "\mathsf{Com}(M')"] & \hat{M}' = C_{X_i}(B') \in FS(i,p) \\
B, M = \mathrm{Des}_{X_i}(B) \arrow[u, "\mathsf{Int}(M, e_{\text{ins}}, 1, e_p, W)"] \arrow[r, "\mathsf{Com}(M')"] & \hat{M} = C_{X_i}(B) \in FS(i,p-1) \arrow[u, "\mathsf{Com}(\mathsf{Int}(\hat{M}, \hat{e}_{\text{ins}}, m, e_p, W))"']
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=large]
& & FS(i,p) \arrow[d, phantom, "\cup"] & \\
B', M' = \mathrm{Des}_{X_i}(B') \arrow[r, "\mathsf{Com}(M')"] & \hat{M}' = C_{X_i}(B') \arrow[r, phantom, "\succ"] & \hat{M}'^{\sharp} = C_{X_i}(B'^{\sharp}) \arrow[r, "\mathsf{Com}(M')"] & B'^{\sharp}, M'^{\sharp} = \mathrm{Des}_{X_i}(B'^{\sharp}) \\
B, M = \mathrm{Des}_{X_i}(B) \arrow[u, "\mathsf{Int}(M, e_{\text{ins}}, 1, e_p, W)"] \arrow[r, "\mathsf{Com}(M')"] & \hat{M} = C_{X_i}(B) \arrow[u, "\mathsf{Com}(\mathsf{Int}(\hat{M}, \hat{e}_{\text{ins}}, m, e_p, W))"] \arrow[r, phantom, "\succ"] & \hat{M}^{\sharp} = C_{X_i}(B^{\sharp}) \arrow[u, "\mathsf{Com}(\mathsf{Int}(\hat{M}^{\sharp}, \hat{e}^{\sharp}_{\text{ins}}, m^{\sharp}, e_p, W))"] \arrow[r, "\mathsf{Com}(M')"] \arrow[d, phantom, "\cap"] & B^{\sharp}, M^{\sharp} = \mathrm{Des}_{X_i}(B^{\sharp}) \arrow[u, "\mathsf{Int}(M^{\sharp}, e^{\sharp}_{\text{ins}}, 1, e_p, W)"] \\
& & FS(i,p-1) &
\end{tikzcd}
$$

Figure 9: A scheme for the proof of Lemma 24.

Moreover, from Lemma 20 we have that

$$C_{X_i}(B'^{\sharp}) \quad = \quad \mathsf{Com}(\mathsf{Int}(M^{\sharp}, e^{\sharp}_{\text{ins}}, 1, e_p, W)). \tag{28}$$

From (28) and algorithm Introduce-edge we have that $C_{X_i}(B'^{\sharp}) \in FS(i,p)$. Finally, from (24), (25), (26), (27), and (28), we have that $C_{X_i}(B'^{\sharp}) \prec C_{X_i}(B')$. $\square$

3.5 A full set for a forget node

We will now consider the case where i is a **forget** node. Let j be the child of i.

Clearly, $G_i = G_j$ and there exists a unique vertex $v \in X_j$ with $v \notin X_i$. We call this vertex v *forgotten*. Given a full set of characteristics $F(j)$ for j, the following algorithm computes a full set of characteristics $F(i)$ at i.

ALGORITHM Forget-Vertex

Input: A full set of characteristics $FS(j)$ for G_j and a forgotten vertex x.
Output: A full set of characteristics $FS(i)$ for G_i.

1: Initialize $FS(i) = \emptyset$.
2: For any $(T, \alpha, \beta, \gamma) \in FS(j)$ set $FS(i) \leftarrow FS(i) \cup \{\mathsf{Com}(T, \alpha - \{x\}, \beta - \{x\}, \gamma)\}$.
3: end.

Lemma 25 *The set $FS(i)$ constructed by the above algorithm is a full set of characteristics for i.*

Proof. We will prove first that $FS(i)$ is a set of characteristics. Let $(\hat{T}', \hat{\alpha}', \hat{\beta}', \hat{\gamma}') \in FS(i)$. We will prove that there is a branch decomposition of G_i with this characteristic. From algorithm Forget-Vertex, there exist some $(\hat{T}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}) \in FS(j)$ such that

$$\mathsf{Com}(\hat{T}, \hat{\alpha} - \{x\}, \hat{\beta} - \{x\}, \gamma) \quad = \quad (\hat{T}', \hat{\alpha}', \hat{\beta}', \hat{\gamma}'). \tag{29}$$

As $(\hat{T}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}) \in FS(j)$ there will exist a branch decomposition $B = (T, \theta)$ of G_j where $C_{X_j}(B) = (\hat{T}, \hat{\alpha}, \hat{\beta}, \hat{\gamma})$. Let $\mathrm{Des}_{X_j}(B) = (T, \alpha, \beta, \gamma)$. Clearly, $(\hat{T}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}) \sqsubset (T, \alpha, \beta, \gamma)$ and it is easy to see that

$$(\hat{T}, \hat{\alpha} - \{x\}, \hat{\beta} - \{x\}, \hat{\gamma}) \quad \sqsubset \quad (T, \alpha - \{x\}, \beta - \{x\}, \gamma). \tag{30}$$

Notice that $B' = B$ is a branch decomposition of G_i as well. We claim that $C_{X_i}(B') = (\hat{T}', \hat{\alpha}', \hat{\beta}', \hat{\gamma}')$. Let $\mathrm{Des}_{X_i}(B') = (T', \alpha', \beta', \gamma')$. From the definition of $\mathrm{Des}_{X_i}(B')$ we have that $T' = T$, $\alpha' = \alpha - \{x\}$, $\beta = \beta' - \{x\}$, and $\gamma' = \gamma$. Therefore, we have that

$$\mathsf{Com}(T', \alpha', \beta', \gamma) \quad = \quad \mathsf{Com}(T, \alpha - \{x\}, \beta - \{x\}, \gamma). \tag{31}$$

Using now (29), (30), and (31) we conclude that $C_{X_i}(B') = \mathsf{Com}(T, \alpha', \beta', \gamma')$.

Next we prove that $FS(i)$ is a full set of characteristics. Let $B' = (T', \theta')$ be a branch decomposition of G_i. We will show that there exists a branch decomposition $B'^{\sharp}$ of G_i such that $C_{X_i}(B'^{\sharp}) \prec C_{X_i}(B')$ and $C_{X_i}(B'^{\sharp}) \in FS(i)$. We set $B =$

$(T,\theta) \leftarrow B'$ and observe that B is also a branch decomposition of G_j. We set $\text{Des}_{X_i}(B) = (T', \alpha', \beta', \gamma')$, $\text{Des}_{X_j}(B) = (T, \alpha, \beta, \gamma)$, and $C_{X_j}(B) = (\hat{T}, \hat{\alpha}, \hat{\beta}, \hat{\gamma})$. Notice that relations (30) and (31) hold as well and finally we obtain

$$C_{X_i}(B') \quad = \quad \mathsf{Com}(\hat{T}, \hat{\alpha} - \{x\}, \hat{\beta} - \{x\}, \hat{\gamma}). \tag{32}$$

As $FS(j)$ is a full set of characteristics, there exist a branch decomposition $B^\sharp = (T^\sharp, \theta^\sharp)$ of G_j such that if $C_{X_j}(l^\sharp) = (T^\sharp, \hat{\alpha}^\sharp, \hat{\beta}^\sharp, \hat{\gamma}^\sharp)$ then $(\hat{T}^\sharp, \hat{\alpha}^\sharp, \hat{\beta}^\sharp, \hat{\gamma}^\sharp) \prec (\hat{T}, \hat{\alpha}, \hat{\beta}, \hat{\gamma})$. It is now easy to verify that

$$(\hat{T}^\sharp, \hat{\alpha}^\sharp - \{x\}, \hat{\beta}^\sharp - \{x\}, \hat{\gamma}^\sharp) \quad \prec \quad (\hat{T}, \hat{\alpha} - \{x\}, \hat{\beta} - \{x\}, \hat{\gamma}). \tag{33}$$

We now set $B'^\sharp = (T'^\sharp, \theta'^\sharp) \leftarrow B^\sharp$ and notice that $B'^\sharp$ is a branch decomposition of G_i as well. We set $\text{Des}_{X_i}(B'^\sharp) = (T'^\sharp, \alpha'^\sharp, \beta'^\sharp, \gamma'^\sharp)$ and $\text{Des}_{X_j}(B^\sharp) = (T^\sharp, \alpha^\sharp, \beta^\sharp, \gamma^\sharp)$. Notice also that relations (30) and (31) hold also for the case of $B^\sharp$, $(T'^\sharp, \alpha'^\sharp, \beta'^\sharp, \gamma'^\sharp)$, $(T^\sharp, \alpha^\sharp, \beta^\sharp, \gamma^\sharp)$, $(\hat{T}^\sharp, \hat{\alpha}^\sharp, \hat{\beta}^\sharp, \hat{\gamma}^\sharp)$ and therefore we obtain the following.

$$C_{X_i}(B'^\sharp) \quad = \quad \mathsf{Com}(T^\sharp, \hat{\alpha}^\sharp - \{x\}, \hat{\beta}^\sharp - \{x\}, \hat{\gamma}^\sharp) \tag{34}$$

From Algorithm $\mathsf{Forget\text{-}Vertex}$ and (34) we have that $C_{X_i}(B'^\sharp) \in FS(i)$. Finally, from relations (32), (33), and (34) we can easily conclude that $C_{X_i}(B'^\sharp) \prec C_{X_i}(B')$.

3.6 Joining branch models

Let $M_i = (T_i, \alpha_i, \beta_i, \gamma_i), i = 1, 2$ be two branch models of a graph G rooted on some set $R \subseteq V(G)$. Assume also that $\sigma \in \mathcal{I}(T(M_1), T(M_2))$. Let T be a tree isomorphic to $T(M_1)$ through a function ρ. We define function $\mathsf{both}_\alpha : E(T) \rightarrow 2^{R-A(G)-I(G)}$ such that $\forall_{e \in E(T)}\ \mathsf{both}_\alpha(e) = \alpha(\rho(e)) \cap \alpha(\sigma(\rho(e)))$. We also define the function $\gamma_1 \otimes_{\sigma,\rho} \gamma_2 : E(T) \rightarrow \hat{\mathcal{S}}$ such that

$$\gamma_1 \otimes_{\sigma,\rho} \gamma_2 = \{\gamma \mid \forall_{e \in E(T)}\ \gamma(e) + |\mathsf{both}_\alpha(e)| \in \gamma_1(\rho(e)) \otimes \gamma_2(\sigma(\rho(e)))\}$$

PROCEDURE $\mathsf{Join}(M_1, M_2, \sigma)$

Input Two dense and typical branch models $M_i = (T, \alpha_i, \beta_i, \gamma_i), i = 1, 2$ of G rooted on R and an isomorphism $\sigma \in \mathcal{I}(T(M_1), M_2$.

Output A collection $\mathcal{M}$ of dense and typical branch models of G rooted on R.

1: (*Interleaving step*)

- Set $\Gamma = \{\gamma \in (\gamma_1 \otimes_{\sigma,\rho} \gamma_2)\}$.
- Set $\mathcal{M} = \{(T, (\alpha_1 \cup_\sigma \alpha_2) \circ \rho, (\beta_1 \cup_\sigma \beta_2) \circ \rho, \tau(\gamma)) \circ \rho \mid \gamma \in \Gamma\}$.

2: (*Normalizing step*) For any $M = (T, \alpha, \beta, \gamma) \in \mathcal{M}$ and any $x \in V(M)$, set $M \leftarrow \mathsf{Norm}(M, x)$.

3: Output $\mathcal{M}$.

4: end.

3.7 A full set for a join node

Let M be a dense and typical branch model of a graph G rooted on some set $R \subseteq V(G)$.

Given an integer d, we call $\mathcal{D}_d(M)$ the set of all the dense branch models that are predecessors of M and have underlying trees with at most d edges. Let $M_i, i = 1, 2$ be two branch models. We call an isomorphism $\sigma \in \mathcal{I}(T(M_1), T(M_2))$ *regular* if $\forall_{t \in \tilde{A}(T(M_1))}\ \beta_1(t) \cap \beta_2(\sigma(t)) = \emptyset$.

Algorithm Join-characteristics

Input: A full set of characteristics $FS(q_1)$ for G_{q_1} and a full set of characteristics $FS(q_2)$ for G_{q_2}.
Output: A full set of characteristics $FS(p)$ for G_p.

1: Initialize $FS(p) = \emptyset$.
2: For any pair of characteristics $\hat{M}_1 \in FS(q_1), \hat{M}_2 \in FS(q_2)$ apply step (**3**).
3: For any pair $\hat{M}_i^* \in \mathcal{D}_{3k\delta(k)}(\hat{M}_i), i = 1, 2$ apply step (**4**).
4: For any regular isomorphism $\sigma \in \mathcal{I}(T(\hat{M}_1^*), T(\hat{M}_2^*))$ apply step (**5**).
5: For any branch model $\hat{M}^* \in \mathsf{Join}(\hat{M}_1^*, \hat{M}_2^*, \sigma)$ where $\mathrm{width}(\hat{M}^*) \leq k$ set $FS(p) \leftarrow FS(p) \cup \{\mathsf{Com}(\hat{M}^*)\}$.
6: end.

In what follows, we will prove that the set $FS(p)$ constructed by the above algorithm is a set of characteristics.

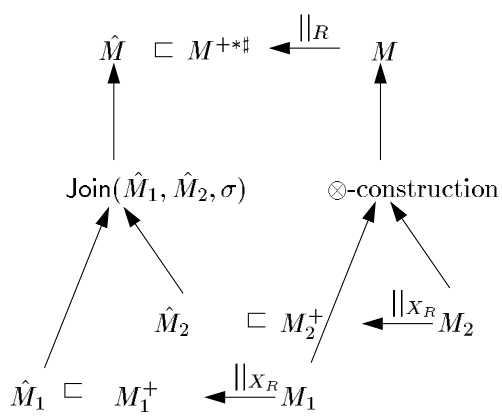


Figure 10: An illustration of Lemma 26

Lemma 26 *Let G be a graph and let G_1, G_2 be graphs such that $V(G_1) \cap V(G_2) = R$ and $E(G_1) \cap E(G_2) = \emptyset$. Suppose that for $i = 1, 2$, $M_i, \hat{M}_i$ are branch models of G_i, rooted on $V(G)$ and R respectively. Then if $M_i, i = 1, 2$ are entire, $\hat{M}_i \sqsubset$*

$M_i\|_R, i = 1,2$, *and* $\hat{M} \in \mathsf{Join}(\hat{M}_1, \hat{M}_2, \sigma)$, *where* σ *is a regular isomorphism in* $\mathcal{I}(T(\hat{M}_1), T(\hat{M}_2))$, *there exist an entire branch model* M *of* G *rooted on* $V(G)$ *such that* $\hat{M} \sqsubset M\|_R$.

Proof We set $M_i^+ = M_i\|_R, i = 1,2$. As $\hat{M}_i \sqsubset M_i^+, i = 1,2$, we can apply lemma 8 and define functions $\psi_{\hat{M}_i, M_i^+}$, $\omega_{\hat{M}_i, M_i^+}$ and we set $\psi_i \leftarrow \psi_{\hat{M}_i, M_i^+}, i = 1,2$ and $\omega_i \leftarrow \omega_{\hat{M}_i, M_i^+}, i = 1,2$. Notice that the underlining tree of M_i is the same with the underlining tree of $M_i^+, i = 1,2$ and, in this way, ψ_i and $\omega_i, i = 1,2$ can also be viewed as an one to one mapping between the same subtree families $\mathcal{T}(\psi_1)$ and $\mathcal{T}(\psi_1)$ of the underlining trees of M_1 and M_2 respectively. We denote $M_i = (T_i, \alpha_i, \beta_i, \gamma_i), i = 1,2$ and $\hat{M}_i = (\hat{T}_i, \hat{\alpha}_i, \hat{\beta}_i, \hat{\gamma}_i), i = 1,2$. Finally, we define

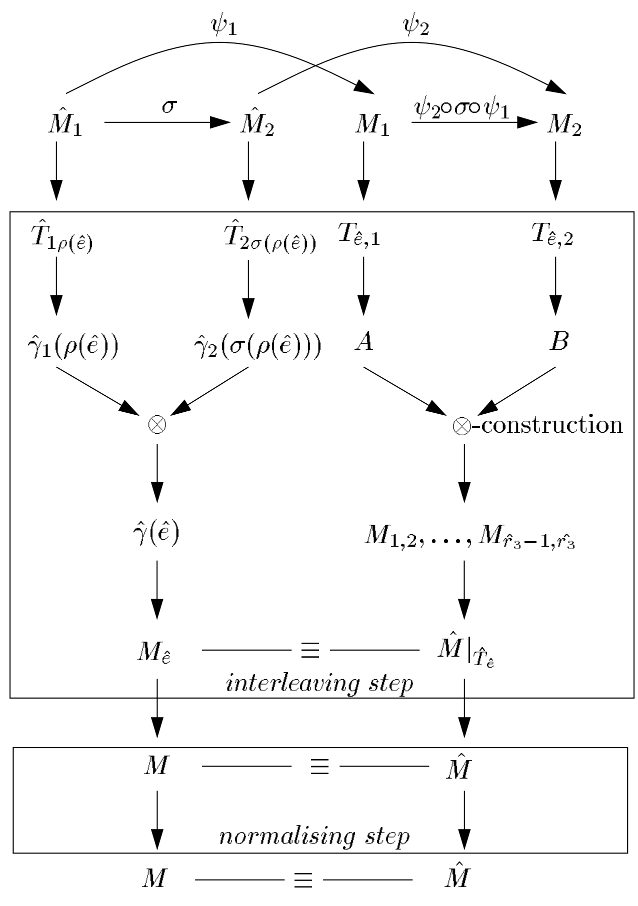


Figure 11: An illustration of the proof of Lemma 26

the function $\theta : \mathcal{T}(\psi_1) \rightarrow \mathcal{T}(\psi_2)$ so that $\theta = \psi_2 \circ \sigma \circ \psi_1^{-1}$. Observe that θ is a one to one function mapping each tree of $\mathcal{T}(\psi_2)$ to a tree of $\mathcal{T}(\psi_1)$.

Let $\hat{M} = (\hat{T}, \hat{\alpha}, \hat{\beta}, \hat{\gamma})$ be the the branch representation that occurred after the application of the *interleaving step* on $\hat{M}_i, i = 1, 2$ (notice that this branch representation is not necessarily a branch model).

Let $\hat{e}$ be an edge of $\hat{T}$. We observe that the values of $\hat{\gamma}$ are formulated by applying τ on some choice of a function γ in Γ and therefore a function $\gamma \in \hat{\gamma}_1 \otimes_{\sigma,\rho} \hat{\gamma}_2$ (i.e. $\hat{\gamma} = \tau(\gamma)$). Clearly, the sequence $\gamma(\hat{e}) + |\mathsf{both}_{\hat{\alpha}}(\hat{e})|$ is a member of $\gamma_1(\rho(\hat{e})) \otimes \gamma_2(\sigma(\rho(\hat{e})))$ (recall that $|\gamma(\hat{e})| = |\hat{\gamma}_1(\hat{e})| + |\hat{\gamma}_2(\hat{e})| - 1$).

We set $T_{\hat{e},i} = \psi_i(\hat{T}_{i\hat{e}}), i = 1, 2,$ and if $\hat{e} = \{\hat{v}, \hat{u}\}$ then for $i = 1, 2$ we set $v_i = \psi_i(\hat{v})$ and $u_i = \psi_i(\hat{u})$. Notice that, the numbers of the (single number) sequences of the values of γ_1 (γ_2) along the edges of the path $P^1_{\hat{e}}$ ($P^2_{\hat{e}}$) connecting u_1 and v_1 (u_2 and v_2) in $T_{\hat{e},1}$ ($T_{\hat{e},2}$) define a sequence A (B) where $\tau(A) = \hat{\gamma}_1(\hat{e})$ ($\tau(B) = \hat{\gamma}_2(\hat{e})$). Let $A = (a_1, \ldots, a_{r_1})$ and $B = (b_1, \ldots, b_{r_2})$. Using now the sequences A and B and the way $\gamma(\hat{e})$ occurs from $\hat{\gamma}_1(\hat{e})$ and $\hat{\gamma}_2(\hat{e})$, we will construct a complete branch representation $M_{\hat{e}}$ of G rooted on $V(M_1|_{T_{\hat{e},1}}) \cup V(M_2|_{T_{\hat{e},2}})$ and such that $C_X(M_{\hat{e}}) \equiv \hat{M}|_{\hat{T}_{\hat{e}}}$.

Let $\hat{\gamma}_1(\hat{e}) = (\hat{a}_1, \ldots, \hat{a}_{\hat{r}_1})$, $\hat{\gamma}_2(\hat{e}) = (\hat{b}_1, \ldots, \hat{b}_{\hat{r}_2})$, and $\gamma(\hat{e}) = (\hat{c}_1, \ldots, \hat{c}_{\hat{r}_3})$ ($\hat{r}_3 = \hat{r}_1 + \hat{r}_2 - 1$). W.l.o.g. we assume that $\hat{c}_1 = \hat{a}_1 + \hat{b}_1$ and $\hat{c}_2 = \hat{a}_1 + \hat{b}_2$ (otherwise, $\hat{c}_1 = \hat{a}_1 + \hat{b}_1$ and $\hat{c}_2 = \hat{a}_2 + \hat{b}_1$, which can be faced in the same way). We set $i_{A,1} = i_{B,1} = 1$. Notice that there exist a subsequence $B_{1,2} = (b_{i_{B,1}}, \ldots, b_{i_{B,2}})$ of B such that $\tau(B_{1,2}) = (\hat{b}_1, \hat{b}_2)$ and $\tau(b_{i_{B,2}+1}, \ldots, b_{r_2}) = (\hat{b}_2, \ldots, \hat{b}_{\hat{r}_2})$. Suppose that $(\check{T}, \check{\alpha}, \check{\beta}, \check{\gamma}) = M_2|_{T_{\hat{e},2}(u_{i_{B,1}}, u_{i_{B,2}+1})}$ where $u_1 = u_{i_{B,1}}$ and $\{u_{i_{B,2}}, u_{i_{B,2}} + 1\}$ is $i_{B,2}$th edge of $P^2_{\hat{e}}$. We set $M_{1,2} = (\check{T}, \check{\alpha} \cup \alpha_1(e_1), \check{\beta}, \check{\gamma} + |\alpha_1(e_1)| - |\mathsf{both}_{\hat{\alpha}}(\hat{e})|)$ where $e_1 = u_{i_{B,1}}$ is the first edge of $P^1_{\hat{e}}$. In the underlining tree of $M_{1,2}$ we call edge $\{v_{i_{B,1}}, v_{i_{B,1}+1}\}$ *first* and edge $\{v_{i_{B,2}-1}, v_{i_{B,2}}\}$ *last*. We also denote them as $\mathsf{first}(M_{1,2})$ and $\mathsf{last}(M_{1,2})$. We now observe that either $\hat{c}_3 = \hat{a}_2 + \hat{b}_2$ or $\hat{c}_3 = \hat{a}_1 + \hat{b}_3$ (in the example of Figure 3.7 we have the first case). In any case, we work as in the case of $M_{1,2}$ and we now construct the branch representation $M_{2,3}$. Moreover we notice that, in both cases, $M_{2,3}|_{T(M_{2,3})_{\mathsf{first}(M_{2,3})}} = M_{1,2}|_{T(M_{1,2})_{\mathsf{last}(M_{1,2})}}$. Going on that way, we construct $M_{2,3}, \ldots, M_{\hat{r}_3-1,\hat{r}_3}$ and observe that $\forall_{2 \leq i \leq \hat{r}_3-1}\ M_{i,i+1}|_{T(M_{i,i+1})_{\mathsf{first}(M_{i,i+1})}} = M_{i-1,i}|_{T(M_{i-1,i})_{\mathsf{last}(M_{i-1,i})}}$. Therefore, for $i = 2, \ldots, \hat{r} - 1$, we can consider edge $\mathsf{first}(M_{i,i+1})$ and edge $\mathsf{last}(M_{i-1,i})$ as the same edge and subsequently we can set $M_{\hat{e}} = M_{1,2} \cup \cdots \cup M_{\hat{r}_3-1,\hat{r}_3}$ (given two branch representations $M_i = (T_i, \alpha_i, \beta_i, \gamma_i), i = 1, 2$ where T_1, T_2 have a common pendant edge e containing an internal leaf and such that $\alpha_1(e) = \alpha_2(e)$ and $\gamma_1(e) = \gamma_2(e)$, we define $M_1 \cup M_2 = (\alpha_1 \cup \alpha_2, \beta_1 \cup \beta_2, \gamma_1 \cup \gamma_2)$). Suppose that $M_{\hat{e}} = (T_{\hat{e}}, \alpha_{\hat{e}}, \beta_{\hat{e}}, \gamma_{\hat{e}})$. Let also $A_1, \ldots, A_{\hat{r}_3}$ ($C_1, \ldots, C_{\hat{r}_3}$)

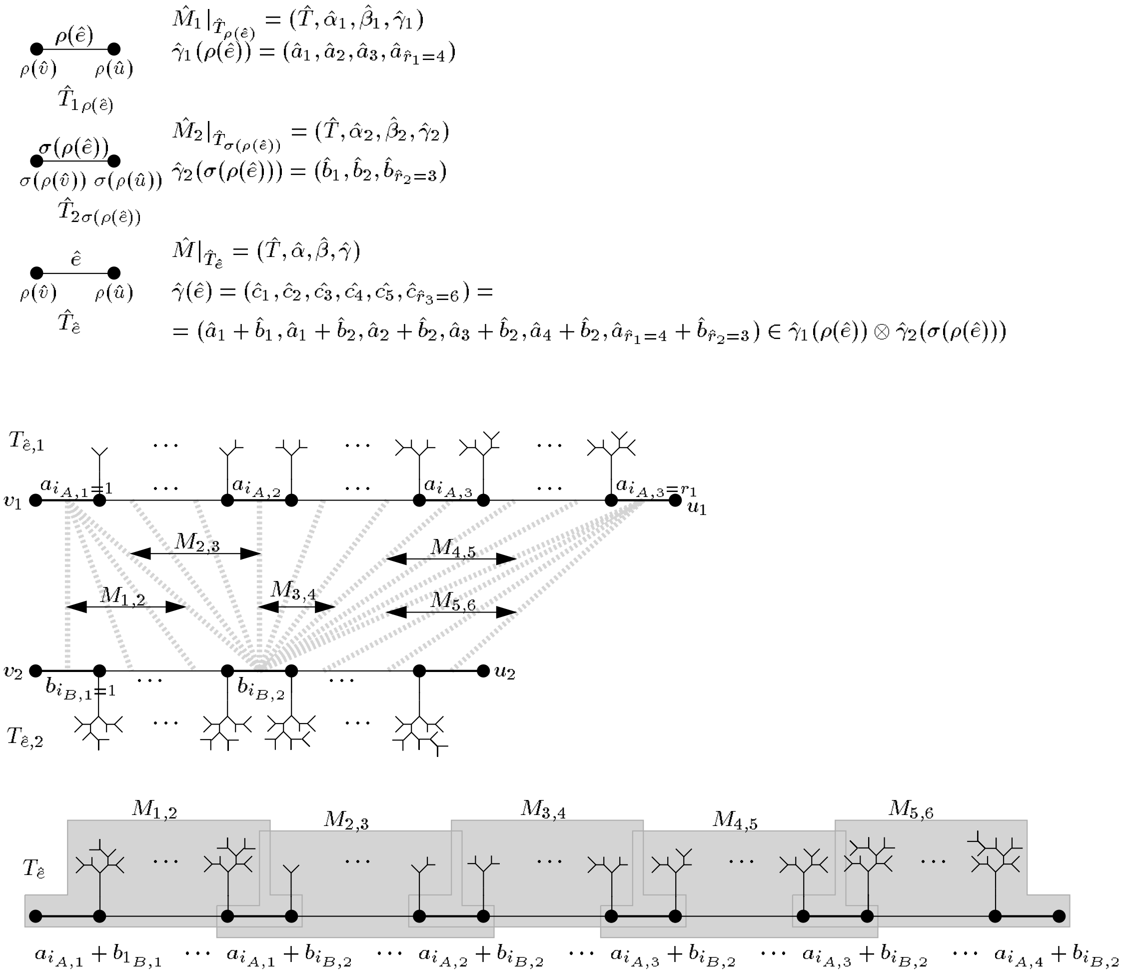


Figure 12: An example of the "$\oplus$-construction" in the proof of Lemma 26

be the values of $\alpha_{\hat{e}}$, $(\gamma_{\hat{e}})$ along the path of $T_{\hat{e}}$ connecting $\mathsf{first}(M_{1,2})$ and $\mathsf{last}(M_{\hat{r}_3-1,\hat{r}_3})$. We notice that $\forall_{1\leq i\leq \hat{r}_3}\ A_i \cap X = \hat{\alpha}(\hat{e})$ and, by the way the sequence $C_1, \ldots, C_{\hat{r}_3}$ has been constructed, one can easily verify that $\gamma(\hat{e}) \sqsubset C_1 \oplus \cdots \oplus C_{\hat{r}_3}$. Using these two facts, it is not hard to see that $\hat{M}|_{\hat{T}_{\hat{e}}} \sqsubset M_{\hat{e}}$. If we now apply the above construction (we call it "$\otimes$-construction" – see Figure 3.7 for an example) for all the edges of $\hat{T}$ we will construct a collection of branch models that, when merged using Lemma 11, result to a complete branch representation M where $\hat{M} \sqsubset M$.

Using Lemma 13 one can prove that $\hat{M} \sqsubset M$ holds also after the application of the *normalizing step* to M and $\hat{M}$ respectively. Moreover, using the facts that σ is regular, G_1, G_2 have not edges in common, and that $M_i, i = 1, 2$ are entire branch models, it can be easily proved that M is also an entire branch model of

G_r rooted on $V(G_r)$.(we omit the details as they are tedious and easy). $\square$

Lemma 27 *Let G be a graph and $M, \hat{M}, \hat{M}^*$ three branch models where $V(M) = V(G)$ and $V(\hat{M}) = V(\hat{M}^*) = R$ for some $R \subseteq V(G)$. Suppose also that $\hat{M} = C_R(M)$ and $\hat{M} \sqsubset \hat{M}^*$. Then, there exist a branch model M^* rooted on $V(G)$ such that $M \sqsubset M^*$ and $M^* \sqsubset M^*\|_R$.*

$$\begin{tikzcd}
\hat{M}^* \arrow[r, phantom, "\sqsubset"] & M^{+*} & M^* \arrow[l, "\|_{X_R}"'] \\
\hat{M} \arrow[u, dash, "\sqcup" description] & M^+ \arrow[l, "\mathsf{Com}"'] \arrow[u, dash, "\sqcup" description] & M \arrow[l, "\|_{X_R}"'] \arrow[u, dash, "\sqcup" description]
\end{tikzcd}$$

Figure 13: A scheme for Lemma 27

Proof Let $M^+ = M\|_R$. Clearly, M^+, $\hat{M}^*$ are both predecessors of $\hat{M}$. Using Lemma 9 we construct a common predecessor M^{+*} of them. As $M^+ \sqsubset M^{+*}$ we use Lemma 10 in order to construct a branch model M^* such that $M^{+*} = M^*\|_R$ and $M \sqsubset M^*$ and the lemma is proved. $\square$

Lemma 28 *The set $FS(p)$ constructed by the algorithm* Join-characteristics *is a set of characteristics.*

Proof. Let $\hat{M} \in FS(p)$. We will show that there exists a branch decomposition B of G_h with width $\leq k$ where $C_{X_p}(B) = \hat{M}$. Clearly, as $\hat{M}$ was constructed by the algorithm Join-characteristics, for $i = 1, 2$ there is a pair of characteristics $\hat{M}_i \in FS(q_i)$ of G_{q_i}, rooted on $X_{q_1} = X_{q_2} = X_p$, that where chosen during step (**1**) in order to construct $\hat{M}$. As $\hat{M}_i \in FS(q_i), i = 1, 2$, there will exist two branch decompositions $B_i, i = 1, 2$ of $G_{q_i}, i = 1, 2$ (both of width$\leq k$) such that $C_X(B_i) = \hat{M}_i, i = 1, 2$. Let $\hat{M}_i^*, i = 1, 2$ be the pair chosen in step (**3**). Using now Lemma 27 we construct two branch models $M_i^*, i = 1, 2$ that have width $\leq k$ and such that $M_i \sqsubset M_i^*$ and $\hat{M}^* \sqsubset M_i^*\|_{X_i}, i = 1, 2$. Let also σ be the regular isomorphism of $\mathcal{I}(T(\hat{M}_1^*), T(\hat{M}_2^*))$ chosen in step (**4**). Applying Lemma 26

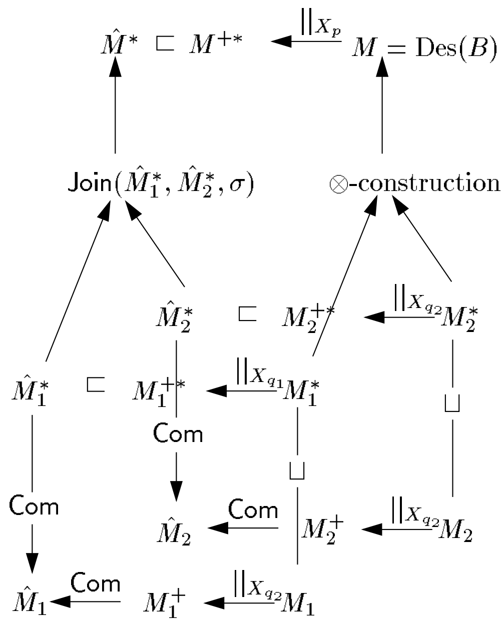


Figure 14: A scheme for Lemma 28

we have that there exist an entire branch model M^* of G rooted on $V(G)$ such that width$(M^*) \leq k$ and $\hat{M}^* \sqsubset M\|_{X_p}$. Clearly, this means that $\mathsf{Com}(\hat{M}^*) = \mathsf{Com}(M^*\|_{X_p})$. Moreover, Using Lemma 15 we construct a branch decomposition B of width $\leq k$ and such that $\mathrm{Des}(B) = M^*$. It is now easy to conclude that $\hat{M} = \mathsf{Com}(\hat{M}^*) = \mathsf{Com}(M^*\|_{X_p}) = C_{X_p}(B)$ and this completes the proof of the lemma. $\square$

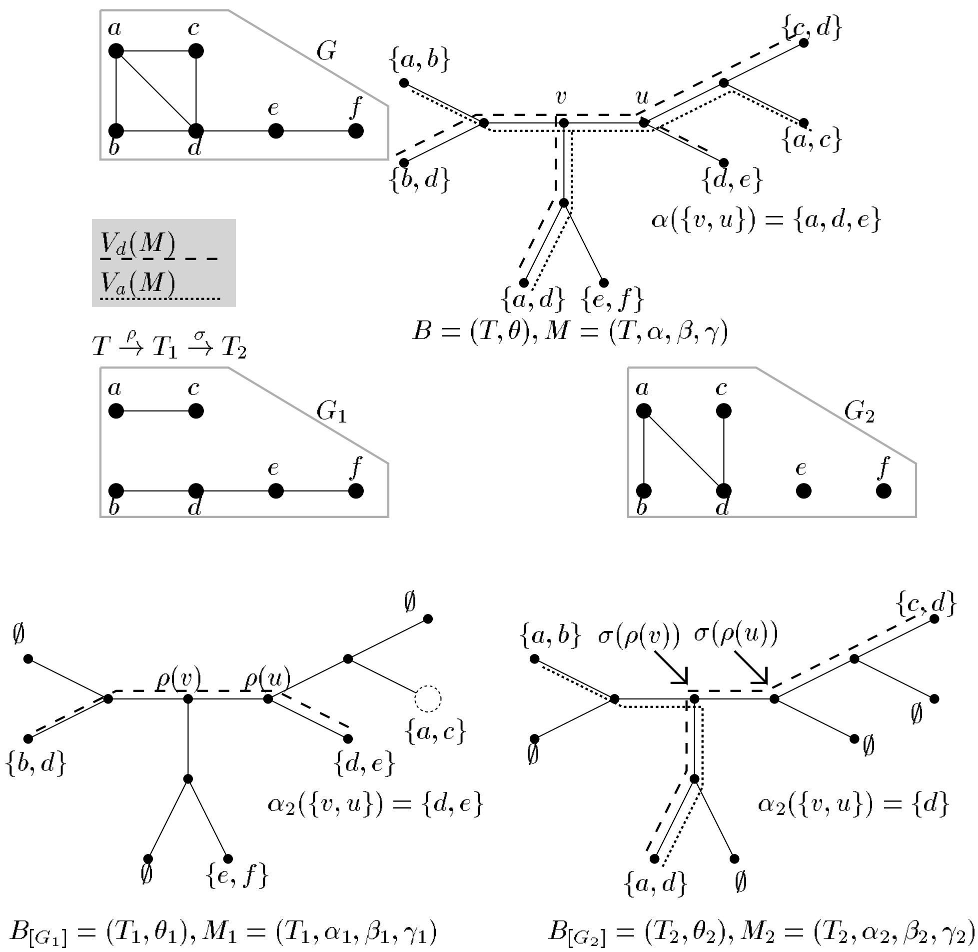
What now remains is to prove that $FS(p)$ is a full set of characteristics. We need first some definitions.

Let $B = (T, \theta)$ be a branch decomposition of a graph G and let $G_i, i = 1, 2$ be two graphs where $E(G_i) \cup E(G_2) = \emptyset$ and $V(G_1) \cap V(G_2) = R$. For $i = 1, 2$, we define the *natural restriction* $B_{[G_i]}$ *of* B *on* G_i as the branch decomposition (T, θ_i) such that

$$\forall_{t \in \tilde{A}(T)}\ \theta(t) \in E(G_i) \Rightarrow \theta_i(t) = \theta(t) \wedge \theta(t) \notin E(G_i) \Rightarrow \theta_i(t) = \emptyset. \tag{35}$$

We observe that T_1, T_2 are nothing more that isomorphic copies of T. In this way, each subtree U_i of T_i corresponds to a subtree U in T. We call U_i, *i-clone* of U and we call U *dote* of U_i. In the natural way, we define the notion of i-clone for any object referring to T_i such as vertices, edges, etc.

Notice that the isomorphism between T_1 and T_2 is regular as any vertex that is pendant in both M_1, M_2 must be a vertex of X that belongs to exactly two edges of G which in turn are mapped to two *different* leaves of T in B.



$\mathsf{Lost}_{M\to M_1,M_2}(\{v,u\}) = \alpha(\{v,u\}) - (\alpha_1(\rho(\{v,u\})) \cup \alpha_2(\sigma(\rho(\{v,u\})))) = \{a\}$

$\mathsf{both}_{\alpha}(\{v,u\}) = \alpha_1(\rho(\{v,u\})) \cap \alpha_2(\sigma(\rho(\{v,u\}))) = \{d\}$

Figure 15: An example of functions $\mathsf{Lost}_{M\to M_1,M_2}(e)$ and $\mathsf{both}_{\alpha}(e)$.

Let $M = \mathrm{Des}(B) = (T, \alpha, \beta, \gamma)$, and $M_i = \mathrm{Des}(B_{G_i}) = (T_i, \alpha_i, \beta_i, \gamma_i), i = 1, 2$. Let e be an edge of T and e_i the i-clone of e in $T_i, i = 1, 2$. Notice that $\alpha(e) \subseteq \alpha_1(e) \cup \alpha_2(e)$ (see figure 15). We define $\mathsf{Lost}_{M\to M_1,M_2}(e) = \alpha(e) - (\alpha_1(e) \cup \alpha_2(e))$. Notice that $\mathsf{Lost}(e) \subseteq X$. We will need the following lemma.

Lemma 29 *Let $B = (T, \theta)$ be a branch decomposition of a graph G and let $G_i, i = 1, 2$ be two graphs where $E(G_i) \cup E(G_2) = \emptyset$ and $V(G_1) \cap V(G_2) = R$. Let also*

$M = \mathrm{Des}(B)$ *and* $M_i = \mathrm{Des}(B_{[G_i]}), i = 1, 2$. *If* $e \in E(T), x \in R$ *such that* $x \in \mathsf{Lost}_{M \to M_1, M_2}(e)$ *then the dote of* $V_x(M_1)$ *and the dote of* $V_x(M_2)$ *are vertex sets each one belonging in different connected components of* $T - e$.

Proof Let L_x ($L_x^i, i = 1, 2$) be the leaves of T ($T_i, i = 1, 2$) that are mapped through θ ($\theta_i, i = 1, 2$) to edges containing to x in G ($G_i, i = 1, 2$). Using Lemma 14 it is easy to see that for any $V_x(M)$ ($V_x(M_i)$) induces the subtree U ($U_i, i = 1, 2$) of T ($T_i, i = 1, 2$) spanned by L_x ($L_x^i, i = 1, 2$). From Relation 35 and the fact that $E(G_i) \cup E(G_2) = \emptyset$, we have that $L_x = L_x^1 \cup L_x^2$ and $L_x^1 \cup L_x^2 = \emptyset$. The fact that $x \in \mathsf{Lost}_{M \to M_1, M_2}(e)$ means that e is not an edge of U_1 or U_2. As $U_i, i = 1, 2$ are connected, we have that their dotes belong in different connected components of $T - e$. $\square$

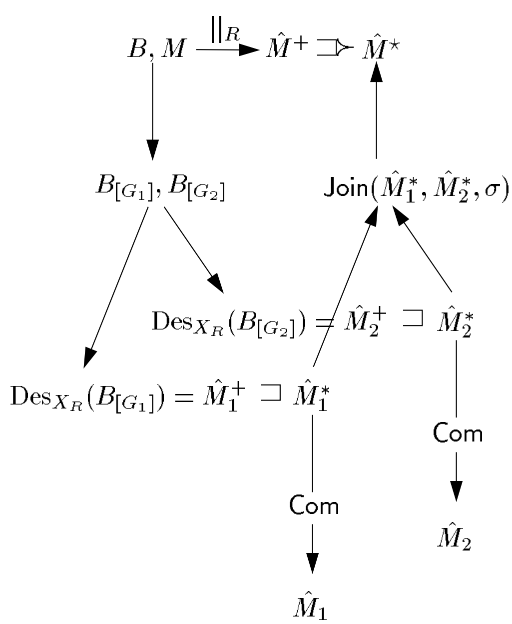


Figure 16: An illustration of Lemma 30

Lemma 30 *Let* $G_i, i = 1, 2$ *be two graphs where* $E(G_1) \cup E(G_2) = \emptyset$ *and* $V(G_1) \cap V(G_2) = R$. *Let* $B = (T, \theta)$ *be a branch decomposition of a graph* G *rooted on a set* $R \subseteq V(G)$. *Finally, let* $\hat{M}_i = C_R(B_{[G_i]}), i = 1, 2$. *Then there exist two branch models* $\hat{M}_i^*, i = 1, 2$ *of* $G_i, i = 1, 2$ *respectively, both rooted on* R *and such that* $\mathcal{I}(T(\hat{M}_1^*), T(\hat{M}_2^*)) \neq \emptyset$, *a regular isomorphism* $\sigma \in \mathcal{I}(T(\hat{M}_1^*), T(\hat{M}_2^*))$, *and a branch model* $\hat{M}^*$ *of* G *rooted on* R *where*

1. $\hat{M}_i^* \in \mathcal{D}_{3k\delta(k)}(\hat{M}_i), i = 1, 2$

2. $\hat{M}^\star \in \mathsf{Join}(\hat{M}_1^*, \hat{M}_2^*, \sigma)$.

3. $\hat{M}^\star \prec\!\sqsubset \mathrm{Des}_R(B)$.

Proof We set $\mathrm{Des}_R(B) = M^+ = (T, \alpha^+, \beta^+, \gamma^+)$, $\mathrm{Des}_R(B_{[G_i]}) = M_i^+ = (T_i, \alpha_i^+, \beta_i^+, \gamma_i^+), i = 1, 2$, $\hat{M} = C_R(B) = (\hat{T}, \hat{\alpha}, \hat{\beta}, \hat{\gamma})$. Notice that T^+ is isomorphic to T_1^+ through an isomorphism ρ. Moreover there exist a regular isomorphism σ from T_1^+ to T_2^+.

As $\hat{M} \sqsubset M^+$ we can apply Lemma 8 and define functions ψ, ω such that $\psi = \psi_{\hat{M},M^+}$ and $\omega = \omega_{\hat{M},M^+}$. As we in the proof of Lemma 26 we observe that M and M^+ have the same underlying tree T, and therefore, the functions ψ and ω can be viewed as mappings of edges/vertices of $\hat{T}$ to trees/vertices of M as well.

We consider an edge $\hat{e} = (\hat{v}, \hat{u}) \in E(\hat{T})$. We set $T_{\hat{e}} = \psi(\hat{T}_{\hat{e}})$ and $v = \omega(\hat{v}), u = \omega(\hat{u})$. Let $P_{\hat{e}} = (e_1, \ldots, e_{r_{\hat{e}}})$ the path in $T_{\hat{e}}$ connecting v and u (we set $r_{\hat{e}} = |P_{\hat{e}}|$ and we assume that $v \in e_1, u \in e_r$). We denote as $R_{\hat{e}}$ all the edges of $T_{\hat{e}}$ that have not common vertices with edges in $P_{\hat{e}}$ and we call them *$\hat{e}$-raked edges*. Notice that these edges are some – but not all – of the edges that should be eliminated by the Com procedure towards constructing $\hat{T}$ from T^+ and, in particular, constructing $\hat{M}\|_{\hat{T}_{\hat{e}}}$ from $M^+\|_{T_{\hat{e}}}$). Taking in mind the definition of M^+, M_1^+, M_2^+,, and Lemma 29 it easy to verify that $\forall_{e_i \in P_{\hat{e}}}$ the following holds.

$$\begin{aligned} \gamma^+(e_i) &= \gamma_1^+(\rho(e_i)) + \gamma_2^+(\sigma(\rho(e_i))) - \\ & \quad |\alpha_1^+(\rho(e_i)) \cap \alpha_2^+(\sigma(\rho(e_i)))| + |\mathsf{Lost}_{M \to M_1, M_2}(e_i) \cap R| \qquad (36) \\ \alpha^+(e_i) &= \alpha_1^+(\rho(e_i)) \cup \alpha_2^+(\sigma(\rho(e_i))) \cup (\mathsf{Lost}_{M \to M_1, M_2}(e_i) \cap R). \qquad (37) \end{aligned}$$

Moreover, as $\mathsf{Com}(M^+\|_{T_{\hat{e}}}) = \hat{T}_{\hat{e}}$, we have the following

$$\forall_{e_i \in P_{\hat{e}}}\ \alpha^+(e_i) = \hat{\alpha}(\hat{e}). \qquad (38)$$

For any $x \in \hat{\alpha}(\hat{e})$, we denote as I_x^i the subinterval $[t_1^i, t_2^i]$ of $[1, r_{\hat{e}}]$ such that $x \in \alpha_1^+(\rho(e_j)) \Leftrightarrow t_1^i \leq j \leq t_2^i$ and $x \in \alpha_2^+(\sigma(\rho(e_j))) \Leftrightarrow t_1^i \leq j \leq t_2^i$. We now set $I_x^{\mathrm{lost}} = \{i \mid x \in \mathsf{Lost}(e_i)\}$ and we finally observe that $\forall_{x \in R}\ I_x^1 \cup I_x^2 \cup I_x^{\mathsf{Lost}} = [1, \ldots, r]$. From Relations (37) and (38) we have that if $I_x^1 \cap I_x^2 \neq \emptyset$ then $I_x^{\mathsf{Lost}} = \emptyset$, otherwise $I_x^1, I_x^2, I_x^{\mathsf{Lost}}$ form a partition of $[1, \ldots, r_{\hat{e}}]$. It is now easy to see that we can partition $P_{\hat{e}} = (e_1, \ldots, e_{r_{\hat{e}}})$ into a collection $\mathcal{E}_{\hat{e}} = \{E_1, \ldots, E_{|\mathcal{E}_{\hat{e}}|}\}$ of at most $3|\hat{\alpha}(\hat{e})| \leq 3|R|$ subpaths such that

$$\forall_{1 \leq m \leq |\mathcal{E}_{\hat{e}}|} \forall_{e, e' \in E_m} \alpha_1^+(\rho(e)) = \alpha_1^+(\rho(e')) \qquad (39)$$

$$\forall_{1\leq m\leq|\mathcal{E}_{\hat{e}}|}\forall_{e,e'\in E_m}\alpha_2^+(\sigma(\rho(e))) = \alpha_2^+(\sigma(\rho(e'))) \text{ and} \quad (40)$$

$$\forall_{1\leq m\leq|\mathcal{E}_{\hat{e}}|}\forall_{e,e'\in E_m}(\mathsf{Lost}_{M\to M_1,M_2}(e)\cap R) = (\mathsf{Lost}_{M\to M_1,M_2}(e')\cap R) \quad (41)$$

i.e. along these subpaths the values of $\alpha_i^+, i = 1,2$ and $\mathsf{Lost}_{M\to M_1,M_2}$ are the same for $i = 1,2$. For the case of the values of $\mathsf{Lost}_{M\to M_1,M_2}$ we set $L(\hat{e},m) = (\mathsf{Lost}_{M\to M_1,M_2}(e)\cap R)$ where e is an arbitrary edge of $P_m \in \mathcal{E}_{\hat{e}}$ i.e. of the mth subpath of $\mathcal{E}_{\hat{e}}$.

Given an edge $\hat{e}\in E(T(\hat{M}))$ we define as $E_{\hat{e},i}$ the ith subpath of $\mathcal{E}_{\hat{e}}$.

We now construct an ancestor $\hat{M}^*$ of M^+ by applying rake and compress operations in two steps. The first allows only rake and compress operations for forks or spines containing only non-central edges. The second step involves only compress operations for spines where all of their edges belong to some set $E_{i,\hat{e}}$ for some $\hat{e}\in E(T(\hat{M}))$ and some $i, 1\leq i\leq|\mathcal{E}_{\hat{e}}|$. Notice that the above rake and compress steps will reduce each subtree $T_{\hat{e}}\in\mathcal{T}(\psi_{\hat{M},M}), \hat{e}\in E(T(\hat{M}))$ to a ternary caterpillar containing $|\mathcal{E}_{\hat{e}}|$ edges. We set $\hat{M}^* = (\hat{T}^*,\hat{\alpha}^*,\hat{\beta}^*,\hat{\gamma}^*)$.

We further construct for $i = 1,2$ and ancestor $\hat{M}_i^*$ of M_i^+ by applying exactly the same order of rake and compress operations that we described before in order to construct $\hat{M}^*$ from M^+. It is easy to see that this construction is doable, as $M_i, i = 1,2$ are the descriptions of the natural restriction of B to $G_i, i = 1,2$ respectively. Moreover, the underlining trees $\hat{T}_i^*, i = 1,2$ of $\hat{M}_i^*, i = 1,2$ remain isomorphic under a regular isomorphism $\hat{\sigma}$ (we omit the details as they are easy and straightforward). Finally it is clear that there exist a isomorphism $\hat{\rho}$ from $\hat{T}^*$ to $\hat{T}_1^*$.

Notice that $\hat{M}\sqsubset\hat{M}^*\sqsubset M^+$ and $\hat{M}_i\sqsubset\hat{M}_i^*\sqsubset M_i^+, i = 1,2$. We set $\hat{M}_i^* = (\hat{T}_i^*,\hat{\alpha}_i^*,\hat{\beta}_i^*,\hat{\gamma}_i^*), i = 1,2$. According to the construction, each edge $\hat{e}$ of $\hat{T}$ corresponds to a subtree of $\hat{T}_i^*, i = 1,2$ that is a ternary caterpillar. Let $P_{\hat{e}}^* = (\hat{e}_1^*,\ldots,\hat{e}_{|\mathcal{E}_{\hat{e}}|}^*)$ be the path formed by the ridge edges of this caterpillar (we assume that the arrangement of the edges in $P_{\hat{e}}^*$ "follows" the ordering of the arrangement of the edges in $P_{\hat{e}}$). Using (39) and (40) we have that each edge $\hat{e}_i^*\in P_{\hat{e}}^*, 1\leq i\leq|\mathcal{E}_{\hat{e}}|$ of such a ternary caterpillar corresponds to some subpath $E_{i,\hat{e}}$ in $\mathcal{E}_{\hat{e}}$. In particular, if $E_{\hat{e},i} = (e_{i_1},\ldots,e_{i_{|E_{\hat{e},i}|}})$, then the following relations hold.

$$\forall_{1\leq j\leq|\mathcal{E}_{\hat{e}}|}\ \forall_{e\in E_{\hat{e},j}}\alpha^+(e) = \hat{\alpha}^*(\hat{e}_j^*) \text{ and} \quad (42)$$

$$\forall_{1\leq j\leq|\mathcal{E}_{\hat{e}}|}\ \hat{\gamma}^*(\hat{e}_j^*) = \tau(\gamma(\rho(e_{i_i}))\oplus\cdots\oplus\gamma^+(e_{i_{|E_{\hat{e},i}|}})). \quad (43)$$

$$\forall_{1\leq j\leq|\mathcal{E}_{\hat{e}}|}\ \forall_{e\in E_{\hat{e},j}}\alpha_1^+(\rho(e)) = \hat{\alpha}_1^*(\hat{\rho}(\hat{e}_j^*)) \text{ and} \quad (44)$$

$$\forall_{1\leq j\leq|\mathcal{E}_{\hat{e}}|}\ \hat{\gamma}_1^*(\hat{\rho}(\hat{e}_j^*)) = \tau(\gamma_1^+(\rho(e_{i_i}))\oplus\cdots\oplus\gamma_1^+(\rho(e_{i_{|E_{\hat{e},i}|}}))). \quad (45)$$

$$\forall_{1\leq j\leq |\mathcal{E}_{\hat{e}}|}\ \forall_{e\in E_{\hat{e},j}} \alpha_2^+(\sigma(\rho(e)) \;=\; \hat{\alpha}_2^*(\hat{\sigma}(\hat{\rho}(\hat{e}_j^*))) \text{ and} \tag{46}$$
$$\forall_{1\leq j\leq |\mathcal{E}_{\hat{e}}|}\ \hat{\gamma}_2^*(\hat{\sigma}(\hat{\rho}(\hat{e}_j^*))) \;=\; \tau(\gamma_2^+(\sigma(\rho(e_{i_i}))\oplus\cdots\oplus\gamma_2^+(\sigma(\rho(e_{i_{|E_{\hat{e},i}|}})))). \tag{47}$$

From Lemma 17 we have that as $\hat{M}$ is dense then $\hat{T}$ has at most $\delta(|R|)$ edges. In the construction above, each edge of $\hat{T}$ correspond to at most $3k$ edges of $\hat{T}^*, i = 1,2$. Therefore, $|E(\hat{T}_i^*)| \leq 3k\delta(k)$ and $\hat{M}_i^* \in \mathcal{D}_{3k\delta(k)}(\hat{M}_i), i = 1,2$. In what follows we will construct a branch model $\hat{M}^\star = (\hat{T}^\star, \hat{\alpha}^\star, \hat{\beta}^\star, \hat{\gamma}^\star)$ such that $\hat{M}^\star \in \mathsf{Join}(\hat{M}_1^*, \hat{M}_2^*, \hat{\sigma})$. and we will prove that $\hat{M}^\star \prec \hat{M}^*$.

We set $\hat{M}^\star = (\hat{T}^*, \hat{\alpha}^\star, \hat{\beta}^\star, \hat{\gamma}^\star)$. As $\hat{M}^\star$ and $\hat{M}^*$ have the same underlying tree we can consider $\hat{\rho}$ as an isomorphism from the underlining tree of $\hat{M}^\star$ to the one of $\hat{M}_1^*$ as required in order to apply $\mathsf{Join}(\hat{M}_1^*, \hat{M}_2^*, \hat{\sigma})$.

Clearly, $\hat{\alpha}^\star = (\hat{\alpha}_1^* \cup_{\hat{\sigma}} \hat{\alpha}_2^*) \circ \hat{\rho}$ and $\hat{\beta}^\star = (\hat{\beta}_1^* \cup_{\hat{\sigma}} \hat{\beta}_2^*) \circ \hat{\rho}$. It remains now to describe, for any edge $\hat{e}_j^*$ of $\hat{T}^*$ that belongs to a set $P_{\hat{e}}^*$ for some $\hat{e} \in E(\hat{T})$, how sequences $\hat{\gamma}_1^*(\hat{\rho}(\hat{e}_j^*))$ and $\hat{\gamma}_2^*(\hat{\sigma}(\hat{\rho}(\hat{e}_j^*)))$ should be interleaved during the interleaving step of $\mathsf{Join}(\hat{M}_1^*, \hat{M}_2^*, \hat{\sigma})$.

We now apply Lemma 6 for the sequences $(\gamma^+(e_1), \ldots, \gamma^+(e_i))$ $(\gamma_1^+(\rho(e_1)), \ldots, \gamma_1^+(\rho(e_{r_{\hat{e}}})))$, and $(\gamma_2^+(\sigma(\rho(e_i))), \ldots, \gamma_2^+(\sigma(\rho(e_{\hat{e}}))))$, and from Relations (36), (43), (42), (45), (44),(47), (46), (39) and (40) we easily have that there exist a sequence $C \in \hat{\gamma}_1^*(\hat{\rho}(\hat{e}_j^*)) \otimes \hat{\gamma}_2^*(\hat{\sigma}(\hat{\rho}(\hat{e}_j^*)))$ such that

$$\tau(C) + |\mathsf{both}_{\hat{\alpha}^*}(\hat{e}_j^*)| \prec \hat{\gamma}^*(\hat{e}_j^*) + |L(\hat{e}, j)|. \tag{48}$$

We set $\hat{\gamma}^\star(\hat{e}_j^*) \leftarrow \tau(\gamma)$ and going on that way we define all the values of $\hat{\gamma}^\star$. It is easy to verify that $\hat{\beta}^* = \hat{\beta}^\star$. Moreover, using Relations (36), (37), (41), and (48) we have that for any $\hat{e}$ and any $\hat{e}_j^* \in P_{\hat{e}}^*$ the following hold.

$$\hat{\alpha}^*(\hat{e}_j^*) \;=\; \hat{\alpha}^\star(\hat{e}_j^*) \cup L(\hat{e}, j) \tag{49}$$
$$\hat{\gamma}^*(\hat{e}_j^*) \;\succ\; \hat{\gamma}^\star(\hat{e}_j^*) + |L(\hat{e}, j)| \tag{50}$$
$$\tag{51}$$

Clearly, $\hat{M}^\star$ is not necessarily a branch model. Let $\hat{e}_j^*$ be an edge of T^* where $\hat{e}_j^* \in P_{\hat{e}}^*$. We will prove that during the normalization step the vertices (numbers) that will be added to the values of $\hat{\alpha}^*$ $\hat{\gamma}^*$ will be exactly those that are required in order relation $\hat{M}^\star \prec \hat{M}^*$ to hold.

Suppose also that $\mathsf{Norm}(\hat{M}^\star, x)$ adds some vertex x in $\hat{\alpha}^\star$ and increases by one all the numbers in $\hat{\gamma}^\star$. Clearly, $\hat{e}_j^* \in E_x(\hat{M}^\star)$ and thus $x \notin \hat{\alpha}^*(\hat{e}_j^*)$. From Relation (42) we have that $\forall_{e\in E_{\hat{e},j}} x \notin \alpha^+(e)$. As $x \in V(M^+)$, we have that $x \in L(\hat{e}, j)$.

Suppose now that a vertex x belongs in some set $L(\hat{e}, j)$. This means that $\rho^{-1}(V_x(T_1^+))$ and $(\rho \circ \sigma)^{-1}(V_x(T_2^*))$ induce two connected trees in T^+ that lay into different connected components of $T - e$ where e is any edge in $E_{\hat{e},j}$. It is not hard to see that this property holds for the ancestors of $M_i^*, i = 1, 2$ in the following way: $\hat{\rho}^{-1}(V_x(\hat{T}_1^*))$ and $(\hat{\rho} \circ \hat{\sigma})^{-1}(V_x(\hat{T}_2^*))$ induce two connected trees in $\hat{T}^+ = \hat{T}^\star$ that lay into different connected components of $\hat{T}^* - \hat{e}_j^*$. This means that $\hat{e}_j^* \in E_x(\hat{M}^\star)$ and therefore $\mathsf{Norm}(\hat{M}^\star, x)$ will add x in $\hat{\alpha}^\star$ and will increase by one all the numbers in $\hat{\gamma}^\star$.

From the above we have that, after the normalization step, $\hat{M}^\star$ is modified in a way that $\hat{M}^\star \prec \hat{M}^*$. We omit the proof of the correctness of $\hat{b}^\star$ as it is easy and do not give any further insight. The lemma now follows from the fact that $\hat{M}^* \sqsubset M^+ = \mathrm{Des}_R(B)$. $\square$

$$\begin{tikzcd}
\hat{M}^* \arrow[r, dash, "\succ"] \arrow[d, "\mathsf{Com}"] & \hat{M}^{*\sharp} \arrow[r, phantom, "\sqsubset"] \arrow[d, "\mathsf{Com}"] & M^{+*\sharp} & M^{*\sharp} \arrow[l, "||_{X_R}"'] \arrow[d, dash, "\sqcup"] \\
\hat{M} \arrow[r, dash, "\succ"] & \hat{M}^\sharp & M^{+\sharp} \arrow[l, "\mathsf{Com}"'] & M^\sharp \arrow[l, "||_{X_R}"']
\end{tikzcd}$$

Figure 17: An illustration of Lemma 31

Lemma 31 *Let $\hat{M}, \hat{M}^*$ be two branch models of a graph G rooted on $R \subseteq V(G)$ and such that $\hat{M} \sqsubset \hat{M}^*$. We also assume that there exist an entire branch model $M^\sharp$ such that if $\hat{M}^\sharp = C_R(M^\sharp)$ then $\hat{M}^\sharp \prec \hat{M}$. Then, there exist two entire branch models $M^{*\sharp}$ and $\hat{M}^{*\sharp}$ rooted on $V(G)$ and R respectively such that*

1. $\hat{M}^\sharp \sqsubset \hat{M}^{*\sharp}$,

2. $M^\sharp \sqsubset M^{*\sharp}$,

3. $\hat{M}^{*\sharp} \sqsubset M^{*\sharp}||_R$,

4. $\hat{M}^{*\sharp} \prec \hat{M}^*$.

Proof Clearly there is a sequence of rake and compress steps that reduce $\hat{M}^*$ to $\hat{M}$. If we now follow these steps in the inverse order and apply Lemma 4 for the cases of compression we can easily construct a branch model $\hat{M}^{*\sharp}$ such that $\hat{M}^{*\sharp} \prec \hat{M}^*$ and $\hat{M}^\sharp \sqsubset \hat{M}^{*\sharp}$. We now observe that $\hat{M}^{*\sharp}$ and $M^{+\sharp} = M^\sharp\|_R$ are common predecessors of $\hat{M}$. Using now Lemma 27 we can construct a branch model $M^{*\sharp}$ that is a predecessor of $M^\sharp$ and such that $\hat{M}^{*\sharp} \sqsubset M^{*\sharp}\|_R$. $\square$

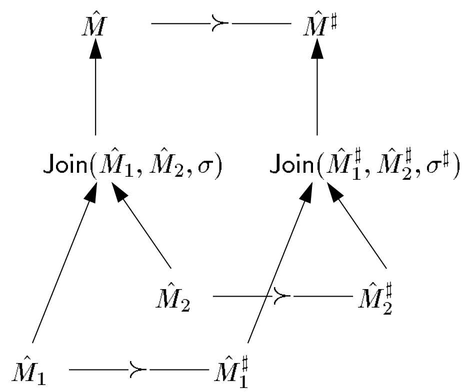


Figure 18: An illustration of Lemma 32

Lemma 32 *Let G be a graph and let $G_i, i = 1, 2$ be graphs such that $V(G_1) \cap V(G_2) = R$ and $E(G_1) \cap E(G_2) = \emptyset$. Let, $M_i, M_i^\sharp, i = 1, 2$ be branch models of G_i rooted on R such that $M_i^\sharp \prec M_i, i = 1, 2$. Let σ be a regular isomorphism in $\in \mathcal{I}(T(M_1), T(M_2))$ and $\sigma^\sharp \in \mathcal{I}(T(M_1^\sharp), T(M_2^\sharp))$. Then, for any $M \in \mathsf{Join}(M_1, M_2, \sigma)$ there exist a regular isomorphism $\sigma^\sharp \in \mathcal{I}(T(M_1^\sharp), T(M_2^\sharp)) \neq \emptyset$ and a branch model $M^\sharp \in \mathsf{Join}(M_1^\sharp, M_2^\sharp, \sigma^\sharp, \sigma^\sharp)$ such that $M^\sharp \prec M$.*

Proof We set $M_i = (T_i, \alpha_i, \beta_i, \gamma_i), i = 1, 2$ and $M_i^\sharp = (T_i^\sharp, \alpha_i^\sharp, \beta_i^\sharp, \gamma_i^\sharp), i = 1, 2$. As $M_i \prec M_i^\sharp$ we can assume that for $i = 1, 2$ there exist a isomorphism $\pi_i \in \mathcal{I}(T_i, T_i^\sharp)$. Suppose that $\mathcal{M}$ and $\mathcal{M}^\sharp$ be the sets created after the interleaving steps of $\mathsf{Join}(M_1, M_2, \sigma)$ and $\mathsf{Join}(M_1^\sharp, M_2^\sharp, \sigma^\sharp)$ respectively. We also assume that each branch model in $\mathcal{M}$ ($\mathcal{M}^\sharp$) has as underlining tree a tree T ($T^\sharp$) isomorphic to T_1 ($T_1^\sharp$) through a function ρ ($\rho^\sharp$). Clearly, M occurs after the normalization step is applied to some member $M' = (T, (\alpha_1 \cup_\sigma \alpha_2) \circ \rho, (\beta_1 \cup_\sigma \beta_2) \circ \rho, \tau(\gamma)) \circ \rho$ of $\mathcal{M}$ where $\gamma \in \gamma_1 \otimes_{\sigma,\rho} \gamma_2$.

Let $M^{\sharp\prime} = (T^{\sharp}, (\alpha_1^{\sharp} \cup_{\sigma^{\sharp}} \alpha_2^{\sharp}) \circ \rho^{\sharp}, (\beta_1^{\sharp} \cup_{\sigma^{\sharp}} \beta_2^{\sharp})\rho^{\sharp}, \tau(\gamma^{\sharp}))$ be the member of $\mathcal{M}^{\sharp}$ where for any edge $e \in E(T^{\sharp})$ $\gamma^{\sharp}(e)$ is defined if we apply Lemma 7 for the sequences $\gamma_1(e)$, $\gamma_2(e)$, $\gamma(e)$, $\gamma_1^{\sharp}(e)$, and $\gamma_2^{\sharp}(e)$ (clearly, $\gamma^{\sharp} \in \gamma_1^{\sharp} \otimes_{\sigma^{\sharp},\rho^{\sharp}} \gamma_2^{\sharp}$). Notice that $\pi = \rho^{\sharp} \circ \pi_1 \circ (\rho^{\sharp})^{-1}$ is an isomorphism in $\mathcal{I}(T, T^{\sharp})$ and $\forall_{e \in E(T)}\ \gamma(e) \prec \gamma^{\sharp}(\pi(e)) \Rightarrow \gamma \prec_{\pi} \gamma^{\sharp}$. Moreover it is clear that $(\alpha_1 \cup_{\sigma} \alpha_2) \circ \rho =_{\pi} (\alpha_1 \cup_{\sigma} \alpha_2) \circ \rho(\alpha_1^{\sharp} \cup_{\sigma^{\sharp}} \alpha_2^{\sharp}) \circ \rho^{\sharp}$ and $(\beta_1 \cup_{\sigma} \beta_2) \circ \rho =_{\pi} (\beta_1^{\sharp} \cup_{\sigma^{\sharp}} \beta_2^{\sharp}) \circ \rho^{\sharp}$ and therefore $M' \prec M^{\sharp\prime}$. Let $M^{\sharp}$ be the branch model occuring after we apply the normalization step on $M^{\sharp\prime}$. Appplying now the "$\prec$" version of Lemma 13 on M' and $M^{\sharp\prime}$ we have that $M \prec M^{\sharp}$ and this completes the proof of the lemma. $\square$

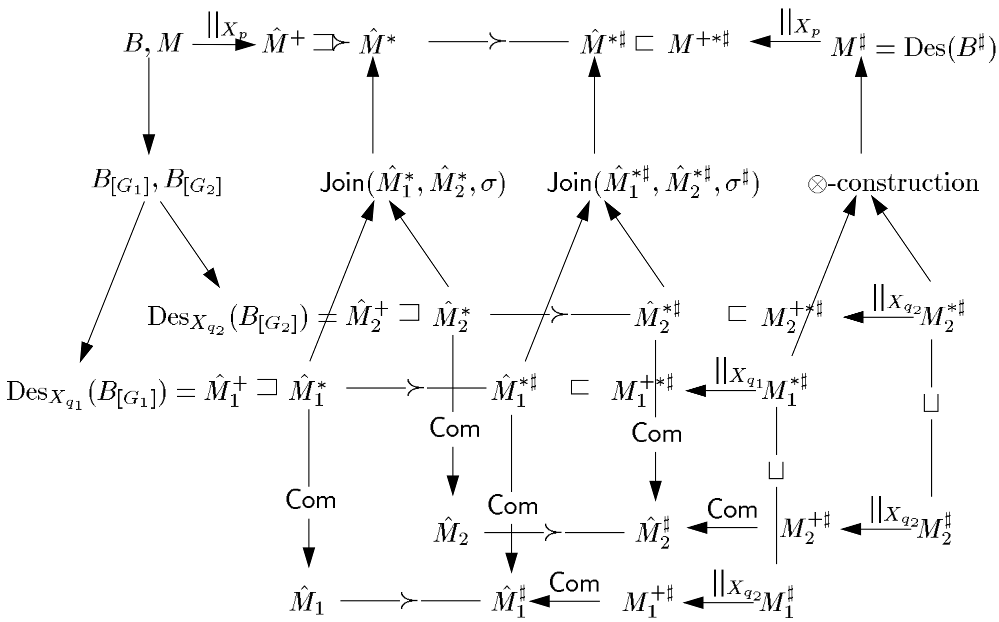


Figure 19: The structure of the proof of Lemma 28

Lemma 33 *The set $FS(p)$ constructed by the above algorithm is a full set of characteristics.*

Proof From 28 we have that $FS(p)$ is a set of characteristics. Ir remains to prove that it is a full set of characteristics. Let B be a branch decomposition of G_p We will prove that there exist a branch decomposition $B^{\sharp}$ of G_p such that $C_{X_p}(B^{\sharp}) \in FS(h)$ and $C_{X_p}(B^{\sharp}) \prec C_{X_p}(B)$. Let $\hat{M}_i = C_{X_{q_i}}(B_{[G_{q_i}]}), i = 1, 2$. We apply Lemma 30 and we have that, for $i = 1, 2$, there exists a branch model

$\hat{M}_i^*$ of G_{q_i} rooted on X_{q_i} (notice that $X_{q_1} = X_{q_2} = X_p$)), a regular isomorphism $\sigma \in \mathcal{I}(T(\hat{M}_1^*), T(\hat{M}_2^*)) \neq \emptyset$, and a branch model $\hat{M}^*$ rooted on X_p where

$$\begin{aligned} \hat{M}_i^* &\in \mathcal{D}_{3k\delta(k)}(\hat{M}_i), i = 1, 2 && (52)\\ \hat{M}^* &\in \mathsf{Join}(\hat{M}_1^*, \hat{M}_2^*, \sigma) && (53)\\ \hat{M}^* &\prec\!\sqsubset \mathrm{Des}_R(B). && (54) \end{aligned}$$

For $i = 1, 2$ we observe that, as $FS(q_i)$ is a full set of characteristics and $\hat{M}_i$ is a characteristic of $B_{[G_{q_i}]}$ there will exist a branch decomposition $B_i^\sharp$ of G_{q_i} such that if $\hat{M}_i^\sharp = C_{X_i}(B_i^\sharp)$ we will have the following.

$$\begin{aligned} \hat{M}_i^\sharp &\in FS(q_i). && (55)\\ \hat{M}_i^\sharp &\prec \hat{M}_i. && (56) \end{aligned}$$

For $i = 1, 2$, we set $M_i^\sharp = \mathrm{Des}(B_i^\sharp)$ and notice that $M_i^\sharp$ is entire. Observe that the following hold for $i = 1, 2$.

$$\hat{M}_i^\sharp = C_{X_{q_i}}(M_i^\sharp). \quad (57)$$

Moreover, from Relation (58) we have that

$$\hat{M}_i \sqsubset \hat{M}_i^*. \quad (58)$$

If now, for $i = 1, 2$ we use Relations (56),(57),(58) and Lemma 31 we have that there exist two branch models $M_i^{*\sharp}$ and $\hat{M}_i^{*\sharp}$ rooted on $V(G)$ and X_{q_i} respectively such that

$$\begin{aligned} \hat{M}_i^\sharp &\sqsubset \hat{M}_i^{*\sharp}, && (59)\\ M_i^\sharp &\sqsubset M_i^{*\sharp}, && (60)\\ \hat{M}_i^{*\sharp} &\sqsubset M_i^{*\sharp}\|_{X_{q_i}}, && (61)\\ \hat{M}_i^{*\sharp} &\prec \hat{M}_i^*. && (62) \end{aligned}$$

Using now relations (62) and (63) for $i = 1, 2$ and Lemma 32 we have that there exist a regular isomorphism $\sigma^\sharp \in \mathcal{I}(T(M_1^\sharp), T(M_2^\sharp)) \neq \emptyset$ and a branch model $\hat{M}^{*\sharp}$ rooted on X_p that satisfies the following relations.

$$\begin{aligned} \hat{M}^{*\sharp} &\in \mathsf{Join}(\hat{M}_1^{*\sharp}, \hat{M}_2^{*\sharp}, \sigma^\sharp) && (63)\\ \hat{M}^{*\sharp} &\prec \hat{M}^*. && (64) \end{aligned}$$

As $M_i^\sharp, i = 1, 2$ are entire, Relation (60) gives that $M_i^{*\sharp}, i = 1, 2$ are entire as well. From Relations (61),(63) and Lemma 26 we have that there exist an entire branch model $M^\sharp$ of G rooted on $V(G)$ such that

$$\hat{M}^{*\sharp} \quad \sqsubset \quad M^\sharp\|_{X_p}. \tag{65}$$

From Lemma 15 we construct a branch decomposition $B^\sharp$ of G_p rooted on $V(G_p)$ such that $\mathrm{Des}(B^\sharp) = M^\sharp$. Clearly, Relation (65) gives that $\hat{M}^{*\sharp} \sqsubset \mathrm{Des}_{X_p}(B^\sharp)$ and therefore $C_{X_p}(B^\sharp) = \mathsf{Com}(\hat{M}^{*\sharp})$. From Relation (64) we have that $\mathsf{Com}(\hat{M}^{*\sharp}) \prec \mathsf{Com}(\hat{M}^*)$ and from Relation (54) we have that $\mathsf{Com}(\hat{M}^*) \prec \mathsf{Com}(\mathrm{Des}_{X_p}(B)) = C_{X_p}(B)$ and therefore $C_{X_p}(B^\sharp) \prec C_{X_p})(B)$ as required. It remains to prove that $C_{X_p}(B^\sharp) \in FS(p)$. Notice that because of Relation (55), algorithm Join-characteristics can choose $\hat{M}^\sharp, i = 1, 2$ in step (**2**). Moreover, from Relation (62) we have that $\hat{M}_i^{*\sharp}, i = 1, 2$ have underlining trees of the same size of those of the underlining trees of $\hat{M}_i^*, i = 1, 2$. Using Relations (52) and (59) we easily have that $\hat{M}_i^{*\sharp} \in \mathcal{D}_{3k\delta(k)}(\hat{M}_i^\sharp), i = 1, 2$ and therefore $\hat{M}_i^{*\sharp}, i = 1, 2$ can be chosen in step (**3**) of algorithm Join-characteristics. Recall that $\sigma^\sharp \in \mathcal{I}(T(M_1^\sharp), T(M_2^\sharp))$ is regular and therefore can be chosen in step (**4**) of algorithm Join-characteristics. Finally, from Relation (63) we have that $\hat{M}^{*\sharp}$ can be chosen during step (**5**) of algorithm Join-characteristics and therefore $\mathsf{Com}(\hat{M}^{*\sharp}) \in FS(p)$. As $C_{X_p}(B^\sharp) = \mathsf{Com}(\hat{M}^{*\sharp})$ we have that $C_{X_p}(B^\sharp) \in FS(p)$ and this completes the proof of the lemma. A diagram for the proof is depicted in Figure 19. □

4 Conclusions

Notice that, because of Lemma 17, the algorithms Introduce-edge, Forget-vertex, and Join-characteristics run in $O(1)$ time when k and l are fixed. We resume the results of the previous subsections in the following (we omit the details of how to transform the decision algorithm to a constructive one as they are an direct consequences of the machinery in the proofs of Lemmata 22,24,25,26,28 and 30).

Theorem 1 *For all $k, l \geq 1$ there exists an algorithm that, given a graph G and a tree decomposition of G with width at most l, computes whether the branchwidth of G is at most k and, if so, constructs a branch decomposition of G with branchwidth at most k and that uses $O(V(G) + |X|)$ time.*

In [18] it is proved that $\mathrm{treewidth}(G) + 1 \leq \lfloor \frac{3}{2}\mathrm{branchwidth}(G) \rfloor$. Combining this fact with Theorem 1 and the result in [5] we have the following:

Theorem 2 *For all k, there exists an algorithm, that given a graph G, computes whether the branchwidth of G is at most k, and if so, constructs a branch decomposition of G with minimum branchwidth in $O(|V(G)|)$ time.*

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