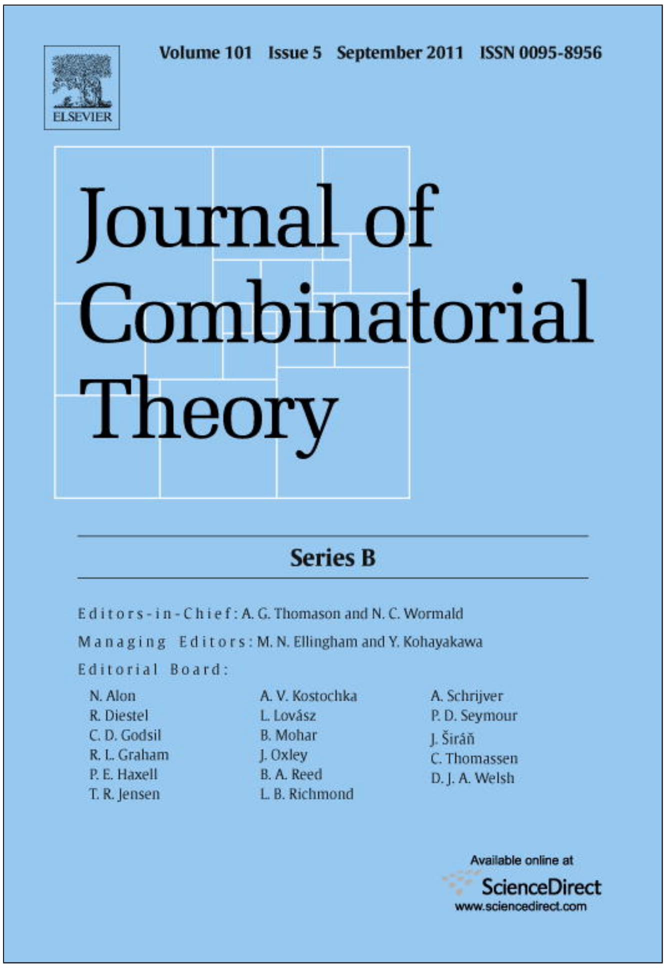
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Contraction obstructions for treewidth ☆

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ABSTRACT

We provide two parameterized graphs Γ_k, Π_k with the following property: for every positive integer k, there is a constant c_k such that every graph G with treewidth at least c_k, contains one of K_k, Γ_k, Π_k as a contraction, where K_k is a complete graph on k vertices. These three parameterized graphs can be seen as "obstruction patterns" for the treewidth with respect to the contraction partial ordering. We also present some refinements of this result along with their algorithmic consequences.

1. Introduction

We say that a graph H is a *contraction* of a graph G if H can be obtained after applying to G a (possibly empty) sequence of edge contractions. We also say that H is a *minor* of G if H is the contraction of some subgraph of G. The minor relation is a partial order relation on graphs that has been studied extensively in the Graph Minors series of papers of Robertson and Seymour. One of the most celebrated results of this project is the following (see Section 2 for the formal definition of treewidth).

Proposition 1. *(See [18] – see also [9,21].) For any positive integer k, there is a $c_k > 0$ such that every graph of treewidth at least c_k contains a $(k \times k)$-grid as a minor.*

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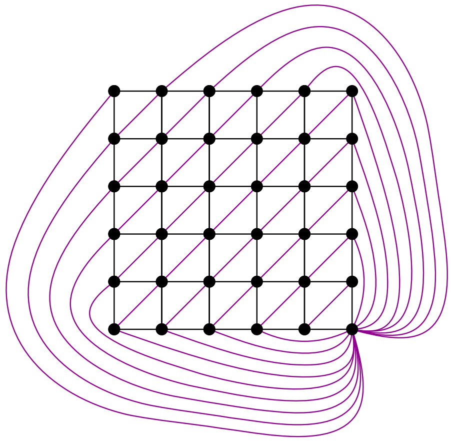

Fig. 1. The graph Γ_6.

Proposition 1 suggests that grids, parameterized by their height h, can be seen as "obstruction patterns" for small treewidth with respect to the minor relation. In this paper we prove an analogue of this result for the contraction relation. In particular, we identify three parameterized graph classes that serve as obstruction patterns for small treewidth with respect to the contraction relation.

Let Γ_k ($k \geqslant 2$) be the graph obtained from the $(k \times k)$-grid by triangulating internal faces of the $(k \times k)$-grid such that all internal vertices become of degree 6, all non-corner external vertices are of degree 4, and then one corner of degree two is joined by edges with all vertices of the external face (the *corners* are the vertices that in the underlying grid have degree two). Graph Γ_6 is shown in Fig. 1. Let Π_k be the graph obtained from Γ_k by adding a new universal vertex adjacent to all vertices of Γ_k. We also denote by K_k the complete graph on k vertices and use the notation $\mathcal{O}_k = \{\Gamma_k, \Pi_k, K_k\}$.

A consequence of our results is the following.

Theorem 1. *For any integer $k > 0$, there is a $c_k > 0$ such that every connected graph of treewidth at least c_k contains a graph from $\mathcal{O}_k$ as a contraction.*

Notice that for any $k, r \geqslant 6$,

- K_6-minor-free graphs Γ_r and Π_r cannot be contracted to K_k;
- K_5-minor-free graph Γ_r cannot be contracted to Π_k which contains K_5 as a minor;
- any contraction of Π_r contains a universal vertex adjacent to all other vertices, and hence Π_r cannot be contracted to Γ_k;
- any contraction of K_r is a complete graphs, and hence K_r cannot be contracted to Γ_k or Π_k.

Since the treewidth of the graphs Γ_r, Π_r, K_r is at least r, this indicates that $\mathcal{O}_k$ is optimal with respect to its size.

Proposition 1 has several refinements. For instance, in [21], it was proved that there is a linear dependence between the treewidth of a planar graph and the maximum height of a grid minor of it (i.e. $c_k = O(k)$). This result has been extended as follows.

Proposition 2. *(See [6].) For every graph H, there is a $c_H > 0$ such that every H-minor free graph of treewidth at least $c_H \cdot k$ contains a $(k \times k)$-grid as a minor.*

As a contraction analogue of Proposition 2, we prove the following.

Theorem 2. *For every graph H, there is a $c_H > 0$ such that every connected H-minor-free graph of treewidth at least $c_H \cdot k^2$ contains either Γ_k or Π_k as a contraction.*

Notice that in the above theorem, the quadratic dependence (on k) is optimal. Indeed, let Z_{k^2} be the graph obtained by adding to Γ_{k^2} a new vertex adjacent to all the k^2 vertices with both coordinates in the underlying grid divisible by k. Then Z_{k^2} excludes K_6, Γ_{k+2}, and Π_{k+2} as contractions and is of treewidth at least k^2.

An *apex graph* is a graph such that the removal of one vertex creates a planar graph. It appears that the linear dependence (on k) in Proposition 2 is possible also for contractions when we consider graphs excluding some apex graph as a minor.

Theorem 3. *For every apex graph H, there is a $c_H > 0$ such that every connected H-minor-free graph of treewidth at least $c_H \cdot k$ contains Γ_k as a contraction.*

The paper is organized as follows. In Section 2 we give the basic definitions and some preliminary results. We dedicate Section 3 to the proof of Lemma 11 which, in turn, is used in Section 4 in order to prove Theorems 1, 2, and 3. Theorem 3 has some meta-algorithmic consequence in the framework of bidimensionality theory that will be presented in Section 5.

2. Basic definitions

We consider finite undirected graphs without loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$ and its edge set by $E(G)$.

Let G be a graph. For a vertex set $U \subseteq V(G)$, we denote by $G[U]$ the subgraph of G induced by U. A set of vertices $K \subseteq V(G)$ is a *clique* of G if vertices of K are pairwise adjacent in G. If $U \subseteq V(G)$ (resp. $E \subseteq E(G)$) then $G - U$ (resp. $G - E$) is the graph obtained from G by the removal of all the vertices of U (resp. the edges of E).

Surfaces. A *surface* Σ is a compact 2-manifold without boundary (we always consider connected surfaces). Whenever we refer to a *Σ-embedded graph G* we consider a 2-cell embedding of G in Σ. To simplify notations, we do not distinguish between a vertex of G and the point of Σ used in the drawing to represent the vertex or between an edge and the line representing it. We also consider a graph G embedded in Σ as the union of the points corresponding to its vertices and edges. That way, a subgraph H of G can be seen as a graph H, where $H \subseteq G$. Recall that $\Delta \subseteq \Sigma$ is an open (resp. closed) disc if it is homeomorphic to $\{(x, y)\colon x^2 + y^2 < 1\}$ (resp. $\{(x, y)\colon x^2 + y^2 \leqslant 1\}$). The *Euler genus* of a non-orientable surface Σ is equal to the non-orientable genus $\tilde{g}(\Sigma)$ (or the crosscap number). The *Euler genus* of an orientable surface Σ is $2g(\Sigma)$, where $g(\Sigma)$ is the orientable genus of Σ. We refer to the book of Mohar and Thomassen [17] for more details on graphs' embeddings. The *Euler genus* of a graph G (denoted by $\mathbf{eg}(G)$) is the minimum integer γ such that G can be embedded on a surface of the Euler genus γ.

Contractions and minors. Given an edge $e = \{x, y\}$ of a graph G, the graph G/e is obtained from G by contracting the edge e, i.e. the endpoints x and y are replaced by a new vertex v_{xy} which is adjacent to the old neighbors of x and y (except x and y). A graph H obtained by a sequence of edge-contractions is called a *contraction* of G. In this work we use contraction with certain topological properties, and for this purpose, it is convenient to give an alternative definition of contraction.

Let G and H be graphs and let $\phi : V(G) \to V(H)$ be a surjective mapping such that

(1) for every vertex $v \in V(H)$, its codomain $\phi^{-1}(v)$ induces a connected graph $G[\phi^{-1}(v)]$;
(2) for every edge $\{v, u\} \in E(H)$, the graph $G[\phi^{-1}(v) \cup \phi^{-1}(u)]$ is connected;
(3) for every $\{v, u\} \in E(G)$, either $\phi(v) = \phi(u)$ or $\{\phi(v), \phi(u)\} \in E(H)$.

We say that *H is a contraction of G via ϕ* and denote it as $H \leqslant_c^{\phi} G$. Let us observe that H is a contraction of G if $H \leqslant_c^{\phi} G$ for some $\phi : V(G) \to V(H)$. In this case we simply write $H \leqslant_c G$. If $H \leqslant_c^{\phi} G$ and $v \in V(H)$, then we call the codomain $\phi^{-1}(v)$ *model of v* in G.

Let G be a graph embedded in some surface Σ and let H be a contraction of G via function ϕ. We say that H is a *surface contraction* of G if for each vertex $v \in V(H)$, $G[\phi^{-1}(v)]$ is embedded in some open disk in Σ. It can be easily noted that if H is a surface contraction of a graph G embedded in Σ then it can be assumed that H is embedded in a surface Σ' homeomorphic to Σ. For simplicity, we always assume in such cases that Σ' and Σ are the same surface.

Let G_0 be a graph embedded in some surface Σ of Euler genus γ and let G^+ be another graph that might share common vertices with G_0. We set $G = G_0 \cup G^+$. Let also H be some graph and let $v \in V(H)$. We say that G *contains a graph H as a v-smooth contraction* if $H \leqslant_c^{\phi} G$ for some $\phi : V(G) \to V(H)$ and there exists a closed disk D in Σ such that all the vertices of G that are outside D are exactly the model of v, i.e. $\phi^{-1}(v) = V(G) \setminus (V(G) \cap D)$.

A graph H is a *minor* of a graph G if H is the contraction of some subgraph of G and we denote it by $H \leqslant_m G$. We say that H is a *surface minor* of a graph G embedded in some surface Σ if H is the surface contraction of some subgraph of G. Observe that H is a graph embedded in Σ.

We say that a graph G is *H-minor-free* when it does not contain H as a minor. We also say that a graph class $\mathcal{G}$ is *H-minor-free* (or, excludes H as a minor) when all its members are H-minor-free. An *apex graph* is a graph obtained from a planar graph G by adding a vertex and making it adjacent to some of the vertices of G. A graph class $\mathcal{G}$ is *apex-minor-free* if $\mathcal{G}$ excludes some fixed apex graph H as a minor.

Grids and their triangulations. Let k and r be positive integers where $k, r \geqslant 2$. The *$(k \times r)$-grid* is the Cartesian product of two paths of lengths $k-1$ and $r-1$ respectively. A vertex of a $(k \times r)$-grid is a *corner* if it has degree 2. Thus each $(k \times r)$-grid has 4 corners. A vertex of a $(k \times r)$-grid is called *internal* if it has degree 4, otherwise it is called *external*.

A *partial triangulation* of a $(k \times r)$-grid is a planar graph obtained from a $(k \times r)$-grid (we call it the *underlying grid*) by adding edges, i.e. if a grid is embedded in a plane then for some faces, we join non-adjacent vertices on the boundary of the face by non-crossing edges inside this face. Let us note that there are many non-isomorphic partial triangulations of an underlying grid. For each partial triangulation of a $(k \times r)$-grid we use the terms *corner*, *internal* and *external* referring to the corners, the internal and the external vertices of the underlying grid.

Let us remind that we define Γ_k (see Fig. 1) as the following (unique, up to isomorphism) triangulation of a plane embedding of the $(k \times k)$-grid: Consider a plane embedding of the $(k \times k)$-grid such that all external vertices are on the boundary of the external face. We triangulate internal faces of the $(k \times k)$-grid such that all the internal vertices have degree 6 in the obtained graph and all non-corner external vertices have degree 4. Finally, one corner of degree two is joined by edges with all the vertices of the external face (we call this corner *loaded*). We also use notation Γ_k^* for the graph obtained from Γ_k if we remove all edges incident to its loaded vertex that do not exist in its underlying grid. We define the graph Π_k as the graph obtained from Γ_k by adding a universal vertex adjacent to all vertices of Γ_k. Let K be a clique of size 3 in Γ_k^*. Notice that exactly two of the edges of $\Gamma_k[K]$ are also edges of the underlying $(k \times k)$-grid of Γ_k. We call the unique vertex of K that is incident to both these two edges *rectangular* vertex of K.

Let G be a partial triangulation of a $(k \times k)$-grid and let m be a positive integer. We denote by $\mathbf{P}_m(G)$ the collection of m^2 vertex disjoint induced subgraphs of G such that each of them is isomorphic to a partially triangulated $(\lfloor k/m \rfloor \times \lfloor k/m \rfloor)$-grid and the union of their vertices induces a partially triangulated $(\lfloor k/m \rfloor \cdot m \times \lfloor k/m \rfloor \cdot m)$-grid.

Suppose that G is a connected graph which contains as an induced subgraph a partially triangulated $((k+2) \times (k+2))$-grid Γ in such a way that internal vertices of Γ are not adjacent to vertices of $V(G) \setminus V(\Gamma)$. We define the *boundary contraction* of G to Γ as the partially triangulated $(k \times k)$-grid $\mathbf{bc}(G, \Gamma)$ obtained as follows: let v be a corner of the subgrid of Γ induced by the internal vertices which has the minimum degree (in this graph), then

- all external vertices of Γ are contracted to v (i.e. all external vertices and v compose the model of one vertex of the resulting graph, and all other models contain exactly one vertex), and
- all vertices of $V(G) \setminus V(\Gamma)$ are contracted to v.

Note that if Γ is embedded in a disk of some surface Σ then $\mathbf{bc}(G, \Gamma)$ is a v-smooth contraction of G.

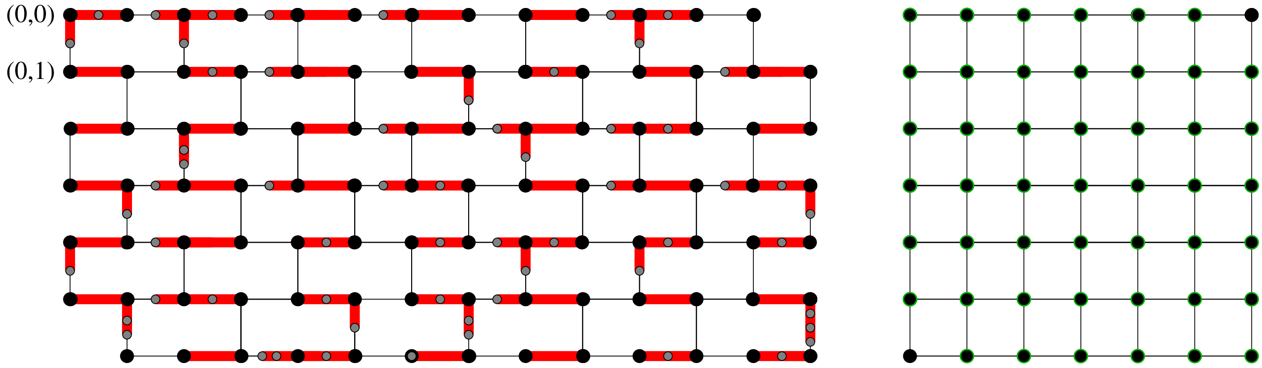

Fig. 2. A subdivided wall of height 6 and a contraction of it to a (7×7)-grid.

Walls. A *wall of height* k, $k \geqslant 1$, is obtained from a $((k+1) \times (2k+2))$-grid with vertices (x, y), $x \in \{0, \ldots, 2k+1\}$, $y \in \{0, \ldots, k\}$, by the removal of the all "vertical" edges $\{(x, y), (x, y+1)\}$ for odd $x + y$, and then the removal of the all vertices of degree 1 (see Fig. 2). We denote such a wall by W_k. A *subdivided wall of height* k is the graph obtained from W_k after some edges of W_k have being replaced by paths without common internal vertices.

Treewidth and pathwidth. A *tree decomposition* of a graph G is a pair $(\mathcal{X}, T)$ where T is a tree and $\mathcal{X} = \{X_i \mid i \in V(T)\}$ is a collection of subsets of $V(G)$ such that:

(1) $\bigcup_{i \in V(T)} X_i = V(G)$;
(2) for each edge $\{x, y\} \in E(G)$, $\{x, y\} \subseteq X_i$ for some $i \in V(T)$, and
(3) for each $x \in V(G)$, the set $\{i \mid x \in X_i\}$ induces a connected subtree of T.

The *width* of a tree decomposition $(\{X_i \mid i \in V(T)\}, T)$ is $\max_{i \in V(T)}\{|X_i| - 1\}$. The *treewidth* of a graph G is the minimum width over all tree decompositions of G. If, in the above definitions, we restrict the tree T to be a path then we define the notions of *path decomposition* and *pathwidth*. We write $\mathbf{tw}(G)$ and $\mathbf{pw}(G)$, respectively, for the treewidth and the pathwidth of a graph G.

Graph minor theorem. The proof of our results is using the Excluded Minor Theorem from the Graph Minor theory. Before we state it, we need some definitions.

Definition 1 *(Clique-sums).* Let G_1 and G_2 be two disjoint graphs, and $k \geqslant 0$ an integer. For $i = 1, 2$, let $W_i \subseteq V(G_i)$, be a clique of size h and let G_i' be the graph obtained from G_i by removing a set of edges (possibly empty) from the graph $G_i[W_i]$. Let $F : W_1 \to W_2$ be a bijection between W_1 and W_2. We define the *h-clique-sum* of G_1 and G_2, denoted by $G_1 \oplus_{h,F} G_2$, or simply $G_1 \oplus G_2$ if there is no confusion, as the graph obtained by taking the union of G_1' and G_2' by identifying $w \in W_1$ with $F(w) \in W_2$, and by removing all the multiple edges. The image of the vertices of W_1 and W_2 in $G_1 \oplus G_2$ is called the *join* of the sum.

Note that some edges of G_1 and G_2 are not edges of G, since it is possible that they had edges which were removed by clique-sum operation. Such edges are called *virtual* edges of G. We remark that $\oplus$ is not well defined; different choices of G_i' and the bijection F could give different clique-sums. A sequence of h-clique-sums, not necessarily unique, which result in a graph G, is called a *clique-sum decomposition* of G.

Definition 2 *(h-nearly embeddable graphs).* Let Σ be a surface and $h > 0$ be an integer. A graph G is *h-nearly embeddable* in Σ if there is a set of vertices $X \subseteq V(G)$ (called *apices*) of size at most h such that graph $G - X$ is the union of subgraphs $G_0, \ldots, G_h$ with the following properties

(i) There is a set of cycles $C_1, \ldots, C_h$ in Σ such that the cycles C_i are the borders of open pairwise disjoint discs Δ_i in Σ;
(ii) G_0 has an embedding in Σ in such a way that $G_0 \cap \bigcup_{i=1,\ldots,h} \Delta_i = \emptyset$;
(iii) graphs $G_1, \ldots, G_h$ (called *vortices*) are pairwise disjoint and for $1 \leqslant i \leqslant h$, $V(G_0) \cap V(G_i) \subset C_i$;
(iv) for $1 \leqslant i \leqslant h$, let $U_i := \{u^i_1, \ldots, u^i_{m_i}\}$ be the vertices of $V(G_0) \cap V(G_i) \subset C_i$ appearing in an order obtained by clockwise traversing of C_i, we call vertices of U_i *bases* of G_i. Then G_i has a path decomposition $\mathcal{B}_i = (B^i_j)_{1 \leqslant j \leqslant m_i}$, of width at most h such that for $1 \leqslant j \leqslant m_i$, we have $u^i_j \in B^i_j$.

The following proposition is known as the Excluded Minor Theorem [20] and is the cornerstone of Robertson and Seymour's Graph Minors theory. We need a stronger version of this theorem, which follows from its proof in [20] (see e.g., [7]).

Proposition 3. *(See [20].) For every non-planar graph H, there exists an integer c_H, depending only on H, such that every graph excluding H as a minor can be obtained by c_H-clique-sums from graphs that can be c_H-nearly embedded in a surface Σ in which H cannot be embedded. Moreover, while applying each of the clique-sums, at most three vertices from each summand other than apices and vertices in vortices are identified.*

Lemmata on treewidth. We need the following two well-known results about treewidth.

Lemma 1. *If G_1 and G_2 are graphs, then $\mathbf{tw}(G_1 \oplus G_2) \leqslant \max\{\mathbf{tw}(G_1), \mathbf{tw}(G_2)\}$.*

Lemma 2. *If G is a graph and $X \subseteq V(G)$, then $\mathbf{tw}(G - X) \geqslant \mathbf{tw}(G) - |X|$.*

The following lemma is implicit in the proofs from [6,3]. Here we give it as it is stated in [4].

Lemma 3. *(See [4, Lemma 4.3].) Let $G = G_0 \cup G_1 \cup \cdots \cup G_h$ be an h-nearly embeddable graph without apices (i.e. where $X = \emptyset$). Then $\mathbf{tw}(G) \leqslant \frac{3}{2}(h+1)^2(\mathbf{tw}(G_0) + 2h + 1)$.*

3. Lemmata on grids and their triangulations

In this section we give a series of auxiliary lemmata used to prove Lemma 11, the most important technical tool in the proofs of Theorems 3 and 2.

The following proposition is a direct consequence of [3, Theorem 4.12] and [19, (5.1)].

Proposition 4. *If G is a graph with treewidth more than $6r(\mathbf{eg}(G) + 1)$, then G has the $(r \times r)$-grid as a minor.*

It is implicit in the proofs in [3, Theorem 4.12] and [19, (5.1)] that if the treewidth of a graph embedded in a surface with Euler genus $\mathbf{eg}(G)$ is large enough then this graph contains $(r \times r)$-grid as a surface minor. We state this with the following lemma.

Lemma 4. *Let G be a graph embedded in a surface Σ of Euler genus γ. If the treewidth of G is more than $12r(\gamma + 1)$, then G has the $(r \times r)$-grid as a surface minor.*

Proof. If $\mathbf{tw}(G) > 12r(\gamma + 1)$, then by Proposition 4, G contains $(2r \times 2r)$-grid as a minor. Therefore it contains a wall W_{r-1} as a minor. Let $\{(x, y) \mid x \in \{0, \ldots, 2r-1\}, y \in \{0, \ldots, r-1\}\}$ be the vertex set of W_{r-1}. Since the maximum vertex degree of the wall is at most 3, we have that G contains a subdivided wall S_{r-1} of height $r-1$ a subgraph. For each edge $e = \{(x, y), (x+1, y)\}$ of W_{r-1}, where x is even, we contract in S_{r-1} the path corresponding to e (even when e has not been subdivided in S_{r-1}, we still contract it). For each edge $e = \{(x, y), (x+1, y)\}$ of W_{r-1} with odd x, we contract in S_{r-1} the corresponding path if its length is more than one. (Thus if e has not been subdivided in H, we do not contract it.) Finally, for each edge $e = \{(x, y), (x, y+1)\}$ of W_k we contract in S_{r-1}, if the length of the corresponding path is more than one, we contract it too. (See Fig. 2.) Notice that the

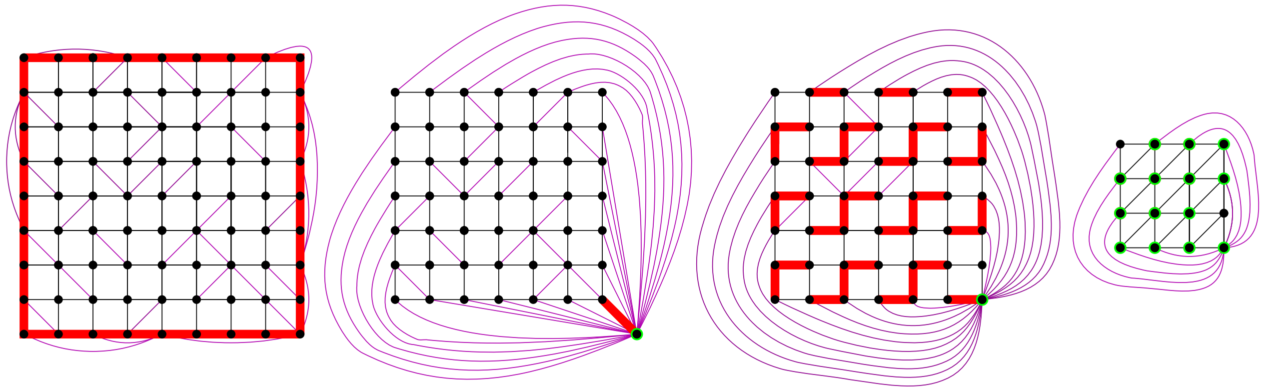

Fig. 3. The steps of the proof of Lemma 5. The two first steps are the boundary contraction of a partial triangulation of a (9×9)-grid. The third step is the contraction to Γ_4.

resulting graph H' is an $(r \times r)$-grid. It remains to observe that for each vertex v of W_{r-1}, model of v induces a tree in S_{r-1}, and hence the described contraction is a surface contraction of S_{r-1}. □

A basic ingredient of our proofs is a result roughly stating that if a graph G containing a big grid as a minor is embedded on a surface Σ of small genus, then there is a disc in Σ with a big enough part of the grid of G. This result is implicit in the work of Robertson and Seymour and there are simpler alternative proofs by Mohar and Thomassen [16,22] (see also [3, Lemma 3.3] and [8, Lemma 4.7]). We use the following variant of this result from Geelen et al. [15].

Proposition 5. *Let γ, l, r be positive integers such that $r \geqslant \gamma(l+1)$ and let G be an $(r \times r)$-grid embedded in a surface Σ of Euler genus at most $\gamma^2 - 1$. Then some $(l \times l)$-subgrid of G is embedded in a closed disc Δ in Σ. In particular, the cycle induced by the boundary vertices of the subgrid can be chosen as the boundary of Δ.*

We also need the following normalization lemma.

Lemma 5. *Let H be a partial triangulation of a $((2k+1) \times (2k+1))$-grid. Then H contains Γ_k as a contraction in a way that all external vertices of H belong to the model of the loaded corner of Γ_k.*

Proof. We first apply a boundary contraction of H to H. This creates a triangulation H' of the $((2k-1) \times (2k-1))$-grid where a single corner vertex v is connected with all the other external vertices. For convenience, we denote the vertices of H' as pairs of the set $\{0, \ldots, 2k-2\}^2$ where $v = (2k-2, 2k-2)$. We denote by $\mathbb{Z}$ the set of all integers and for each $(x, y) \in \mathbb{Z}^2$ we define

$$E(x, y) = \Big\{\big\{(2x-1, 2y), (2x, 2y)\big\}, \\ \big\{(2x, 2y), (2x, 2y-1)\big\}, \\ \big\{(2x, 2y-1), (2x+1, 2y-1)\big\}\Big\}.$$

In H', we contract the edges in $E(H') \cap \{E(x, y) \mid (x, y) \in \mathbb{Z}^2\}$. The resulting graph is isomorphic to Γ_k (see Fig. 3). □

The proof of the following lemma is based on Lemmata 4 and 5 and Proposition 5.

Lemma 6. *Let G be a graph embedded in a surface of Euler genus γ and let k be a positive integer. If the treewidth of G is more than $12 \cdot (\gamma+1)^{3/2} \cdot (2k+4)$, then G contains Γ_k as a v-smooth contraction with v being one of the corners of Γ_k.*

Proof. Applying Lemma 4 for $r = (\gamma+1)^{1/2} \cdot (2k+4)$, we deduce that G contains an $(r \times r)$-grid H as a surface minor. This implies that after a sequence of vertex/edge removals or contractions G, can be transformed to H. If we apply only the contractions in this sequence, we end up with some graph G' in Σ which contains H as a subgraph. The embedding of G' in Σ induces an embedding of H in this surface. By Proposition 5, some $((2k+3)\times(2k+3))$-subgrid H' of H is embedded in a close disk D of Σ in a way that the boundary cycle of H' is the boundary of D. For each internal face F of H' in D we do the following: contract each component of the graph induced by vertices of G laying inside F into a single vertex, choose an edge which joins this vertex with a vertex of H and contract it. Let G'' be the obtained graph. Notice that $G' \cap D$ is contracted to some partial triangulation H'' of the grid H'. Then we perform the boundary contraction of the graph G'' to H''. Thus we have contracted G'' to a $((2k+1)\times(2k+1))$-grid Γ and the described contraction is a v-smooth contraction, where v is a corner of Γ. It remains to apply Lemma 5 to conclude the proof of the lemma. □

Lemma 7. *There is a constant c such that if G is a graph h-nearly embedded in a surface of Euler genus γ without apices, where $\mathbf{tw}(G) \geqslant c \cdot \gamma^{3/2} \cdot h^{5/2} \cdot k$, then G contains as a v-smooth contraction the graph Γ_k with the loaded corner v.*

Proof. We choose c such that $c \cdot \gamma^{3/2} \cdot h^{5/2} \cdot k \geqslant (12 \cdot (\gamma+1)^{3/2} \cdot (2 \cdot \lceil h^{1/2} \rceil \cdot (k+2) + 4) + 2h + 1) \cdot \frac{3}{2} \cdot (h+1)^2$. Let Σ be a surface of Euler genus γ with cycles $C_1, \ldots, C_h$, such that each cycle C_i is the border of an open disc Δ_i in Σ and such that G is h-nearly embedded in Σ. Let also $G = G_0 \cup G_1 \cup \cdots \cup G_h$, where G_0 is embedded in Σ and $G_1, \ldots, G_h$ are vortices. We assume that $G_0 \cap \bigcup_{i=1,\ldots,h} \Delta_i = \emptyset$ and $V(G_0) \cap V(G_i) \subset C_i$ for $i \in \{1, \ldots, h\}$. By Lemma 3, $\mathbf{tw}(G_0) \geqslant 12 \cdot (\gamma+1)^{3/2} \cdot (2 \cdot \lceil h^{1/2} \rceil \cdot (k+2) + 4)$ and by Lemma 6, G_0 contains graph $\Gamma' = \Gamma_{\lceil h^{1/2} \rceil \cdot (k+2)}$ as a v-smooth contraction with the loaded corner v. We apply the same contractions to the graph G and denote the obtained graph by G'. Each disk Δ_i lies inside some face of G_0. Since Γ' is a triangulated grid, by the definition of the smooth contraction for each cycle C_i, vertices of $V(G_0) \cap C_i$ are contracted either to one vertex, or to two adjacent vertices, or to three pairwise adjacent vertices which lay on the boundary of some face of Γ'. Consider now the collection $\mathbf{P}_{\lceil h^{1/2} \rceil}(\Gamma')$ of $\lceil h^{1/2} \rceil^2 \geqslant h$ vertex disjoint induced $(k+2, k+2)$-subgrids of Γ'. There is a graph Γ'' in this collection such that models of internal vertices of Γ'' do not contain vertices of vortices $G_1, \ldots, G_h$. Note that Γ'' is inside some disk Δ in Σ and Γ'' is a partially triangulated $((k+2)\times(k+2))$-grid which is a subgraph of Γ'. We apply the boundary contraction of G' to Γ''. Then the resulting graph $\mathbf{bc}(G', \Gamma'') = \Gamma_k$ and the described contraction is smooth. □

Let $\mathcal{C} = \{K_1, \ldots, K_r\}$ be a sequence of (not necessary different) cliques in a graph G and let $E \subseteq \bigcup_{i=1,\ldots,r} E(G[K_i])$. We define the $\mathbf{cl}(G, \mathcal{C}, E)$ to be the graph constructed from $G - E$ by adding, for each non-empty K_i, a new vertex $z^{(i)}_{\text{new}}$ and making it adjacent to all vertices in K_i.

Lemma 8. *Let G_0 be a graph embedded in surface Σ of Euler genus γ and let G^+ be another graph that might share common vertices with G_0. We set $G' = G_0 \cup G^+$. Let $\mathcal{C} = \{K_1, \ldots, K_r\}$ be a collection of cliques in G' such that each of them shares at most 3 vertices with G_0. Let $E \subseteq \bigcup_{i=1,\ldots,r} E(G'[K_i])$ and let $\hat{G}' = \mathbf{cl}(G', \mathcal{C}, E)$. If G' contains Γ_{2k+5} with the loaded corner v as a v-smooth contraction, then $\hat{G}'$ contains Γ_k as a contraction.*

Proof. Let Γ_{2k+5} with loaded corner v be a v-smooth contraction of G' via mapping ϕ. We consider the subgrid $\Gamma' = \Gamma_{2k+3}$ obtained from Γ_{2k+5} by removal all neighbors of v. Observe that if G' has a clique of size at least 4, then it contains a vertex of G^+ and therefore it intersects the model of the loaded corner $\phi^{-1}(v)$ of Γ_{2k+5}. This implies that the vertices of this clique can belong only to models $\phi^{-1}(u)$ for vertices u of Γ_{2k+5}, which are neighbors of v. Thus they do not belong to models of vertices of Γ'. Therefore, only cliques of size at most 3 can intersect models of vertices of Γ'.

For each clique K_i intersecting models for the mapping ϕ of vertices of Γ', we examine the following cases and apply corresponding contractions to $\hat{G}'$:

Case 1. K_i intersects only one model $\phi^{-1}(u)$ of vertices Γ_{2k+5}. Then we contract all edges of $\hat{G}'$ incident to $z_{\text{new}}^{(i)}$ in $\hat{G}'$.
Case 2. K_i intersects two models $\phi^{-1}(u)$ and $\phi^{-1}(w)$ of vertices of Γ_{2k+5}. Assume that $|\phi^{-1}(u) \cap K_i| \geqslant |\phi^{-1}(w) \cap K_i|$. Then we contract all edges of $\hat{G}'$ joining $z_{\text{new}}^{(i)}$ and $\phi^{-1}(u) \cap K_i$.
Case 3. K_i intersects three models $\phi^{-1}(u), \phi^{-1}(w)$ and $\phi^{-1}(x)$ of vertices of Γ_{2k+5}. Then $|\phi^{-1}(u) \cap K_i| = |\phi^{-1}(w) \cap K_i| = |\phi^{-1}(x) \cap K_i| = 1$. Therefore $\{u, w, x\}$ induces a triangle in Γ^*_{2k+5}. Assume that u is the rectangular vertex of this triangle. Then we contract the edge of $\hat{G}'$ which joins $z_{\text{new}}^{(i)}$ and the unique vertex of $\phi^{-1}(u) \cap K_i$.
Finally for each edge $\{x, y\} \in G' - E$ such that $x, y \in \phi^{-1}(u)$ for some $u \in V(\Gamma')$, we contract this edge. Denote the obtained graph by $\hat{G}''$.

Observe that after applying all the above contractions, for each vertex $u \in V(\Gamma')$, vertices of $\phi^{-1}(u)$ are contracted into a single vertex. To prove it for $u \in V(\Gamma')$, let us consider two adjacent (in G') vertices $x, y \in \phi^{-1}(u)$. If $\{x, y\} \notin E(\hat{G}')$, then there is a clique K_i such that $x, y \in K_i$ and edges $\{x, z_{\text{new}}^{(i)}\}, \{y, z_{\text{new}}^{(i)}\}$ are contracted.

Let $W = \bigcup_{u \in V(\Gamma')} \phi^{-1}(u)$. It should be noted that the subgraph of $\hat{G}'$ induced by the set of vertices $W \cup \{z_{\text{new}}^{(i)}\colon\ K_i \cap W \neq \emptyset\}$ is contracted to a partially triangulated $((2k+3) \times (2k+3))$-grid Γ''. To prove this claim we consider two vertices $u, w \in V(\Gamma')$ which are adjacent in the underlying grid of Γ'. Suppose that sets $\phi^{-1}(u)$ and $\phi^{-1}(w)$ are not joined by edges in $\hat{G}'$. Then there is a clique K_i such that $K_i \cap \phi^{-1}(u) \neq \emptyset$ and $K_i \cap \phi^{-1}(w) \neq \emptyset$. If K_i does not intersect models of other vertices, then, by Case 2, $z_{\text{new}}^{(i)}$ is either included in the model which contains $\phi^{-1}(u)$ or added in the model containing $\phi^{-1}(w)$. If K_i intersects a model of some other vertex w, then one of the vertices u, w is a rectangular vertex of the triangle induced by u, w and x. Now by Case 3, we again have that $z_{\text{new}}^{(i)}$ is either included in the model which contains $\phi^{-1}(u)$ or added in the model containing $\phi^{-1}(w)$.

Finally we apply the boundary contraction of $\hat{G}''$ to Γ''. The graph $\mathbf{bc}(\hat{G}'', \Gamma'')$ is a partially triangulated $((2k+1) \times (2k+1))$-grid which, by Lemma 5, contains Γ_k as a contraction. $\square$

Lemma 9. *Let G be a graph and let $\mathcal{C} = \{K_1, \ldots, K_r\}$ be a sequence of cliques in G, let $E \subseteq \bigcup_{i=1,\ldots,r} E(G[K_i])$ and let $\hat{G} = \mathbf{cl}(G, \mathcal{C}, E)$. Let also $G' = G - X$ for some $X \subseteq V(G)$, where $|X| \leqslant h$. We set $\mathcal{C}' = \{K_1 \setminus X, \ldots, K_r \setminus X\}$, E' be the edges of E without endpoints in X and let $\hat{G}' = \mathbf{cl}(G', \mathcal{C}', E')$. Then if $\hat{G}'$ can be contracted to Γ_k, then $\hat{G}$ can be contracted to a graph H containing a vertex subset Y, $|Y| \leqslant h$, where $H - Y = \Gamma_k$.*

Proof. Let Γ_k be a contraction of $\hat{G}'$. We apply the same contraction for the graph $\hat{G}$. Note that if $K_i \setminus X \neq \emptyset$, then $z_{\text{new}}^{(i)}$ is contracted to some vertex of Γ_k and if $\hat{G}$ contains an edge $\{z_{\text{new}}^{(i)}, x\}$, where $x \in X$, then the resulting edge joins x with some vertex of Γ_k. For every K_i such that $K_i \setminus X = \emptyset$, vertex $z_{\text{new}}^{(i)}$ is adjacent only to vertices of X and we contract all edges incident $z_{\text{new}}^{(i)}$. Let H be the resulting graph and let Y be the set of vertices obtained after these contractions from X. Then $|Y| \leqslant |X| \leqslant h$ and $H - Y = \Gamma_k$, which concludes the proof of the lemma. $\square$

Lemma 10. *Let G be a connected graph obtained from $\Gamma_{2^r k + 4(2^r - 1)}$ by adding $r \geqslant 1$ new vertices and an arbitrary number of edges incident to these vertices. Then G can be contracted to an apex graph which contains Γ_k and at most one additional vertex which is adjacent to some vertices of Γ_k.*

Proof. Let $f(r,k) = 2^r k + 4(2^k - 1)$ and observe that $f(r,k) = 2(f(r-1,k) + 2)$. We denote by X the set of additional vertices. Recall that Γ^*_k is the graph obtained from Γ_k by removing all edges incident to its loaded corner that do not exist in its underlying grid.

We may assume that X is an independent set, otherwise we just contract all edges in each of the connected components of $G[X]$. We prove the lemma by making use of induction on r. For $r = 1$ the graph G is an apex graph itself and in this case lemma is trivial. Now we assume that it is correct for $|X| < r$, for some $r \geqslant 2$. Let A_i, $i = 1, \ldots, 4$, be four vertex disjoint copies of $\Gamma^*_{f(r-1,k)+2}$ in $\mathbf{P}_4(\Gamma_{f(k,r)})$. Let x_1 and x_2 be two vertices of X and let u_1, u_2 be neighbors of x_1 and x_2 in $\Gamma_{f(r,k)}$. We choose $i \in \{1, 2, 3, 4\}$ such that $u_1, u_2 \notin V(A_i)$. Then we contract the following edges of G: First contract all edges with both endpoints in $\bigcup_{j\in\{1,2,3,4\}\setminus\{i\}} A_j$ to some vertex w. Then contract the edges $\{x_1, w\}$ and $\{x_2, w\}$ to a vertex x. Then contract all boundary vertices of A_i to a single vertex z. This creates a graph H that is isomorphic to $\Gamma^*_{f(r,k)}$ together with the vertex z, connected with all its boundary vertices and the vertices in $X' = X \setminus \{x_1, x_2\} \cup \{x\}$ adjacent with some vertices of $\Gamma^*_{f(r,k)}$ or z. Then, there is a corner w of $\Gamma^*_{f(r,k)}$ such that if we further contract $\{w, z\}$ in H, the resulting graph will be $\Gamma_{f(r-1,k)}$ together with the vertices in X' adjacent to some of the vertices in $\Gamma_{f(r-1,k)}$. Now we use the induction assumption for $|X'| \leqslant r-1$, and obtain that H contains Γ_k as a contraction. □

The following lemma is the most crucial technical result used in the proofs of Theorems 3 and 2.

Lemma 11. *Let G be a connected graph excluding a graph H as a minor. Then there exists some constant c_H such that if $\mathbf{tw}(G) \geqslant c_H \cdot k$, then G contains as a contraction a graph where the removal of at most one of its vertices results in Γ_k.*

Proof. Let G be a connected H-minor-free graph. If H is a planar graph then G has bounded treewidth [21] and the claim of the theorem is trivial. Assume that H is not planar. By Proposition 3, G can be represented as h-clique-sum $G = G_1 \oplus \cdots \oplus G_m$ such that each graph G_i can be h-nearly-embedded in a surface Σ (on which H cannot be embedded) where h is a constant which depends only on H. Let $F = G_i$ such that $\mathbf{tw}(F) = \max_{j=1,\dots,m} \mathbf{tw}(G_j)$. By Lemma 1,

$$\mathbf{tw}(G) \leqslant \mathbf{tw}(F). \tag{1}$$

Assume that F is h-nearly-embedded in Σ and denote by X the set of apices of F. Recall that $|X| \leqslant h$. Let $F' = F - X$. By Lemma 2,

$$\mathbf{tw}(F) - |X| \leqslant \mathbf{tw}\bigl(F'\bigr). \tag{2}$$

Observe that F' is h-nearly embedded in Σ without apices. By combining Lemma 7 with (1) and (2), we conclude that there is a constant c_H which depends only on H such that if $\mathbf{tw}(G) \geqslant c_H \cdot k$, then F' contains as a v-smooth contraction the graph Γ_r where v is the loaded corner of Γ_r and $r = 2^{h+1} \cdot k + 8(2^h - 1) + 5$.

Denote by $S_1, \dots, S_t$ components of the graph $G - V(F)$. For each S_i let K_i be the set of vertices of F which are adjacent to some vertex of S_i and let $\mathcal{C} = \{K_1, \dots, K_t\}$. By the definition of h-clique-sum each K_i is a clique of F. Denote by E the set of virtual edges of F. We assume that for any virtual edge $\{u, v\}$, there is a clique $K_i \in \mathcal{C}$ such that $u, v \in K_i$ (otherwise it is easy to redefine h-clique-sums in the representation of G and exclude such an edge). For every component S_i, all vertices of it are contracted into a single vertex $z^{(i)}_{\text{new}}$. Denote by $\hat{F}$ obtained from G by these contractions. It can be easily seen that $\hat{F}$ is the graph $\mathbf{cl}(F, \mathcal{C}, E)$. We set $\mathcal{C}' = \{K_1 \setminus X, \dots, K_t \setminus X\}$, E' be the edges of E without endpoints in X and let $\hat{F}' = \mathbf{cl}(F', \mathcal{C}', E')$. Since F' can be contracted to Γ_r, it follows immediately from Lemma 8 that $\hat{F}'$ contains Γ_s as a contraction for $s = (r-5)/2 = 2^h \cdot k + 4(2^h - 1)$. Then by Lemma 9, $\hat{F}$ (and consequently the graph G) can be contracted to a graph R containing a vertex subset Y, $|Y| \leqslant h$ such that $R - Y = \Gamma_s$. It remains to use Lemma 10 and note that R can be contracted to an apex graph which consists of Γ_k and at most one apex vertex which is adjacent to some vertices of Γ_k. The graph R is a contraction of G, so G contains as a contraction a graph which after the removal of at most one of its vertices results to Γ_k. □

4. Proofs of theorems

Proof of Theorem 3. Let H be an apex graph. It was shown by Robertson et al. [21], that every planar graph on $\lceil h/14 \rceil$ vertices is a minor of an $(h \times h)$-grid and without loss of generality, we can assume that H is a graph constructed from an $(h \times h)$-grid by adding one apex vertex adjacent to all vertices of the grid. By Lemma 11, if $\mathbf{tw}(G) \geqslant c_H \cdot k$, for some constant c_H, then G contains as a contraction a graph F such that the removal of at most one of its vertices results in $\Gamma = \Gamma_{h\cdot(k+2)}$. If $F = \Gamma$, then the theorem follows trivially. Thus we assume that F has an additional vertex u adjacent to some vertices of Γ. We consider the collection $\mathbf{P}_h(\Gamma)$ of h^2 vertex disjoint induced subgraphs of Γ. We claim that there is a subgraph in $\mathbf{P}_h(\Gamma)$ such that none of its vertices is adjacent to u. Indeed, if each subgraph in $\mathbf{P}_h(\Gamma)$ contains a vertex adjacent to u, then Γ contains an $(h \times h)$-grid as a minor such that the nodes of this grid are the neighbors of u. But this contradicts the assumption that G is H-minor-free.

Thus there is a subgraph Γ^*_{k+2} in $\mathbf{P}_h(\Gamma)$ such that none of the vertices of Γ^*_{k+2} is adjacent to u. The graph Γ^*_{k+2} can be seen as a graph obtained from Γ_{k+2} after the removal all the edges adjacent to the loaded corner of Γ_{k+2} that are not edges of the underlying grid. Therefore, after applying the boundary contraction of F to Γ^*_{k+2}, the resulting graph $\mathbf{bc}(F, \Gamma')$ is Γ_k. $\square$

Proof of Theorem 2. Let us assume that $\mathbf{tw}(G) \geqslant c_H \cdot k^2$, where c_H is the constant from Lemma 11. By the same lemma, G can be contracted to a graph H such that by the removal of at most one vertex of H the result is isomorphic to Γ_{k^2}. If H is itself isomorphic to Γ_{k^2} then we are done as Γ_{k^2} contains Γ_k as a contraction. Suppose then that G has an additional vertex x and let $S = N_G(x)$. Let $\mathcal{P}$ be a collection of k disjoint copies of Γ^*_k in Γ_{k^2}. In case, where for some $A \in \mathcal{P}$, $V(A) \cap S = \emptyset$, we contract all edges with both endpoints in $\cup_{H \in \mathcal{P}\setminus\{A\}} V(H)$. The obtained graph is Γ^*_k with one more vertex adjacent to all its external vertices and this graph can be further contracted to Γ_k. Suppose now that each graph in $\mathcal{P}$ intersects some neighbor of x. Then contract all edges of all graphs in $\mathcal{P}$ and the resulting graph is Π_k. $\square$

Proof of Theorem 1. Suppose that G does not contain $H = K_k$ as a contraction. Then G is an H-minor-free graph. By Theorem 2, there exists some constant c_H such that if $\mathbf{tw}(G) \geqslant c_H \cdot k^2$, then G contains as a contraction either Γ_k or Π_k. We put $c_k = c_H \cdot k^2$, which concludes the proof of the theorem. $\square$

5. Contraction bidimensionality revised

The theory of bidimensionality is a meta algorithmic framework for designing subexponential fixed-parameter algorithms, kernelization and approximation algorithms for a broad range of graph problems [2,8,5,6,3,14]. In this section we present a simplification effect of Theorem 3 to this theory. In particular, the theorem simplifies the applications of bidimensionality theory to contraction closed parameters on planar graphs [2], graphs of bounded genus [8], and apex-minor-free graphs [2].

A graph parameter $\mathbf{p}$ is a function mapping graphs to nonnegative integers. We say that $\mathbf{p}$ is *minor* (*contraction*)*-closed* if for every two graphs H and G where H is a minor (a contraction) of G, it holds that $\mathbf{p}(H) \leqslant \mathbf{p}(G)$.

The decision problem associated with $\mathbf{p}$ asks, for a given graph G and nonnegative integer k, whether $\mathbf{p}(G) \leqslant k$. Intuitively, a parameter is bidimensional if its value depends on the "area" of a grid and not on its width or height.

For minor-closed parameters, the definition of bidimensionality is simple to define. According to [3], a parameter $\mathbf{p}$ is *minor bidimensional* if

(a) $\mathbf{p}$ is closed under taking of minors, and
(b) for the $(k \times k)$-grid G_k, $\mathbf{p}(G_k) = \Omega(k^2)$.

Examples of minor bidimensional parameters are sizes of a vertex cover, a feedback vertex set, or a minimum maximal matching in a graph.

For contraction-closed parameters, the definition of bidimensionality is much more complicated and depends on the class of graphs it is used for. A parameter $\mathbf{p}$ is *contraction bidimensional* if the following hold:

(a) $\mathbf{p}$ is closed under taking of contractions, and
(b) for a "$(k \times k)$-grid-like graph" Γ, $\mathbf{p}(\Gamma) = \Omega(k^2)$.

According to the current state of the art, the property of being a "$(k \times k)$-grid-like graph" is different for different graph classes and is defined as follows.

(b1) For planar graphs and single-crossing-minor-free graphs, a "$(k \times k)$-grid-like graph" is a partially triangulated $(k \times k)$-grid;
(b2) For graphs of Euler genus γ, this is a partially triangulated $(k \times k)$-grid with up to γ additional handles (introduced in [8]);
(b3) For apex-minor-free graphs, this is $(k \times k)$-augmented grid, i.e. partially triangulated grid augmented with additional edges such that each vertex is incident to $O(1)$ edges to non-boundary vertices of the grid (introduced in [2]).

Typical examples of contraction bidimensional parameters are sizes of a dominating, clique-transversal, or edge domination sets.

Unfortunately, there is a drawback in the above contraction-bidimensionality framework which was inherited by the "excluding-grid" theorem for contractions. The problem is that the number of augmented grids is huge. Even the number of planar augmented grids, i.e. graphs obtained by triangulating some faces of a $(k \times k)$-grid, is at least $2^{(k-1)^2}$. As a result, to verify if a parameter is apex-contraction bidimensional, one has to estimate its value on a graph family of exponential size.

The main contribution of Theorem 3 to contraction bidimensionality is that the notions of "grid-like" graphs (b1), (b2), and (b3) can be replaced by the following simpler one

(b′) $\mathbf{p}(\Gamma_k) = \Omega(k^2)$.

This unification is justified by the following theorem, which can be seen as a (meta) algorithmic consequence of this paper.

Theorem 4. *Let $\mathbf{p}$ be a graph parameter which satisfies conditions* (a) *and* (b′)*. Let G be an n-vertex graph excluding an apex graph H as a minor. Then if $\mathbf{p}$ is computable in time $2^{O(\mathbf{tw}(G))} \cdot n^{O(1)}$, then deciding $\mathbf{p}(G) \leqslant k$ can be done in time $2^{O(\sqrt{k})} \cdot n^{O(1)}$.*

Proof. Because $\mathbf{p}$ satisfies condition (b′), there is a constant $\alpha > 0$ such that $\mathbf{p}(\Gamma_\ell) > \alpha \ell^2$ for every $\ell \geqslant 1$. Feige, et al. [10] gave a polynomial constant factor approximation algorithm computing the treewidth of a graph excluding some fixed graph H as a minor. We run this β-approximation algorithm on G, where $\beta > 1$ is a constant depending only on H and let t be the value given by this algorithm. Thus $t \leqslant \beta \cdot \mathbf{tw}(G)$. If $t > \beta \cdot c_H \cdot \sqrt{k/\alpha}$, we can deduce that $\mathbf{p}(G) > k$. Indeed, in this case, the treewidth of G is at least $c_H \cdot \sqrt{k/\alpha}$ and by Theorem 3, G contains $\Gamma_{\sqrt{k/\alpha}}$ as a contraction. Because of condition (a), $\mathbf{p}(G) \geqslant \mathbf{p}(\Gamma_{\sqrt{k/\alpha}}) > \alpha \cdot (\sqrt{k/\alpha})^2 = k$. If $t \leqslant \beta \cdot c_H \cdot \sqrt{k/\alpha}$, then $t \leqslant c_H \sqrt{k/\alpha}$ and by conditions of the theorem, we can compute $\mathbf{p}(G)$ in time $2^{O(\sqrt{k})} \cdot n^{O(1)}$. □

We stress that Theorem 4 is not the only algorithmic application of our results. Theorem 3 has already being used in parameterized algorithm design [1,14], approximation algorithms [12,13], and the study of other partial orderings on graphs [23].

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