A strong "abc-conjecture" for certain partitions a+b of c

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Abstract We prove that for any positive integer c and any $\varepsilon > 0$ there are representations of c as a sum $a+b$ of two co-prime positive integers a, b, such that the respective radicals $R(abc)$ satisfy

$$k_\varepsilon \, R(c)^{1-\varepsilon} \, c^2 \;<\; R(abc),$$

where k_ε is an absolute constant depending only on ε. For the representations in question this is a stronger result than the abc-conjecture

$$\kappa_\varepsilon \, c^{\frac{1}{1+\varepsilon}} \;<\; R(abc).$$

An upper bound, depending on the number of prime factors of c, is also established.

Preliminaries. Let $c = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_\omega^{\alpha_\omega}$, q_i different primes, $\alpha_i \geq 1$, be a positive integer. Denote its radical $q_1 q_2 \cdots q_\omega$, by $R(c)$, and similarly $R(n)$ for any integer n. Consider the positive solutions of the Diophantine equation $x + y = c$, $(x, y) = 1$, $x < y$. Their number is $\varphi(c)/2$. Denoting them in some order by a_i, b_i, $1 \leq i \leq \varphi(c)/2$, and listing them with their respective radicals, one has

$$\begin{array}{cc} a_1 + b_1 = c & R(a_1 b_1 c) \\ a_2 + b_2 = c & R(a_2 b_2 c) \\ \cdots & \cdots \\ a_{\frac{\varphi(c)}{2}} + b_{\frac{\varphi(c)}{2}} = c & R(a_{\frac{\varphi(c)}{2}} b_{\frac{\varphi(c)}{2}} c). \end{array} \tag{1}$$

Form the product of above radicals

$$G_c = \prod_{1 \leq i \leq \frac{\varphi(c)}{2}} R(a_i b_i c).$$

$G_c^{2/\varphi(c)}$ is the geometric mean of the radicals.

The function $E_c(x)$ defined for any real $x \neq 0$ by

$$E_c(x) = \left[\frac{1}{x}\frac{c}{1}\right] - \sum_{1 \leq i \leq \omega} \left[\frac{1}{x}\frac{c}{q_i}\right] + \sum_{1 \leq i,j \leq \omega} \left[\frac{1}{x}\frac{c}{q_i q_j}\right] - \cdots + (-1)^\omega \left[\frac{1}{x}\frac{c}{q_1 \cdots q_\omega}\right],$$

plays a key role in our investigations. If the numbers in the integral part brackets are integers the function reduces to

$$\frac{1}{x}\frac{c}{1} - \sum_{1\le i\le\omega} \frac{1}{x}\frac{c}{q_i} + \sum_{1\le i,j\le\omega} \frac{1}{x}\frac{c}{q_iq_j} - \cdots + (-1)^{\omega}\frac{1}{x}\frac{c}{q_1\cdots q_\omega} = \frac{1}{x}\varphi(c).$$

Throughout the paper, p designates prime numbers.

Theorem 1.

$$G_c = R(c)^{\frac{\varphi(c)}{2}} \prod_{\substack{2\le p<c \\ (p,\,c)=1}} p^{E_c(p)}.$$

Proof. Consider *all* positive solutions of the Diophantine equation $x+y=c, \quad x\le y,$ and their corresponding radicals $R(xyc)$

$$\begin{array}{cc} 1+(c-1)=c & R(1(c-1)c) \\ 2+(c-2)=c & R(2(c-2)c) \\ \cdots & \cdots \\ \left[\frac{c}{2}\right]+\left(c-\left[\frac{c}{2}\right]\right)=c & R\left(\left[\frac{c}{2}\right]\left[c-\frac{c}{2}\right]c\right) \end{array} \tag{2}$$

Above equalities comprise (1), but include also those for which $(x,\ y)>1$. That all q_i appear $\frac{\varphi(c)}{2}$ times in G_c is obvious. Consequently, so does their product $R(c)$. For primes $p\neq q_i$, we apply the inclusion-exclusion principle. As all numbers $<c$ do occur in the equalities (2), the number of times p appears in the radicals (2) is $\left[\frac{c}{p}\right]$. The number of times pq_i appears in the radicals (2) is $\sum_i^{\omega}\left[\frac{c}{pq_i}\right]$. The number of times pq_iq_j appears in the radicals (2) is $\sum_{i,\,j}^{\omega}\left[\frac{c}{pq_iq_j}\right]$, e.t.c. Inserting these values in the inclusion-exclusion formula we get for the total number of times p appears in the radicals (1)

$$\left[\frac{1}{p}\frac{c}{1}\right] - \sum_{1\le i\le\omega}\left[\frac{1}{p}\frac{c}{q_i}\right] + \sum_{1\le i,j\le\omega}\left[\frac{1}{p}\frac{c}{q_iq_j}\right] - \cdots + (-1)^{\omega}\left[\frac{1}{p}\frac{c}{q_1\cdots q_\omega}\right] = E_p(c),$$

as stated in the theorem.

Corollary. If d denotes the divisors of c, then

$$\prod_{d|c} G_d \ = \ q_1^{\Theta(q_1)} \ldots \ q_\omega^{\Theta(q_\omega)} \prod_{\substack{2\le p<c, \\ (p,\,c)=1}} p^{\left[\frac{c}{p}\right]},$$

where

$$\Theta(q_i) = \sum_{\substack{d=0(q_i)\\ d|c}} \frac{\varphi(d)}{2} + \sum_{\substack{d\neq 0(q_i)\\ d|c}} E_d(q_i).$$

Proof. Applying Theorem 1 to the divisors d of c one has

$$G_d = R(d)^{\frac{\varphi d}{2}} \prod_{\substack{2\leq p<d\\ (p,\, d)=1}} p^{E_d(p)}.$$

Multiplying over all $\tau(c)$ divisors and using the, easily established, fact that

$$\prod_{\substack{x+y=c\\ x\leq y}} R(xyc) = R(c)^{\left[\frac{c}{2}\right]} \prod_{\substack{2\leq p<c\\ (p,\, c)=1}} p^{\left[\frac{c}{p}\right]},$$

gives for $\prod_{d|c} G_d$ the result.

The corollary will not be used in the sequel.

We shall now prove certain Lemmas regarding the function $E_c(x)$, and state, without proof, some well known facts from the elementary theory of primes, so as not to interrupt the main body of the proof. Absolute constants will be denoted by k_i, indexed in the order they first appear.

Lemma 1. For $x > 0$ (actually for any $x \neq 0$)

$$Max\left(0,\ \frac{\varphi(c)}{x} - 2^{\omega-1}\right) < E_c(x) < \frac{\varphi(c)}{x} + 2^{\omega-1}.$$

Proof. By definition one has

$$\frac{1}{x}\frac{c}{1} - \binom{\omega}{0} < \left[\frac{1}{x}\frac{c}{1}\right] \leq \frac{1}{x}\frac{c}{1}$$

$$-\sum_{1\leq i\leq\omega} \frac{1}{x}\frac{c}{q_i} \leq -\sum_{1\leq i\leq\omega} \left[\frac{1}{x}\frac{c}{q_i}\right] < \binom{\omega}{1} - \sum_{1\leq i\leq\omega} \frac{1}{x}\frac{c}{q_i}$$

$$\sum_{1\leq i\ j\leq\omega} \frac{1}{x}\frac{c}{q_iq_j} - \binom{\omega}{2} < \sum_{1\leq i,j\leq\omega} \left[\frac{1}{x}\frac{c}{q_iq_j}\right] \leq \sum_{1\leq i,j\leq\omega} \frac{1}{x}\frac{c}{q_iq_j}$$

$$\cdots \qquad\qquad \cdots \qquad\qquad \cdots$$

$$\frac{1}{x}\frac{c}{q_1\cdots q_\omega} - \binom{\omega}{\omega} < (-1)^\omega \left[\frac{1}{x}\frac{c}{q_1\cdots q_\omega}\right] \le \frac{1}{x}\frac{c}{q_1\cdots q_\omega}, \qquad \omega \equiv 0(2)$$

$$-\frac{1}{x}\frac{c}{q_1\cdots q_\omega} \le -1)^\omega \left[\frac{1}{x}\frac{c}{q_1\cdots q_\omega}\right] < \binom{\omega}{\omega} - \frac{1}{x}\frac{c}{q_1\cdots q_\omega}, \qquad \omega \equiv 1(2).$$

Adding term-wise above inequalities, we have

$$\frac{1}{x}\left\{c - \sum_{1\le i\le\omega}\frac{c}{q_i} + \sum_{1\le i,j\le\omega}\frac{c}{q_iq_j} - \cdots\right\} - \sum_{\nu\equiv 0(2)}\binom{\omega}{\nu} < E_c(x) <$$

$$\frac{1}{x}\left\{c - \sum_{1\le i\le\omega}\frac{c}{q_i} + \sum_{1\le i,j\le\omega}\frac{c}{q_iq_j} - \cdots\right\} + \sum_{\nu\equiv 1(2)}\binom{\omega}{\nu}.$$

Considering that

$$\sum_{\nu\equiv 0(2)}\binom{\omega}{\nu} = \sum_{\nu\equiv 1(2)}\binom{\omega}{\nu} = 2^{\omega-1},$$

and since $E_c(x)$ is the # of numbers $n \le \frac{c}{x}$, $(n,c)=1$, i.e. always ≥ 0, we have, as required,

$$Max\left(0,\ \frac{\varphi(c)}{x} - 2^{\omega-1}\right) < E_c(x) < \frac{\varphi(c)}{x} + 2^{\omega-1}.$$

Lemma 2.

$$E_c(x) \quad > \quad \begin{cases} \dfrac{\varphi(c)}{x} - 2^{\omega-1} & \text{for} \quad 0 < x < \dfrac{\varphi(c)}{2^{\omega-1}} \\ \quad 1 & \text{for} \quad \dfrac{\varphi(c)}{2^{\omega-1}} \le x < c. \end{cases}$$

Proof. The expression $\frac{\varphi(c)}{x} - 2^{\omega-1}$ is positive for all $0 < x < \frac{\varphi(c)}{2^{\omega-1}}$. For $x \ge \frac{\varphi(c)}{2^{\omega-1}}$ the lowest limit of $E_c(x)$ is 1, since $x < c$.

Lemma 3. If one of the prime factors q_i, has exponent $\boxed{\alpha_i \geq 2}$ then

$$E_c(q_i) = \frac{\varphi(c)}{q_i}.$$

Proof. Writing q_i instead of x in $E_c(x)$, and renaming the running indexes, one has

$$E_c(q_i) = \left[\frac{1}{q_i}\frac{c}{1}\right] - \sum_{1\leq j\leq \omega}\left[\frac{1}{q_i}\frac{c}{q_j}\right] + \sum_{1\leq j,k\leq \omega}\left[\frac{1}{q_i}\frac{c}{q_j q_k}\right] - \cdots + (-1)^{\omega}\left[\frac{1}{q_i}\frac{c}{q_1\cdots q_\omega}\right].$$

Since by supposition the exponent of q_i is ≥ 2, the numbers within the integral part brackets are integers so that we can skip the brackets. This gives

$$\frac{1}{q_i}\frac{c}{1} - \sum_{1\leq j\leq \omega}\frac{1}{q_i}\frac{c}{q_j} + \sum_{1\leq j,k\leq \omega}\frac{1}{q_i}\frac{c}{q_j q_k} - \cdots + (-1)^{\omega}\frac{1}{q_i}\frac{c}{q_1\cdots q_\omega} = \frac{\varphi(c)}{q_i},$$

as stated.

Lemma 4. If one of the prime factors q_i has exponent $\boxed{\alpha_i = 1}$, then

$$E_c(q_i) = \frac{\varphi(c)}{q_i - 1} - E_{c/q_i}(q_i).$$

Proof. For convenience, putting $\overline{c} = c/q_i$, renaming indexes as above, and writing on the left hand side the terms of the $E_c(q_i)$ function vertically, we have following equalities

$$\left[\frac{1}{q_i}\frac{c}{1}\right] = \left[\frac{1}{q_i}\frac{q_i\overline{c}}{1}\right]$$

$$-\sum_{1\leq j\leq \omega}\left[\frac{1}{q_i}\frac{c}{q_j}\right] = -\left[\frac{1}{q_i}\frac{q_i\overline{c}}{q_i}\right] - \sum_{2\leq j\leq \omega}\left[\frac{1}{q_i}\frac{q_i\overline{c}}{q_j}\right]$$

$$+\sum_{1\leq j,\, k\leq \omega}\left[\frac{1}{q_i}\frac{c}{q_j q_k}\right] = +\sum_{2\leq j\leq \omega}\left[\frac{1}{q_i}\frac{q_i\overline{c}}{q_i q_j}\right] + \sum_{2\leq j,\, k\leq \omega}\left[\frac{1}{q_i}\frac{q_i\overline{c}}{q_j q_k}\right]$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$(-1)^{\omega-1}\sum_{1\le j_\nu\le\omega}\left[\frac{1}{q_i}\frac{c}{q_{i_1}q_{j_1}\cdots q_{j_{\omega-1}}}\right]=(-1)^{\omega-1}\sum_{1\le j_\nu\le\omega}\left[\frac{1}{q_i}\frac{q_i\overline{c}}{q_iq_{i_1}q_{j_1}\cdots q_{j_{\omega-1}}}\right]+$$

$$(-1)^{\omega-1}\left[\frac{1}{q_i}\frac{q_i\overline{c}}{q_2\cdots q_\omega}\right]$$

$$(-1)^{\omega}\left[\frac{1}{q_i}\frac{c}{q_1q_2\cdots q_\omega}\right]=(-1)^{\omega}\left[\frac{1}{q_i}\frac{q_i\overline{c}}{q_iq_2\cdots q_\omega}\right].$$

Adding above equalities, the sum of the left hand side terms is, as stated, $E_c(q_i)$. The right hand side is equal to

$$\Bigg\{-\left[\frac{1}{q_i}\frac{q_i\overline{c}}{q_i}\right]+\sum_{2\le j\le\omega}\left[\frac{1}{q_i}\frac{q_i\overline{c}}{q_iq_j}\right]+\dots$$

$$+(-1)^{\omega-1}\sum_{1\le j_\nu\le\omega}\left[\frac{1}{q_i}\frac{q_i\overline{c}}{q_iq_{i_1}q_{j_1}\cdots q_{j_{\omega-2}}}\right]+(-1)^{\omega}\left[\frac{1}{q_i}\frac{q_i\overline{c}}{q_iq_2\cdots q_\omega}\right]\Bigg\}+$$

$$\Bigg\{\left[\frac{1}{q_i}\frac{q_i\overline{c}}{1}\right]-\sum_{2\le j\le\omega}\left[\frac{1}{q_i}\frac{q_i\overline{c}}{q_j}\right]+\sum_{2\le j,\ k\le\omega}\left[\frac{1}{q_i}\frac{q_i\overline{c}}{q_jq_k}\right]-\dots+(-1)^{\omega-1}\left[\frac{1}{q_i}\frac{q_i\overline{c}}{q_2\cdots q_\omega}\right]\Bigg\}.$$

Cancelling q_i within the brackets, the first brace is clearly equal to $-E_{\overline{c}}(q_i)$. In the second brace, as all $q_j,\ q_k,\ \dots\ ,2\le j,\ k\le\omega,\ \dots$ divide $\overline{c}$, the numbers within the integral part brackets are integers. We therefore can write it

$$\overline{c}\left(1-\sum_{2\le j\le\omega}\frac{1}{q_j}-\sum_{2\le j\ k\le\omega}\frac{1}{q_jq_k}+\dots+(-1)^{\omega-1}\frac{1}{q_2\cdots q_\omega}\right)=\varphi(\overline{c})$$

Substituting the braces by their found values and taking into account that $\varphi(\overline{c})=\varphi(c)/(q_i-1)$, since $(c,\overline{c})=1$, we get

$$E_c(q_i)=\frac{\varphi(c)}{q_i-1}-E_{c/q_i}(q_i),$$

which proves the Lemma.

Lemma 5.

$$e^{k_1 c} < \prod_{2 \le p \le c} p \; < e^{k_2 c}.$$

Proof. This is the multiplicative form of Tchebycheff 's estimate for $\sum\limits_{2 \le p \le c} \log p$, namely,

$$k_1 c < \sum_{2 \le p \le c} \log p < k_2 c,$$

for $c \ge 2$, with k_1 k_2 positive absolute constants.

Lemma 6.

$$e^{-k_3} c < \prod_{2 \le p \le c} p^{\frac{1}{p}} < e^{k_3} c.$$

Proof. This is the multiplicative form of Merten's estimate for $\sum\limits_{2 \le p \le c} \frac{1}{p} \log p$, namely,

$$\log c - k_3 < \sum_{2 \le p \le c} \frac{1}{p} \log p < \log c + k_3,$$

for $c \ge 2$, with k_3 a positive absolute constant.

We now state a result which gives a lower bound for the geometric mean $G_c^{\frac{2}{\varphi(c)}}$ in terms of the prime factors $q_i(c)$ and their exponents α_i.

Theorem 2.

$$G_c^{\frac{2}{\varphi(c)}} > k_4 \prod_{1 \le i \le \omega} q_i^{2\alpha_i - 1 - \frac{2}{\varphi(c)} E_c(q_i)} \left(\frac{q_i - 1}{2} \right)^2,$$

where k_4 is a positive absolute constant.

Proof. We transform the expression given for G_c in Theorem 1, as follows:

$$G_c = \prod_{1 \le i \le \omega} q_i^{\frac{\varphi(c)}{2}} \prod_{\substack{2 \le p < c \\ (p,\, c) = 1}} q_i^{E_c(p)}$$

$$= \prod_{1\le i\le\omega} q_i^{\frac{\varphi(c)}{2}} \prod_{1\le i\le\omega} q_i^{-E_c(q_i)} \prod_{2\le p<c} p^{E_c(p)}.$$

Joining the first two products into one and splitting the third product as indicated, we have

$$G_c = \prod_{1\le i\le\omega} q_i^{\frac{\varphi(c)}{2}-E_c(q_i)} \prod_{2\le p<\frac{\varphi(c)}{2^{\omega-1}}} p^{E_c(p)} \prod_{\frac{\varphi(c)}{2^{\omega-1}}<p<c} p^{E_c(p)}.$$

Applying Lemma 2 to the second and third product and splitting the products in an obvious way, we get successively

$$G_c > \prod_{1\le i\le\omega} q_i^{\frac{\varphi(c)}{2}-E_c(q_i)} \prod_{2\le p<\frac{\varphi(c)}{2^{\omega-1}}} p^{\frac{\varphi(c)}{p}-2^{\omega-1}} \prod_{\frac{\varphi(c)}{2^{\omega-1}}<p<c} p$$

$$> \prod_{1\le i\le\omega} q_i^{\frac{\varphi(c)}{2}-E_c(q_i)} \left\{\prod_{2\le p<\frac{\varphi(c)}{2^{\omega-1}}} p^{\frac{1}{p}}\right\}^{\varphi(c)} \left\{\prod_{2\le p<\frac{\varphi(c)}{2^{\omega-1}}} p\right\}^{-2^{\omega-1}} \left\{\prod_{2\le p<\frac{\varphi(c)}{2^{\omega-1}}} p\right\}^{-1} \prod_{2\le p<c} p.$$

Joining the third and the fourth product into one, we have

$$G_c > \prod_{1\le i\le\omega} q_i^{\frac{\varphi(c)}{2}-E_c(q_i)} \left\{\prod_{2\le p<\frac{\varphi(c)}{2^{\omega-1}}} p^{\frac{1}{p}}\right\}^{\varphi(c)} \left\{\prod_{2\le p<\frac{\varphi(c)}{2^{\omega-1}}} p\right\}^{-(2^{\omega-1}+1)} \prod_{2\le p<c} p.$$

Applying Lemma 6 to the second product, Lemma 5 to the third and fourth product, we have

$$G_c > \prod_{1\le i\le\omega} q_i^{\frac{\varphi(c)}{2}-E_c(q_i)} \left\{e^{-k_3}\frac{\varphi(c)}{2^{\omega-1}}\right\}^{\varphi(c)} \left\{e^{k_2\frac{\varphi(c)}{2^{\omega-1}}}\right\}^{-(2^{\omega-1}+1)} e^{k_1 c}.$$

Summing the exponents of e, we have

$$G_c > \prod_{1\le i\le\omega} q_i^{\frac{\varphi(c)}{2}-E_c(q_i)} \left(\frac{\varphi(c)}{2^{\omega-1}}\right)^{\varphi(c)} e^{-k_3\varphi(c)+k_1c-k_2\varphi(c)-k_2\frac{\varphi(c)}{2^{\omega-1}}}.$$

Raising this inequality to the power $\frac{2}{\varphi(c)}$ we get for the geometric mean $G_c^{\frac{2}{\varphi(c)}}$,

$$G_c^{\frac{2}{\varphi(c)}} > \prod_{1\le i\le\omega} q_i^{1-\frac{2}{\varphi(c)}E_c(q_i)} \left(\frac{\varphi(c)}{2^{\omega-1}}\right)^2 e^{-2k_3+2k_1\frac{c}{\varphi(c)}-2k_2\left(1+\frac{1}{2^{\omega-1}}\right)}. \tag{3}$$

Evaluating the second parenthesis, we have

$$\left(\frac{\varphi(c)}{2^{\omega-1}}\right)^2 = 4 \prod_{1\le i\le\omega} q_i^{2\alpha_i-2}\left(\frac{q_i-1}{2}\right)^2.$$

On the other hand, since $\dfrac{c}{\varphi(c)} > 1$ and $\left(1+\dfrac{1}{2^{\omega-1}}\right) \le 2$, the entire exponent of e appearing in (3) is $> -2k_3+2k_1-4k_2$. Setting $k_4 = 4e^{-2k_3+2k_1-4k_2}$, as a new absolute constant and substituting in (3), we finally get

$$G_c^{\frac{2}{\varphi(c)}} > k_4 \prod_{1\le i\le\omega} q_i^{2\alpha_i-1-\frac{2}{\varphi(c)}E_c(q_i)}\left(\frac{q_i-1}{2}\right)^2,$$

which was to be proved.

Following main theorem gives a lower bound for the geometric mean $G_c^{\frac{2}{\varphi(c)}}$ in terms of c and its radical.

Theorem 3. For any given $\varepsilon > 0$

$$G_c^{\frac{2}{\varphi(c)}} \;>\; k_\varepsilon\; R(c)^{1-\varepsilon}\; c^2,$$

where k_ε is a positive absolute constant, depending on ε.

Proof. Denote the expressions within the product of theorem 2 by $F(q_i,\alpha_i)$, $1 \le q_i \le \omega$. The proof is in three steps. The first step gives a lower bound of $F(q_i,\alpha_i)$ for $\alpha_i \ge 2$. The second step gives a lower bound of $F(q_i,\alpha_i)$ for $\alpha_i = 1$. The third step combines these results to prove the theorem.

Step 1. For $\boxed{\alpha_i \ge 2}$ we have by Lemma 3

$$\begin{aligned} F(q_i,\alpha_i) &= q_i^{2\alpha_i-1-\frac{2}{\varphi(c)}E_c(q_i)}\left(\tfrac{q_i-1}{2}\right)^2 \\ &= q_i^{2\alpha_i-1-\frac{2}{q_i}}\left(\tfrac{q_i-1}{2}\right)^2 \\ &= q_i^{2\alpha_i+1}\left(q_i^{\frac{1}{q_i}}\right)^{-2}\left(\tfrac{1}{2}-\tfrac{1}{2q_i}\right)^2. \end{aligned}$$

Since for $q_i \ge 2$ the product $\left(q_i^{\frac{1}{q_i}}\right)^{-2}\left(\frac{1}{2}-\frac{1}{2q_i}\right)^2$, increasing monotonically, tends to $1/4$ for $q_i \to \infty$, we can write, for any given $\varepsilon > 0$

$$F(q_i,\alpha_i) > q_i^{2\alpha_i+1} q_i^{-\varepsilon} = q_i^{2\alpha_i+1-\varepsilon},$$

for all primes greater than N_ε, where N_ε is a number depending only on ε.

For primes smaller than N_ε, we can write

$$F(q_i,\alpha_i) \;>\; q_i^{2\alpha_i+1}\left(q_i^{\frac{1}{q_i}}\right)^{-2}\left(\frac{1}{2}-\frac{1}{2q_i}\right)^2 q_i^{-\varepsilon} \;>\; \frac{1}{32}\; q_i^{2\alpha_i+1-\varepsilon},$$

since, as said before, the product $\left(q_i^{\frac{1}{q_i}}\right)^{-2}\left(\frac{1}{2}-\frac{1}{2q_i}\right)^2$ is monotonically increasing, and therefore its minimum is at $q_i=2$, i.e. is equal to $\left(2^{\frac{1}{2}}\right)^{-2}\left(\frac{1}{2}-\frac{1}{4}\right)^2=\frac{1}{32}$.

N_ε can be calculated. It is the abscissa where the two curves $y=\left(x^{\frac{1}{x}}\right)^{-2}\left(\frac{1}{2}-\frac{1}{2x}\right)^2$ and $y=x^{-\varepsilon}$ cut each other. Consequently it is the positive root of the equation

$$x-1=2x^{\frac{1}{x}+\frac{2-\varepsilon}{2}},$$

which results after setting

$$\left(x^{\frac{1}{x}}\right)^{-2}\left(\frac{1}{2}-\frac{1}{2x}\right)^2=x^{-\varepsilon}.$$

By elementary analysis $N_\varepsilon\to\infty$ for $\varepsilon\to 0$.

Step 2. For $\boxed{\alpha_i=1}$ we have by Lemma 4 and Lemma 1

$$\begin{aligned}
F(q_i,\alpha_i) &= q_i^{2\alpha_i-1-\frac{2}{\varphi(c)}E_c(q_i)}\left(\tfrac{q_i-1}{2}\right)^2\\
&= q_i^{2\alpha_i-1-\frac{2}{\varphi(c)}\left[\frac{\varphi(c)}{q_i-1}-E_{c/q_i}(q_i)\right]}\left(\tfrac{q_i-1}{2}\right)^2\\
&= q_i^{2\alpha_i-1-\frac{2}{q_i-1}+\frac{2}{\varphi(c)}E_{c/q_i}(q_i)}\left(\tfrac{q_i-1}{2}\right)^2\\
&> q_i^{2\alpha_i-1-\frac{2}{q_i-1}}\left(\tfrac{q_i-1}{2}\right)^2 = q_i^{2\alpha_i+1}\left(q_i^{\frac{1}{q_i-1}}\right)^{-2}\left(\tfrac{1}{2}-\tfrac{1}{2q_i}\right)^2.
\end{aligned}$$

Since for $q_i\geq 2$ the product $\left(q_i^{\frac{1}{q_i-1}}\right)^{-2}\left(\frac{1}{2}-\frac{1}{2q_i}\right)^2$, increasing monotonically, tends to $\frac{1}{4}$ for $q_i\to\infty$, we can write

$$F(q_i,\alpha_i) > \; q_i^{2\alpha_i+1}q_i^{-\varepsilon}=q_i^{3-\varepsilon},$$

for all primes greater than M_ε, where M_ε is a certain number depending only on ε.

For primes smaller than M_ε we can write

$$
\begin{aligned}
F(q_i,\alpha_i) &> q_i^{2\alpha_i+1}\left(q_i^{\frac{1}{q_i-1}}\right)^{-2}\left(\tfrac{1}{2}-\tfrac{1}{2q_i}\right)^2 q_i^{-\varepsilon}\\
&> \tfrac{1}{64}q_i^{2\alpha_i+1-\varepsilon}=\tfrac{1}{64}q_i^{3-\varepsilon},
\end{aligned}
$$

since the product $\left(q_i^{\frac{1}{q_i-1}}\right)^{-2}\left(\frac{1}{2}-\frac{1}{2q_i}\right)^2$ is monotonically increasing, and therefore its minimum is at $q_i=2$, i.e. is equal to $(2^{-2})\left(\frac{1}{2}-\frac{1}{4}\right)^2=\frac{1}{64}$.

M_ε can be calculated as a function of ε. It is the abscissa where the two curves $y=\left(x^{\frac{1}{x-1}}\right)^{-2}\left(\frac{1}{2}-\frac{1}{2x}\right)^2$ and $y=x^{-\varepsilon}$ cut each other. It is therefore the positive root of the equation

$$
x-1=2x^{\frac{1}{x-1}+\frac{2-\varepsilon}{2}},
$$

which results after setting

$$
\left(x^{\frac{1}{x-1}}\right)^{-2}\left(\frac{1}{2}-\frac{1}{2x}\right)^2=x^{-\varepsilon}.
$$

By elementary analysis $M_\varepsilon\to\infty$ for $\varepsilon\to 0$.

Step 3. Let p run through the primes q_i, $1\le i\le\omega$ of c, and denote by $D(p)$ the corresponding exponents α_i.

All the integers $q_i^{\alpha_i}$, $1\le i\le\omega$, fall into one of the following, mutually exclusive, classes specified under Steps 1 and Step 2, namely

$2\le p\le[N_\varepsilon]$	$D(p)\ge 2$	$\omega_1(\varepsilon)$
$[N_\varepsilon]+1\le p$	$D(p)\ge 2$	$\omega_2(\varepsilon)$
$2\le p\le[M_\varepsilon]$	$D(p)=1$	$\omega_3(\varepsilon)$
$[M_\varepsilon]+1\le p$	$D(p)=1$	$\omega_4(\varepsilon)$.

The numbers $\omega_1(\varepsilon)$, $\omega_2(\varepsilon)$, $\omega_3(\varepsilon)$, $\omega_4(\varepsilon)$, are the number of elements in each class. Obviously, they depend only on ε and their sum is equal to ω.

Accordingly, the product $\prod\limits_{1\le i\le\omega}F(q_i,\alpha_i)$ extended over above classes can be written

$$\prod_{1\le i\le\omega} F(q_i,\alpha_i) = \prod_{\substack{2\le p<[N_\varepsilon]\\ D(p)\ge 2}} F(p,D(p)) \prod_{\substack{[N_\varepsilon]+1\le p\\ D(p)\ge 2}} F(p,D(p))$$

$$\prod_{\substack{2\le p\le[M_\varepsilon]\\ D(p)=1}} F(p,D(p)) \prod_{\substack{[M_\varepsilon]+1\le p\\ D(p)=1}} F(p,D(p)).$$

Applying to these four products the lower bounds found in Step 1 and Step 2 for the respective expressions $F(q_i,\alpha_i)$, we obtain

$$\prod_{1\le i\le\omega} F(q_i,\alpha_i) > \frac{1}{32^{\omega_1(\varepsilon)}} \prod_{\substack{2\le p<[N_\varepsilon]\\ D(p)\ge 2}} p^{2D(p)+1-\varepsilon} \prod_{\substack{[N_\varepsilon]+1\le p\\ D(p)\ge 2}} p^{2D(p)+1-\varepsilon}$$

$$\frac{1}{64^{\omega_2(\varepsilon)}} \prod_{\substack{2\le p\le[M_\varepsilon]\\ D(p)=1}} p^{2D(p)+1-\varepsilon} \prod_{\substack{[M_\varepsilon]+1\le p\\ D(p)=1}} p^{2D(p)+1-\varepsilon}.$$

Since, as said above, the four classes cover, by definition, the whole range of the integers $q_i^{\alpha_i}$, $1\le i\le\omega$, this inequality can be written

$$\prod_{1\le i\le\omega} F(q_i,\alpha_i) > \frac{1}{2^{5\omega_1(\varepsilon)+6\omega_2(\varepsilon)}} \prod_{1\le i\le\omega} q_i^{2\alpha_i+1-\varepsilon} = \frac{1}{2^{5\omega_1(\varepsilon)+6\omega_2(\varepsilon)}}\ R(c)^{1-\varepsilon}\ c^2.$$

By Theorem 2 we therefore have

$$G_c^{\frac{2}{\varphi(c)}} = k_4 \prod_{1\le i\le\omega} F(q_i,\alpha_i) > \frac{k_4}{2^{5\omega_1(\varepsilon)+6\omega_2(\varepsilon)}}\ R(c)^{1-\varepsilon}\ c^2.$$

Setting $k_\varepsilon = \frac{k_4}{2^{5\omega_1(\varepsilon)+6\omega_2(\varepsilon)}}$ as an absolute constant depending on ε, we finally get

$$G_c^{\frac{2}{\varphi(c)}} > k_\varepsilon\ R(c)^{1-\varepsilon}\ c^2,$$

as claimed by Theorem 3.

From above theorem results immediately

Theorem 4. For any positive integer there are partitions $c=a+b$, with positive coprime integers a and b, such that

$$k_\varepsilon\ R(c)^{1-\varepsilon}\ c^2 < R(abc).$$

Proof. Because of Theorem 3, all radicals of equations (1) which are greater than $G_c^{\frac{2}{\varphi(c)}}$ satisfy a fortiori the condition of Theorem 4.

For said partitions this is a substantially stronger result than the abc-conjecture

$$\kappa_\varepsilon \, c^{\frac{1}{1+\varepsilon}} \quad < \quad R(abc), \qquad \varepsilon > \; 0.$$

We shall now obtain an upper bound for $G_c^{\frac{2}{\varphi(c)}}$. This will depend on ω, the number of prime factors of c. To this end we prove following

Theorem 5.

$$G_c^{\frac{2}{\varphi(c)}} < \; k_5 \, k_6^{3^\omega} \; R(c) \, c^2,$$

where $k_5 > 0$ and $k_6 > 1$ are absolute constants.

Proof. Applying Lemma 1, Lemma 5 and Lemma 6, we get successively

$$\begin{aligned}
G_c &= R(c)^{\frac{\varphi(c)}{2}} \prod_{\substack{2 \le p < c \\ (p,\, c)=1}} p^{E_c(p)} && \text{by Theorem 1} \\
&= \prod_{1 \le i \le \omega} q_i^{\frac{\varphi(c)}{2} - E_c(q_i)} \prod_{2 \le p \le c} p^{E_c(p)} \\
&< \prod_{1 \le i \le \omega} q_i^{\frac{\varphi(c)}{2} - E_c(q_i)} \prod_{2 \le p \le c} p^{\frac{\varphi(c)}{p} + 2^{\omega-1}} && \text{by Lemma 1} \\
&< \prod_{1 \le i \le \omega} q_i^{\frac{\varphi(c)}{2} - E_c(q_i)} \left\{ \prod_{2 \le p \le c} p^{\frac{1}{p}} \right\}^{\varphi(c)} \left\{ \prod_{2 \le p \le c} p \right\}^{2^{\omega-1}} \\
&< \prod_{1 \le i \le \omega} q_i^{\frac{\varphi(c)}{2} - E_c(q_i)} \left(e^{k_3} c \right)^{\varphi(c)} \left(e^{k_2 c} \right)^{2^{\omega-1}} && \text{by Lemma 5, 6} \\
&< e^{k_3 \varphi(c) + k_2 2^{\omega-1} c} \;\; c^{\varphi(c)} \prod_{1 \le i \le \omega} q_i^{\frac{\varphi(c)}{2} - E_c(q_i)} && .
\end{aligned}$$

Raising the last inequality to the power $\frac{2}{\varphi(c)}$ we have

$$G_c^{\frac{2}{\varphi(c)}} \; < \; e^{2k_3 + k_2 2^\omega \frac{c}{\varphi(c)}} \;\; c^2 \prod_{1 \le i \le \omega} q_i^{1 - \frac{2}{\varphi(c)} E_c(q_i)}.$$

Since $\dfrac{2q_i}{q_i - 1} \le 3$ for all $q_i \ge 3$ and $\dfrac{2q_i}{q_i - 1} = 4$ for $q_i = 2$, we have

$$2^\omega \frac{c}{\varphi(c)} = \prod_{1 \le i \le \omega} \frac{2q_i}{q_i - 1} \; < \; 2 \cdot 3^\omega.$$

Hence

$$G_c^{\frac{2}{\varphi(c)}} \;<\; e^{2k_3+2k_2 3^\omega}\; c^2 \prod_{1\le i\le\omega} q_i^{\,1-\frac{2}{\varphi(c)}E_c(q_i)}$$
$$\;<\; e^{2k_3+2k_2 3^\omega}\; c^2 \prod_{1\le i\le\omega} q_i.$$

Setting $k_5 = e^{2k_3}$ and $k_6 = e^{2k_2}$ as absolute constants, we obtain

$$G_c^{\frac{2}{\varphi(c)}} \;<\; k_5\, k_6^{3^\omega}\, R(c)\, c^2,$$

as required.

Remark. Combining Theorem 4 and Theorem 5 we have

$$k_\varepsilon\, R(c)^{1-\varepsilon}\, c^2 \;<\; G_c^{\frac{2}{\varphi(c)}} \;<\; k_5\, k_6^{3^\omega}\, R(c)\, c^2,$$

and dividing by $R(c)c^2$ we get

$$k_\varepsilon\, R(c)^{-\varepsilon} \;<\; \frac{G_c^{\frac{2}{\varphi(c)}}}{R(c)c^2} \;<\; k_5\, k_6^{3^\omega}.$$

Letting c run through the numbers $c_x = q_1^{x_1}\cdots q_\omega^{x_\omega}$, $1 \le x_i < \infty$, which all have the same radical $R(c) = R$ we have

$$k_\varepsilon\, R^{-\varepsilon} \;<\; \liminf_{1\le x_i<\infty} \frac{G_{c_x}^{\frac{2}{\varphi(c_x)}}}{R(c)c_x^2} \;<\; \limsup_{1\le x_i<\infty} \frac{G_{c_x}^{\frac{2}{\varphi(c_x)}}}{R(c)c_x^2} \;<\; k_5\, k_6^{3^\omega}.$$

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