Mathematical Structures Defined by Identities

Constantin M. Petridi,
11 Apollonos Street, 151 24 Maroussi (Athens), Greece, Fax+30-1-6129250
In part - collaboration with
Peter B, Krikelis, Mathematics Dep., Athens University,
e-mail pkrikel@cc.uoa.gr

Abstract

We propound the thesis that there is a "limitation" to the number of possible structures which are axiomatically endowed with identities involving operations. In the case of algebras with a binary operation satisfying a formally irreducible (to be explained) identity between two n-iterates of the operation, it is established that the frequency of such algebras goes to zero like $e^{-n/16}$ as $n \to \infty$. This is proved by a suitable ordering and labeling of the expressions (words) of the corresponding free algebra and the formation of a series of tableaux whose entries are the labels. The tableaux reveal surprising symmetry properties, stated in terms of the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$ and their partitions. As a result of the defining identity and the tableaux all iterates of order higher than n fall into equivalence classes of semantically equal ones. Classnumbers depending on the tableaux are calculated for all algebras of order $n = 3$ (and partially for $n = 4$). Certain classnumbers are invariants in the sense that for algebras of same order they are equal. Algebras with several operations of any arity are considered. A generalization of Catalan numbers depending on homomorphisms of the structure is proposed and corresponding generating functions set up. As an example of this, a kind of skein polynomials are constructed characterizing the formal build-up of iterates. As no distinction is made between the various algebras, isomorphic or not, which are models of the identity the results can also be formulated in terms of varieties with signature the identity in question. [1]

1. Introduction

1.1. Our basic thesis is that in the ocean of mathematical structures defined by identities the essential ones, the ones which are liable to give rise to important developments, are few. Such an assertion may sound presumptuous. We believe, however, that our findings, exposed in this paper, constitute the first steps towards validating our claim.

Above statement necessitates of course to make clear the intuitive meaning of the words structure, ocean, essential and few. We think that every mathematician, irrespective of his special area of interest, has the same notion in mind of the word structure, codified in any of the various formulations which evolved in the course of time. Ocean refers to the multitude resulting from our freedom to imagine and write down formally any combinations of elements and concepts entering in the definition of structure. Structures axiomatically endowed with identities involving operations within the underlying universe are called, as usual *algebras*, resp. *varieties* for the totality of algebras of same signature. The meaning of few is harder to describe. But let us say it entails a kind of measure for a class of algebras having a specific property in relation to the totality of pertinent possible algebras. The establishment of such a specific property and of the relevant measure will become clear in the sequel at

[1] This remark was pointed out to me by Prof. Tom C. Brown, Simon Fraser University, Burnaby, B.C., Canada.

least for the algebras which will be considered. As to the term essential structure this is actually the great unknown and only conjectures can be made if one is to be honest. To illustrate this characterization we would outright say that an algebra with a binary law Vxy satisfying $VVxyz = VxVyz$ is undeniably essential. This meager functional equation over a specified universe (set) for its variables is simply astounding. It pervades the greater part of mathematics, running like Ariadne's thread through the labyrinth of mathematical theories, old and modern. The rich harvest of deep results which have been obtained thanks to its validity - not to say anything, platonically speaking, of those which are still out there waiting to be unveiled - bears out the assertion. Is this only due to the fact that its basic models are the natural numbers and their extensions, as well as the various groups? Or is there a much *deeper reason* that more complicated axiom systems involving identities, are rare among all possible structures we can think of? Dieudonné, in another context, has voiced similar doubts by saying in his introduction to the *"Abrégé d' histoire des Mathematiques 1700-1900"* that "... l' étude des algébres non - associatives les plus générales, n'ont guère justifié par des applications à des problèmes anciens les espoirs de leurs auteurs" .

Groups aside (associative identity), the importance of identities for breaking new ground in mathematics can hardly be emphasized enough. With the exception perhaps of parts of pure geometry, it is a task indeed to seek out mathematical theories where identities are not present either in their axioms or in their developments. Just consider whether the theory of Lie groups and algebras would at all have been possible were it not for the Jacobi identity. Another case in point is the Burnside problem for groups with generators whose elements satisfy the identity $x^n = 1$. The celebrated Euler identity in eight variables, crucial in proving that every positive integer is a sum of four squares, offers another example.

Our investigations have been motivated by our belief, corroborated by examination of low order algebras, that there is a limitation restricting the number of novel algebras we can postulate. Unless we are cunning and lucky to pick the right ones, the ones we propose to call *essential*. The search for these and their classification is the main goal. Our paper, in four sections, is a first step in this direction. Section 2 deals with algebras whose binary operation Vx_1x_2 satisfies an identity of order n, i.e. an equality between two n-iterates of V, which cannot be *formally reduced* (to be explained) to an identity of lower order. It will be shown that the number of such algebras is of the order of $S_n^2 e^{-n/16}$ as $n \to \infty$, where $S_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number. Since formally there are S_n^2 different n-identities, this of course means that the frequency of such algebras goes to zero as $n \to \infty$. The proof is by a novel method we introduce namely that of tableaux, which reveal unexpected properties yielding this result. Section 3 expounds the semantic consequences of the defining identity. All iterates of order higher than n fall into equivalence classes of *semantically equal* ones (to be explained). For the associative identity equivalence classes of iterates of same order collapse into a single class. Depending on the tableaux used, all identities of order n have the same classnumbers whose values are explicitly determined as functions of certain partitions of the Catalan numbers. If natural unicity constraints are imposed some essential algebras have been found. For example, for $n = 3$ the only essential algebras are $VVVx_1x_2x_3\, x_4 = Vx_1\, VVx_2x_3x_4$ and $VVx_1Vx_2x_3\, x_4 = Vx_1\, Vx_2Vx_3x_4$, all others reducing to semigroups. Section 4 in form of notes and remarks, advances the idea that the *method of tableaux* can be extended to any structure provided with an equivalence relation and generating formally new elements from elements already in the structure. Generalizations of Catalan numbers depending on the homomorphisms of the structure, extension of the concept of homomorphism to several variables, skein polynomials associated to the iterates and other allied material are included at the end. Annexed are Exhibits of tables used in the text.

2. Formal Part

2.1. Ordering of iterates of a binary operation. Construction of tableaux A_n and B_n. The number of iterates of order n (n-iterates) of a binary operation $V(x_1x_2)$ is the same as the number of completely parenthesizing a product of $n+1$ letters, with two factors in each set of parentheses. As known, it is equal to the nth Catalan number $S_n = \frac{1}{n+1}\binom{2n}{n}$. We designate the n-iterates by $J^n(x_n \dots x_{n+1}), i = 1, \dots, S_n$, putting in evidence the $n+1$ variables $x_1, \dots, x_{n+1}$.

Suppose now we already have an ordering of the S_{n-1} iterates Vx_1x_2 of order $n-1$. Write them in a line in that order and, omitting indexes, form below table by substituting successively each variable x by Vxx:

$$\begin{array}{cccc}
J_1^{n-1}(x_1 \dots x_n) & J_2^{n-1}(x_1 \dots x_n) & \cdots & J_{S_{n-1}}^{n-1}(x_1 \dots x_n) \\
\hline
J_1^{n-1}(Vxx, x \dots x) & J_2^{n-1}(Vxx, x \dots x) & \cdots & J_{S_{n-1}}^{n-1}(Vxx, x \dots x) \\
J_1^{n-1}(x, Vxx \dots x) & J_2^{n-1}(x, Vxx \dots x) & \cdots & J_{S_{n-1}}^{n-1}(x, Vxx \dots x) \\
\cdots & \cdots & \cdots & \cdots \\
J_1^{n-1}(x, x \dots Vxx) & J_2^{n-1}(x, x \dots Vxx) & \cdots & J_{S_{n-1}}^{n-1}(x, x \dots Vxx)
\end{array}$$

Above table comprises all S_n n-iterates $J_i^n(x_1 \dots x_{n+1})$, or J_i^n for short, $i = 1, \dots, S_n$, each with a certain multiplicity. Tableau A_n (see annexed Exhibit 1, for $n = 4$) is formed by assigning to each iterate a label number from 1 to S_n, starting with 1 for $J_1^{n-1}(Vxx, x \dots x_n)$ and going horizontally through all the lines of the table, from left to right, assigning the same label whenever the same iterate is encountered. We thus get a well-defined ordering of the n-iterates, uniquely induced by the ordering of the $(n-1)$-iterates.

n-Iterates can be generated from $(n-1)$-iterates not just by inserting Vxx in the variable places of $(n-1)$-iterates, as done above, but also by forming

$$\begin{array}{cccc}
V(J_1^{n-1}, x) & V(J_2^{n-1}, x) & \cdots & V(J_{S_{n-1}}^{n-1}, x) \\
V(x, J_1^{n-1}) & V(x, J_2^{n-1}) & \cdots & V(x, J_{S_{n-1}}^{n-1}).
\end{array}$$

Assigning to the iterates of this table the same labels as for A_n we obtain tableau B_n (see Exhibit 3).

Starting with tableaux A_1 and B_1 we can thus *recursively* form tableaux A_n and B_n, for any n. Tableau A_n has nS_{n-1} entries, tableau B_n has $2S_{n-1}$ entries. Besides above two procedures leading to A_n and B_n there is no other operation generating n-iterates from $(n-1)$-iterates. There is a method based on arithmetical properties of A_n and B_n to write them out, but this is quite involved so that at this stage we omit it. Labels have the advantage that it is much more convenient to peruse tables with numbers and find their properties, than tables with expressions like $VVxxVxVxx$ (see Exhibit 2). Whenever we refer to tableaux A_n and B_n we shall indiscriminately use J_i^n or its label $i^{(n}$, or just i, if n is fixed.

2.2. Properties of tableaux A_n. Let L_i, $i = 1, \dots n$ be the n sets figuring respectively in the lines of tableau A_n. If $|M|$ denotes the cardinality of set M and $M_1 \cap M_2$ the intersection of M_1 and M_2 then

$$|L_i \cap L_j| = \begin{cases} S_{n-1} & \text{if} \quad i = j \\ 0 & \text{if} \quad |i-j| = 1 \\ S_{n-2} & \text{otherwise} \end{cases}$$

$$|L_i \cap L_j \cap L_k| = \begin{cases} S_{n-1} & \text{if} \quad i = j = k \\ 0 & \text{if at least one} \quad |i-j|, \dots, |j-k| \quad \text{is} = 1 \\ S_{n-3} & \text{otherwise} \end{cases}$$

and generally for $k = 1, 2, \dots, n$

$$|L_{i_1} \cap L_{i_2} \cap \dots \cap L_{i_k}| = \begin{cases} S_{n-1} & \text{if } i_1 = i_2 = \dots = i_k \\ 0 & \text{if at least one} \quad |i_1 - i_2|, \dots, |i_{k-1} - i_k| \quad \text{is} = 1 \\ S_{n-k} & \text{otherwise} \end{cases}$$

S_n are the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$.

Definition. The *multiplicity* $M(J_i^n)$, $i = 1, ...S_n$ of an iterate J_i^n is the number of times it occurs in tableau A_n.

The number of iterates with multiplicity k is

$$T_{nk} = 2^{n-2k+1}\binom{n-1}{2k-2} S_{k-1}, \quad k = 1, \dots, \left[\frac{n+1}{2}\right].$$

Above facts have been found by induction from low values of n, but we have no proofs. Maybe some competent combinatorialist can prove them.

Part of the tableau A_n, starting from the right, is a Young tableau, being a partition of S_n into n parts. The parts are the respective lengths in each line comprising sections of consecutive numbers of $1, 2, \dots, S_n$. For example for $n = 4$ (Exhibit 3) we have by A_4:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ & & 11 & 12 & 13 \\ & & & & 14 \end{array} \qquad S_n = 14 = 5 + 5 + 3 + 1$$

In the general case $S_n = c_{n,1} + c_{n,2} + \cdots + c_{n,n}$ where the $c_{n,j}$ can be calculated recursively from following "Pascal" triangle

$n\backslash j$	1	2	3	4	5	$\cdots$		n	$\cdots$
1	1								$S_1 = 1$
2	1	1							$S_2 = 1 + 1 = 2$
3	2	2	1						$S_3 = 2 + 2 + 1 = 5$
4	5	5	3	1					$S_4 = 5 + 5 + 3 + 1 = 14$
5	14	14	9	4	1				$S_5 = 14 + 14 + 9 + 4 + 1 = 42$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
n	$c_{n,1}$	$c_{n,2}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$c_{n,n}$		$S_n = \sum_{j=1}^{n} c_{n,j}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

The recursion rule is

$$c_{n,j} = c_{n-1,j-1} + c_{n-1,j} + \cdots + c_{n-1,n-1}.$$

The numbers $c_{n,j}$ will be used in the semantic section of the paper. An extended "Pascal" triangle up to $n = 10$ is appended (Exhibit 4). Out of curiosity we looked up the vertical sequences in Sloane's Handbook of Integer Sequences and were surprised to see that e.g. $1, 3, 9, 28, 90, 297, \ldots$ are Laplace transforms coefficients; $1, 4, 14, 48, 165, 572, \ldots$ are partitions of a polygon by a number of parts. The numbers are known in the literature as ballot numbers, their values being $c_{n,j} = \frac{j}{n}\binom{2n-j-1}{n-1}$ (see Riordan, [1968], Aigner [2001]).

2.3. Formal reducibility of an identity and incidence matrix. An identity $J_i^n = J_j^n$ between two n-iterates is said to be of order n. It is *formally reducible* or simply *reducible* to an identity of order $n-1$ if Vxx appears on both sides at the *same variable place*. For example, $VVx\underline{Vxx}\ x = Vx\ V\underline{Vxx}\ x$ of order 3 is actually formally reducible to $VVxx\,x = Vx\,Vxx$ of order 2. Repeating the process, an identity of order n is reducible to an identity of lower order if there is an iterate J^k, $k < n$, appearing at the same variable place on both sides of the identity. If there is no such J^k the identity is called *formally irreducible,* or simply *irreducible.*

In order to calculate the number of reducible resp. irreducible identities we first form following incidence matrix relative to tableau A_n of all possible S_n^2 identities of order n, including $J_i^n = J_i^n$ and counting $J_i^n = J_j^n$ and $J_j^n = J_i^n$ as different:

$i\backslash j$	J_1^n	J_2^n	$\cdots$	$J_{S_n}^n$
J_1^n	$\cdots$	$\cdots$	$\cdots$	$\cdots$
J_2^n	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\delta(J_i^n, J_j^n)$	$\cdots$
$J_{S_n}^n$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

where

$$\delta(J_i^n, J_j^n) = \begin{cases} 1 & \text{if} \quad J_i^n = J_j^n \quad \text{reducible} \\ 0 & \text{if} \quad J_i^n = J_j^n \quad \text{irreducible.} \end{cases}$$

The $\delta(J_i^n, J_j^n)$ are determined by going successively through the lines of tableau A_n whose construction is such that all identities between n-iterates on the same line are reducible to identities of order $n-1$ or lower. As an example we write out the incidence matrix relative to tableau A_3 for all possible 25 ($S_3 = 5$) identities of order 3:

	J_1^3	J_2^3	J_3^3	J_4^3	J_5^3	$\sum_j 1$
J_1^3	1	1	0	0	0	2
J_2^3	1	1	0	0	1	3
J_3^3	0	0	1	1	0	2
J_4^3	0	0	1	1	0	2
J_5^3	0	1	0	0	1	2
						$I_3 = 11$

The incidence matrix of order $n = 4$ relative to tableau A_4 is given in Exhibit 5, and is of size 14×14 since $S_4 = 14$.

The sum

$$I_n = \sum_{i,j=1,\ldots,S_n} \delta(J_i^n, J_j^n)$$

gives the total number of reducible identities and therefore $S_n^2 - I_n$ is the number of irreducible identities. For $n = 3$, $I_3 = 11$ as shown above. One of the main goals is to calculate I_n.

2.4. Connection between multiplicity of J_i^n and $\sum_{j=1}^{S_n} \delta(J_i^n, J_j^n)$. The multiplicity $M(J_i^n)$ of J_i^n was defined in subsection 2.1, as the number of times J_i^n occurs in tableau A_n. $\sum_{j=1}^{S_n} \delta(J_i^n, J_j^n)$ is the sum of 1's in the i-th line of the incidence matrix. We shall prove following theorem which is fundamental in our calculation of I_n.

Theorem. Let $M(J_i^n) = k$. Then

$$\sum_{j=1}^{S_n} \delta(J_i^n, J_j^n) = \sum_{\nu=1}^{k} (-1)^{\nu-1} \binom{k}{\nu} S_{n-\nu}.$$

In other words $\sum_{j=1}^{S_n} \delta(J_i^n, J_j^n)$ does not depend on J_i^n but only on the multiplicity k of J_i^n.

Proof. J_i^n occurs in tableau A$_n$, k times, say once and only once on lines $L_{\alpha_1}, L_{\alpha_2}, \ldots, L_{\alpha_k}$, $1 \le \alpha_\nu \le n,\ \alpha_\nu \ne \alpha_\mu$. Because of the construction of A_n it can never occur more than once on the same line as all iterates of the line are formally different from each other. Applying the inclusion - exclusion principle we get

$$\begin{aligned}\sum_{j=1}^{S_n} \delta(J_i^n, J_i^n) \quad = \quad & |L_{\alpha_1}| + |L_{\alpha_2}| + \cdots + |L_{\alpha_k}| - \\ & |L_{\alpha_1} \cap L_{\alpha_2}| - \cdots - |L_{\alpha_{k-1}} \cap L_{\alpha_k}| + \\ & |L_{\alpha_1} \cap L_{\alpha_2} \cap L_{\alpha_3}| + \cdots + |L_{\alpha_{k-2}} \cap L_{\alpha_{k-1}} \cap L_{\alpha_k}| - \\ & \cdots + (-1)^{k-1} |L_{\alpha_1} \cap L_{\alpha_2} \cap \cdots \cap L_{\alpha_k}|.\end{aligned}$$

The case of one of the $|L_{\alpha_\nu} \cap L_{\alpha_\mu}|$ being 0 is impossible. This would mean according to 2.2 that J_i^n occurs in tableau A_n in two consecutive lines. It can be proved however by induction from A_{n-1} to A_n that this cannot happen. In fact A_n is a "Schachtelung" of Young tableaux distanced vertically by at least 2 lines from each other as seen e.g. for A_n on annexed Exhibit 3. As a consequence all $|L_{\alpha_1} \cap L_{\alpha_2} \cap \ldots|$ are also different from 0. Substituting their values as per 2.2 considering that $\alpha_\nu \neq \alpha_\mu$ we obtain $\sum_{j=1}^{S_n} \delta(J_i^n, J_j^n) = \sum_{\nu=1}^{k} (-1)^{\nu-1} \binom{k}{\nu} S_{n-\nu}$ as required.□

2.5. Calculation of I_n. We can now calculate $I_n = \sum_{i=1}^{S_n} \sum_{j=1}^{S_n} \delta(J_i^n, J_j^n)$. Since $\sum_{j=1}^{S_n} \delta(J_i^n, J_j^n)$ is the number of times $J_i = J_j$ is reducible in the i-th line of the incidence matrix and, as indicated in 2.2, there are

$$T_{n,k} = 2^{n-2k+1} \binom{n-1}{2k-2} S_{k-1} \qquad \binom{n-1}{2k-2} \geq 1$$

iterates J_i with multiplicity k we get for I_n:

$$I_n = \sum_{k=1}^{[\frac{n+1}{2}]} T_{n,k} \left(\sum_{j=1}^{S_n} \delta(J_i^n, J_j^n) \right).$$

The upper limit $[\frac{n+1}{2}]$ for k is obtained from $n-1 \geq 2k-2$, all $T_{n,k}$ with $k > [\frac{n+1}{2}]$ being zero.

Inserting for $\sum_{j=1}^{S_n} \delta(J_i^n, J_j^n)$ its value found in 2.4 we get

$$I_n = \sum_{k=1}^{[\frac{n+1}{2}]} T_{n,k} \left[\binom{k}{1} S_{n-1} - \binom{k}{2} S_{n-2} + \cdots + (-1)^{k-1} \binom{k}{k} S_{n-k} \right]$$

and by changing the order of summation

$$I_n = \sum_{k=1}^{\frac{n+1}{2}} (-1)^{k-1} S_{n-k} \left[\binom{k}{k} T_{n,k} + \binom{k+1}{k} T_{n,(k+1)} + \cdots + \binom{\left[\frac{n+1}{2}\right]}{k} T_{n,\left[\frac{n+1}{2}\right]} \right].$$

The evaluation of the brackets in this formula has transcended our efforts but straightforward calculations of I_n based on tableaux A_n for $n = 3, 4, 5, 6, 7$ and their respective incidence matrices have led us to surmise that

$$\sum_{\nu=0}^{[\frac{n+1}{2}]-k} \binom{k+\nu}{k} T_{n,k+\nu} = \binom{n-k+1}{k} S_{n-k}.$$

For $k = 1$ this formula reduces to

$$1T_{n,1} + 2T_{n,2} + \cdots + \left[\frac{n+1}{2}\right] T_{n,\left[\frac{n+1}{2}\right]} = nS_{n-1},$$

which is certainly correct considering that tableau A_n has nS_{n-1} entries. Maybe application of the "Snake Oil Method" or the "Wilf-Zeilberger Method" which we have not tried will succeed in proving this identity (see Wilf [1990]).

Substituting the brackets in above formula by their values $\binom{n-k+1}{k}S_{n-k}$ and skipping the upper limit $[\frac{n+1}{2}]$ in the summation sign, as all $\binom{n-k+1}{k}$ are zero for $k > [\frac{n+1}{2}]$, the expression for I_n takes finally the form

$$I_n = \sum_{k=1}^{\infty} (-1)^{k-1} \binom{n-k+1}{k} S_{n-k}^2.$$

2.6. Asymptotic evaluation of I_n. As there are S_n^2 formally different algebras, counting them as in section 2.3, defined by an identity $J_i^n = J_j^n$, and the number of reducible algebras is I_n,

$$\frac{I_n}{S_n^2} = \sum_{k=1}^{\infty} (-1)^{k-1} \binom{n-k+1}{k} \left(\frac{S_{n-k}}{S_n}\right)^2$$

is the probability (measure) for an algebra to be reducible and hence

$$1 - \frac{I_n}{S_n^2}$$

is the probability that it be irreducible.

From the recurrence $S_n = 2\frac{2n-1}{n+1}S_{n-1}$ for the Catalan numbers it can be easily shown by induction from n to $n+1$ that

$$\tfrac{S_{n-k}}{S_n} \rightarrow \tfrac{1}{4^k} \quad \text{for} \quad n \rightarrow \infty.$$

On the other hand $\binom{n-k+1}{k}$ is $\sim \frac{n^k}{k!}$ for $n \rightarrow \infty$. The general term therefore of above series behaves like

$$(-1)^{k-1} \frac{1}{k!} \frac{1}{4^{2k}} n^k.$$

Summing over k we can approximate I_n/S_n^2 by

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k!} \left(\frac{n}{16}\right)^k = 1 - e^{-n/16},$$

and hence,

$$\lim_{n\to\infty} \frac{I_n}{S_n^2} \sim \lim_{n\to\infty} \left(1 - e^{-n/16}\right) = 1.$$

Above reasoning is of course not rigorous needing either a direct evaluation of the series or a "Fehlerabschätzung" of the difference with $1 - e^{-n/16}$.

Stronger results can be obtained if instead of counting only the reducible identities of tableau A_n we do also take into account the reducible identities of tableau B_n, which do not already occur in tableau A_n . For example from tableau B_4 (Exhibit 3) we infer that $J_1^4 = J_8^4$ and $J_9^4 = J_{13}^4$ are formally reducible, a fact *which can not be deduced from tableau* A_4. By "adding" tableau A_n and B_n to form $A_n \oplus B_n$ with $(n+2)S_{n-1}$ entries, we can again evaluate its corresponding incidence matrix which remarkably has the same properties as the incidence matrix of tableau A_n. The theorem that the sum of 1's on a line of the incidence matrix is the same for all J_i^n having the same multiplicity remains unchanged. The occurrence, however, of an iterate with multiplicity k has now been found to be

$$T_{n,k}^{A_n \oplus B_n} = T_{n,k} + 2\left(T_{n-1,k-1} - T_{n-1,k}\right),$$

with obvious notation. We have not pursued the calculations to find an expression for

$$I_n^{A_n \oplus B_n} = \sum_{k=1}^{\left[\frac{n+1}{2}\right]} T_{n,k}^{A_n \oplus B_n} \left[\binom{k}{1} S_{n-1} - \binom{k}{2} S_{n-2} + \cdots + (-1)^{k-1}\binom{k}{k} S_{n-k}\right],$$

but evaluations up to $n = 7$ show as expected a much faster convergence of $I_n^{A_n \oplus B_n}/S_n^2$ to 1, than that of I_n/S_n^2.

2.7. Excursus on the implications of $I_n/S_n^2 \to 1$. The fact that $I_n^{A_n}/S_n^2 \to 1$ shows that almost all n-identities reduce formally to identities of lower order. Since

$$\frac{S_n^2 - I_n}{S_n^2} \sim e^{-n/16}$$

this means that *there is almost no formally irreducible identity as $n \to \infty$.* This explicates the word *few* we used in the introduction regarding the possibility of stipulating structures axiomatically provided with an identity.

Formal irreducibility is an imperative requirement of an identity figuring in an axiom system: nobody would define a semigroup by $VVVx_1x_2, x_3, x_4 = VVx_1x_2, Vx_3x_4$ (Vx_1x_2 occurring in the same variable space on both sides) and not by $VVx_1x_2, x_3 = Vx_1, Vx_2x_3$. If we ask therefore a mathematician to postulate an algebra with an identity of say order $n = 35$ and claim it is a new structure, chances are high that his new product is redundant, as being reducible to a lower order, maybe even to the associative law. If the question to turn out a kilometer long identity is put to a computer, theoretically speaking, it is almost certain that the brainchild will be a hydrocephalus.

At this point we cannot resist the temptation to digress somehow from the mainstream of our exposition. We have always wondered why mathematical axiom systems, theorems and formulas are so short, rarely taking more than a quarter page of a book to be *formally* expressed within a certain alphabet. Obviously, this is due to our anthropomorphic yardstick which is limiting our capacity to consider and manipulate long expressions. With the help of computers the limits are just receding farther off. Unavoidably, however, this raises a major question which, to our knowledge we haven't seen addressed to before. Are there mathematical "truths" which would require at least, say $100^{100^{100}}$ pages of an ordinary book to be expressed in some formal language, not including proofs? Clearly, abbreviations like $\sum$ and formulations depending on a parameter n and increasing in length with increasing n cannot be considered to yield such statements if a specific large number is inserted for n. And of what nature and content would such monster theorems be which we wouldn't be able to read, let alone grasp?

But back to terra ferma and our smallish theorems until such a monster is discovered - or not.

3. Semantic Part

3.1. Classes of semantically equal iterates resulting from an identity. Semantic equality means that in an identity $J_i^n(x_1 \ldots x_{n+1}) = J_j^n(x_1 \ldots x_{n+1})$ the two sides are formally different but are the same (equal) for all x_ν running over the prescribed space for the variables. The usual properties of an equivalence relation hold.

The results of section 2 were mainly obtained by investigating the lines of tablaux A_n (occasionally also those of tablaux B_n). In the following, we will investigate the columns of tableaux A_n and B_n and derive semantic information regarding the iterates of Vx_1x_2 of order higher than n, as a consequence of the defining identity.

To illustrate the general case we will take an example drawn from tableaux A_4 and B_4 (Exhibit 3). Suppose we postulate a 3-algebra defined by the identity $J_2^3 = J_4^3$ or written out $V(Vxx, Vxx) = V(x, V(Vxx, x))$. Using the enumerative labeling described in 2.1, $(2^{(3}, 4^{(3}, 10^{(4}, \ldots$ stand respectively for $J_2^3, J_4^3, J_{10}^4, \ldots)$ and going down columns 2 and 4 of tableaux A_4 and B_4 we deduce following identities:

From Tableau A_4	From Tableau B_4
$2^{(3} = 4^{(3}$	$2^{(3} = 4^{(3}$
$2^{(4} = 4^{(4}$	$3^{(4} = 8^{(4}$
$7^{(4} = 9^{(4}$	$10^{(4} = 13^{(4}$
$4^{(4} = 12^{(4}$	
$5^{(4} = 10^{(4}$	

This means that the 14 iterates of order 4 fall into 10 classes of semantically equal iterates if tableau A_4 is used, namely (skipping upper indices):

$$\{1\}, \{2, 4, 12\}, \{3\}, \{5, 10\}, \{6\}, \{7, 9\}, \{8\}, \{11\}, \{13\}, \{14\}.$$

If B_4 is also taken into account the number of classes is reduced to 8 because of the equivalence properties of semantic equality:

$$\{1\}, \{2, 4, 12\}, \{3, 8\}, \{5, 10, 13\}, \{6\}, \{7, 9\}, \{11\}, \{14\}.$$

We can now repeat the process by using either of above sets as defining identities and obtain via tableaux A_5 and B_5 the equivalence classes of order 5 and their classnumbers (number of classes). This procedure, which can be continued indefinitely, *splits the set of all iterates of any order into equivalence classes of semantically equal ones,* as a consequence of the initial defining identity $2^{(3} = 4^{(3}$.

In the general case we start with an algebra postulated by an identity $J^n = \overline{J^n}$ between two formally different iterates - several identities can also be considered as we shall see in section 4. We can apply above procedure by using tableaux A_{n+k} and B_{n+k}, $k \geq 1$, in a certain prescribed order, in which case the corresponding class sets and classnumbers will depend on the chosen order. We shall restrict ourselves to using exclusively either tableaux A_{n+k} or both tableaux A_{n+k} and B_{n+k}, in conjunction, which as said in 2.6 we denote by $A_{n+k} \oplus B_{n+k}$. In the first case we have found a formula to calculate the classnumbers for any k, (see following theorem), without actually going through the laborious procedure of forming the successive class sets. This however is not giving the best results as shown by above example. The actual minima of the classnumbers are obtained by using consistently tableaux $A_{n+k} \oplus B_{n+k}$. We shall denote by h^A_{n+k}, h^B_{n+k} and $h^{A\oplus B}_{n+k}$ respectively the classnumbers in reference to the tableaux employed. As $h^{A\oplus B}_{n+k}$ is the minimal number of the equivalence classes we shall omit the indication $A \oplus B$ and simply call $h_{n+k} = h_{n+k}(J^n = \overline{J^n})$ the classnumbers of the identity, for $k \geq 0$. Below n the classnumbers are of course the Catalan numbers since the identity is of order n.

Writing H^{n+k}_i, $i = 1, ..., h_{n+k}$ for the classes of order $n+k$ we can list them as follows together with their h_{n+k}:

$$
\begin{array}{lllllllllll}
H^0_1 & & & & & & & & & & h_0 = 1 \\
H^1_1 & & & & & & & & & & h_1 = 1 \\
H^2_1 & H^2_2 & & & & & & & & & h_2 = 2 \\
H^3_1 & H^3_2 & H^3_3 & H^3_4 & H^3_5 & & & & & & h_3 = 5 \\
\cdots & \cdots & \cdots & \cdots & \cdots & & & & & & \cdots \\
H^{n-1}_1 & H^{n-1}_2 & \cdots & \cdots & \cdots & H^{n-1}_{S_{n-1}} & & & & & h_{n-1} = S_{n-1} \\
H^n_1 & H^n_2 & \cdots & \cdots & \cdots & \cdots & H^n_{S_n-1} & & & & h_n = S_n - 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
H^{n+k}_1 & H^{n+k}_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & H^{n+k}_{h_{n+k}} & & h_{n+k} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

The main objective is to calculate the classnumbers h_{n+k} for a given identity of order n and to classify the identities according to their classnumbers. A major find, conjectured, towards this aim, if we restrict ourselves to using only the sequence of the tableaux $A_1, A_2, \ldots, A_k, \ldots$ is the

Theorem. All identities $J^n = \overline{J^n}$ of order n have the same classnumber h^A_{n+k} namely,

$$h^A_{n+k}(J^n = \overline{J^n}) = S_{n+k} - c_{n+k+1,n+1} \qquad k \geq 0$$

where the $c_{i,,j}$ are the numbers of the "Pascal" triangle defined in 2.2.

Since $h_{n+k} < h^A_{n+k}$, as any equivalence class from tableau $A \oplus B$ comprises all iterates of the corresponding equivalence class of tableau A, we have

$$h_{n+k}(J^n = \overline{J^n}) < S_{n+k} - c_{n+k+1,n+1} \qquad k \geq 0$$

which of course is stronger than the trivial $h_{n+k} < S_{n+k}$, especially for big k.

Using tableaux B_n only it is easily seen that $h^B_{n+k} = S_{n+k} - 2^k$, as all entries in the 2 lines of B_{n+k} appear only once.

3.2. The algebra of classes. The classes H^p_i can be taken as elements of a new algebra with a binary composition law $W(H^p_i, H^q_j)$ defined as follows:

Definition. If J^p_i is a representative of the class H^p_i and J^q_j is a representative of the class H^q_j then the class $W(H^p_i, H^q_j)$ is defined as the class of order $p + q + 1$ to which the iterate $V(J^p_i, J^q_j)$ of order $p + q + 1$ belongs.

The class $W(H_i^p, H_j^q) = H_\lambda^{p+q+1}$ is uniquely defined as any other pair $J_\mu^p,\ J_\nu^q$, of representatives will give an iterate $V(J_\mu^p, J_\nu^q)$ semantically equal to $V(J_i^p, J_j^q)$. The new law $W(H_1, H_2)$ defined over the classes satisfies the same identity as the original algebra. It is a Löwenheim - Skolem denumerable model of the identity. The classnumbers h_{n+k} now play the same role as the Catalan numbers for the original algebra and corresponding tableaux can be formed. The process can be repeated giving a series of algebras.

3.3 Semantically reducible algebras. By *semantic reducibility* we mean that $V(x, y)$ besides satisfying the defining formally irreducible identity, does also satisfy, because of inherent reasons of the structure, an identity of lower order. In the opposite case the identity is called *semantically irreducible* or *essential.*

So far the algebras considered were subjected to no constraint whatsoever besides the axiom of the defining identity. In the rest of this subsection we shall additionally provide that if $V(x, a) = V(y, a)$ then $x = y$ and similarly $V(a, x) = V(a, y) \to x = y$. These are natural restrictions and practically fulfilled in many structures of importance (unicity of solutions of $V(x, a) = b$ resp. $V(a, x) = b$ in case they exist).

We have made a thorough investigation of all 3-algebras subject to above unicity property of $V\,xy$ which led to the determination of all essential structures of that order as well as of their classnumbers.

The following theorem summarizes these results and the proof straightforward but tedious, clarifies the concept of semantic reducibility.

Theorem. There are only 2 algebras of order 3, semantically irreducible i.e.

$$V(V(Vx_1x_2, x_3), x_4) = V(x_1, V(Vx_2x_3, x_4)), \qquad (1^{(3} = 4^{(3})$$

and

$$V(V(x_1, Vx_2x_3), x_4) = V(x_1, V(x_2, Vx_3x_4)), \qquad (3^{(3} = 5^{(3}).$$

All other 3-algebras are reducible, formally or semantically to algebras of lower order, i.e. to semigroups or to the free algebra of Vxy with no identity. This theorem is true provided $V(x, y)$ fulfills above unicity conditions.

Proof. We form the incidence matrix (see 2.3) 5×5 of all 3-order identities resulting *from both tableaux* A_3 *and* B_3, i.e. from $A_3 \oplus B_3$, in order to include also the reducible identities which result from tableau B_3 but not from tableau A_3:

	J_1	J_2	J_3	J_4	J_5
J_1	1	1	1	0	0
J_2		1	0	0	1
J_3			1	1	0
J_4				1	1
J_5					1

Because of symmetry only the upper triangle is considered.

We see that the only identities which are not formally reducible are the following five, indicated by 0's in the matrix:

$$1^{(3} = 4^{(3}, \quad 1^{(3} = 5^{(3}, \quad 2^{(3} = 3^{(3}, \quad 2^{(3} = 4^{(3}, \quad 3^{(3} = 5^{(3}.$$

Consider f.e. $1^{(3} = 5^{(3}$ and construct the equivalence classes of iterates of order 4 and 5, by applying successively tableaux $A_4 \oplus B_4$ and $A_5 \oplus B_5$ as outlined in 3.1. From tableau $A_4 \oplus B_4$ we get successively (writing $i^{(n}$ or i for J_i^n), by collection of classes having common elements, following 6 identities which yield 8 ($h_4 = 8$) equivalence classes for the 14 ($S_4 = 14$) iterates of order 4:

$$1^{(3} = 5^{(3}$$

$$1^{(4} = 5^{(4}$$
$$6^{(4} = 10^{(4}$$
$$3^{(4} = 13^{(4}$$
$$2^{(4} = 14^{(4}$$
$$1^{(4} = 11^{(4}$$
$$9^{(4} = 14^{(4}$$

{1,5,11}, (2,9,14}, {3,13}, {4}, {6,10}, {7}, {8}, {12}.

Repeating the same procedure with above classes we get from tableaux $A_5 \oplus B_5$ (Exhibit 3) -singletons can be omitted- following 14 classes for the 5-iterates (the equality sign between the resulting 5-iterates has been omitted)

$1^{(4} = 5^{(4} = 11^{(4}$			$2^{(4} = 9^{(4} = 14^{(4}$			$3^{(4} = 13^{(4}$		$6^{(4} = 10^{(4}$	
1	5	11	2	9	14	3	13	6	10
15	19	25	16	23	28	17	27	20	24
6	10	34	7	32	37	8	36	29	33
3	13	38	4	26	41	11	40	17	27
2	14	30	5	24	42	7	37	16	28
1	11	29	3	22	38	6	34	15	25
23	28	39	24	36	42	26	41	32	37

{1, 5, 8, 11, 20, 24, 29, 33, 36, 40, 42}, {2, 9, 14, 30}, {3, 13, 22, 38}
{4, 26, 41}, (6, 10, 34}, {7, 32, 37}, {12}, {15, 19, 25}, {16, 23, 28, 39}
{17, 27}, {18}, {21}, {31}, {35}

As we see one of the 14 5-order iterate classes is (1, 5, 8, 11, 20, 24, 29, 33, 36, 40, 42) which contains both $8^{(5}$and $11^{(5}$ as semantically equal. Let us write out in full the identity $8^{(5} = 11^{(5}$ by substituting for the labels their corresponding 5-iterates (see Exhibit 2):

$$V(V(Vx_1x_2, V(Vx_3x_4, x_5)), x_6) = V(V(Vx_1x_2, V(x_3, Vx_4x_5)), x_6).$$

Because of the unicity assumption stipulated in the beginning of this section, we get successively

$$V(Vx_1x_2, V(Vx_3x_4, x_5)) = V(Vx_1x_2, V(x_3, Vx_4x_5))$$
$$V(Vx_3x_4, x_5) = V(x_3, Vx_4x_5).$$

The last identity is of course the associative law so that the algebra $1^{(3} = 5^{(3}$ which is formally irreducible, is actually a semigroup.□

The same can be shown to be true for the algebras $2^{(3} = 3^{(3}$ and $2^{(3} = 4^{(3}$, leaving as only semantically irreducible (essential!) algebras, the cases $1^{(3} = 4^{(3}$, $3^{(3} = 5^{(3}$, as stated in the theorem. We would like to stress that above theorem is true at the present stage of the investigations as for the proof it suffices to form the equivalence classes of order up to 5. But who knows whether by proceeding to equivalence classes of orders 6, 7, ..., n_0, we won't find a pair of n_0 -iterates which would similarly also eliminate the cases $1^{(3} = 4^{(3}$, and $3^{(3} = 5^{(3}$? We have arithmetic evidence that this is rather unlikely.

It is possible to spare to trouble of writing out the iterates, which is extremely cumbersome, if we know beforehand which pairs of n-iterates automatically imply the semantic equality of iterates of order $2, 3, ..., n-1$, respectively. This can be done with the help again of tableau A_n. The number of such pairs can be shown to be

$$(n-k+1)S_{n-k}\frac{S_k(S_k-1)}{2} \qquad 2 \leq k \leq n-1.$$

Annexed here-with (Exhibit 6), by way of examples, are the tables of such iterates for $n = 5$, $k = 2$ (couples) and for $n = 5$, $k = 3$ (triples). They figure respectively in the two tableaux A_5, connected by the sign $\updownarrow$ and should be read off vertically. Thus in tableaux A_5, $8^{(5}$ and $11^{(5}$ figure in same column connected by $\updownarrow$, implying associativity, as shown above. For $n = 5$, $k = 3$ all pairs of 5-order iterates in same column connected by $\updownarrow$, if semantically equal, automatically implicate the semantic equality of two 3-order iterates.

The rules of placement of the relational sign $\updownarrow$, which we won't expound, involve again the Catalan numbers and the numbers c_{nj} of the "Pascal" triangle defined in 2.2. This is further evidence of the wealth of information encoded in the tableaux A_n.

3.4. The series of classnumbers h_n for identities of order 3 and 4. Equality in this subsection means semantic equality.

As shown in 3.1 the partition of all N-iterates into h_N classes of equal iterates induces as a result of tableaux $A_{N+1} \oplus B_{N+1}$ a partition of all $(N+1)$-iterates into h_{N+1}classes of equal iterates. The problem of one defining n-identity, is starting with $h_n = S_n - 1$, to calculate the series $h_{n+1}, ..., h_{n+k}, ...$. Generating functions $\sum_0^\infty h_n t^n$ can be defined and calculated in some cases, for instance for $3^{(3} = 5^{(3}$.

We have carried out the investigation for all 3-identities and calculated the respective class-numbers up to $n = 7$, as well as the number of classes containing only 1 element (singletons). Beyond that the process gets unwieldy because of the fast growth of the Catalan numbers S_n. The following list summarizes our results for the 10 possible identities of order 3, obviously excluding $J_i^{(3} = J_i^{(3}$ and counting $J_i^{(3} = J_j^{(3}$ and $J_j^{(3} = J_i^{(3}$ as one identity. We have also included the results for the formally reducible algebras because of the surprising fact that for some of them the number of singletons are the Fibonacci numbers F_n:

Algebra	Formally reducible/ irreducible	Classnumber h_n		Number of singletons	
$1^{(3} = 2^{(3}$	reducible	2^{n-1}	$n \geq 3$	F_n	$n \geq 3$
$1^{(3} = 3^{(3}$	reducible	2^{n-1}	$n \geq 3$	F_n	$n \geq 3$
$1^{(3} = 4^{(3}$	irreducible	2^{n-1}	$n \geq 3$	n	$n \geq 3$
$1^{(3} = 5^{(3}$	irreducible	4	$n = 3$	n	$n \geq 3$
		8	$n = 4$		
		14	$n = 5$		
		20	$n = 6$		
		16	$n = 7$		
$2^{(3} = 3^{(3}$	irreducible	4	$n = 3$	n	$n \geq 3$
		8	$n = 4$		
		14	$n = 5$		
		20	$n = 6$		
		24	$n = 7$		
$2^{(3} = 4^{(3}$	irreducible	4	$n = 3$	n	$n \geq 3$
		8	$n = 4$		
		14	$n = 5$		
		20	$n = 6$		
		24	$n = 7$		
$2^{(3} = 5^{(3}$	reducible	2^{n-1}	$n \geq 3$	F_n	$n \geq 3$
$3^{(3} = 4^{(3}$	reducible	2^{n-1}	$n \geq 3$	n	$n \geq 3$
$3^{(3} = 5^{(3}$	irreducible	2^{n-1}	$n \geq 3$	n	$n \geq 3$
$4^{(3} = 5^{(3}$	reducible	2^{n-1}	$n \geq 3$	F_n	$n \geq 3$

The behavior of $1^{(3} = 5^{(3}$ shows something which wouldn't expect - repeated checks have not revealed any error - namely that $h_7 = 16$ is less than $h_6 = 20$. Is this an indication that for all higher n the series $h_8, h_9, \ldots$ is bounded upwards? A phenomenon like that should not be excluded. For example for the algebra defined by the two identities $1^{(3} = 4^{(3}$ and $3^{(3} = 5^{(3}$ the classnumbers are $h_3 = 3$, $h_4 = 3$, $h_n = 1$ for $n \geq 5$. In final analysis the method of tableaux, as employed here, boils down to *set theory.* Namely substituting two sets M_1, M_2 of label numbers by the single set $M_1 \cup M_2 - M_1 \cap M_2$ *provided* $M_1 \cap M_2$ *is not empty.* Surprises are to be expected!

For algebras of order 4 we have conducted some investigations to determine their classnumbers up to order 7. Since $S_4 = 14$ there are $\frac{14 \times 13}{2} = 91$ such identities, reducible and irreducible. We checked 10 of them, taken at random, and found that for 8 of them (f.ex. $11^{(4} = 14^{(4}$) the classnumbers seem to follow the rule

$$h_n = S_n - 2\frac{4^{n-3} - 1}{3} + \frac{(n-2)(n-3)}{2}, \qquad 3 \leq n \leq 7.$$

As this expression becomes negative after $n = 14$ it certainly cannot give a general formula for h_n, unless strange phenomena appear for $n > 14$, similar to those observed for the algebra $1^{(3} = 5^{(3}$. A computer search is imperative to decide the ambivalence.

4. Notes and Remarks

4.1. Algebras with several operations of any arity. The process of labeling iterates and the construction of the relevant tableaux can be used for any algebra with any finite number of operations of any arity. The order n of an iterate is the number of operation symbols figuring in each expression. The iterate ($a_i \geq 0, \quad p_i \geq 0$):

$$J \left\{ \begin{array}{cccc} a_1\text{-}V_1 & a_2\text{-}V_2 & \dots & a_k\text{-}V_k \\ p_1 & p_2 & \dots & p_k \end{array} \right\}$$

containing the a_1-ry operation V_1, p_1 times, the a_2-ry operation V_2, p_2 times, ... has order

$$n = p_1 + p_2 + \cdots + p_k$$

and the number of variable places is

$$(a_1 - 1)p_1 + (a_2 - 1)p_2 + ... + (a_k - 1)p_k + 1.$$

It is possible to calculate the corresponding "Catalan" numbers for the nunber of iterates via generating functions as done in the classical case $k = 1$, $a_1 = 2$. Their generating function $\phi(t)$ satisfies the equation

$$\frac{\phi - 1}{t} = \phi^{a_1} + \phi^{a_2} + ... + \phi^{a_k}$$

which reduces for $k = 1$, $a_1 = 2$ to the classical equation $\frac{\phi-1}{t} = \phi^2$, giving the Catalan numbers S_n, and similarly to the equations $\frac{\phi-1}{t} = \phi^a$, for the higher "Catalan" numbers:

$$S_n^a = \frac{1}{(a-1)n+1}\binom{an}{n}, \qquad a \geq 3.$$

The proof of above functional equation for the generating function is basically the same as in the case of the ordinary Catalan numbers, but explicit formulas for the coefficients are only possible in exceptional cases (see Berndt [1965], p. 71).

We believe that the method of tableaux used in section 2 to calculate the number of reducible identities can be extended to the general case. Will the end result be the same, namely that the number of irreducible identities divided by the number of possible identities goes to zero with increasing n ? We do not know. But if such is the case we are confronted again with a limitation in postulating new algebras of some order without risking to fall back, formally or semantically, on lower order algebras already established as essential. A preliminary check in the case of two binary operations $V_1x_1x_2$ and $V_2x_1x_2$ seems to confirm this contention.

4.2. Tableaux and Formal Languages. The method of tableaux we used to prove the results of sections 2 and 3 is a special case of a general method applicable to formal languages.

Roughly described, it consists in assigning a "length" n to the formulas (n-formulas) of the system and label them recursively from n to $n+1$ in a specific order proper to the system. A formal deduction rule operating in the system generates $(n+1)$-formulas from n-formulas. This allows the recursive formation of tableaux by listing the labels of the n-formulas in a line, in increasing order, and beneath it the labels of the corresponding deducible $(n+1)$-formulas.

A $(n+1)$-formula may be deducible from several n-formulas and thus have its label appear in several places of the tableau. The tableaux may be square, orthogonal, of unequal column lengths, finite or infinite, depending on the system. An equivalence relation defined over the Cartesian product of the set of n-formulas, subject to the rule that its validity extends to its validity for the corresponding deducible $(n+1)$-formulas, partitions all pairs of n-formulas into two classes. One class will comprise the pairs which imply the validity of the equivalence for *at least one pair* of $(n-1)$-formulas. The other class will comprise the pairs which do not have this property. This introduces a concept of *reducibility* of a formal system equipped with an equivalence relation, to a system of lower order. In the case of finite tableaux we can consequently count the number of pairs in either class.

Above procedure applied to the formal language of expressions x, Vxx, $VVxxx$, $VxVxx$, $VVVxxxx$, ... generated by iterating the binary operation Vxy yields the tableaux of our paper and leads to the asymptotic formula for the number of formally irreducible algebras and the other results, the equivalence relation in question being the identity between two n -iterates of the operation.

4.3. Commutativity. In section 3 we have investigated n-iterates where the variable spaces are filled respectively by the same variable on both sides of the identity. But what happens if for example we have a structure (algebra) where following holds:

$$V(x_1, VVx_2x_3, x_4) = V(x_3, Vx_2, Vx_4x_1)$$

identically in all variables? The simplest case is of course commutativity $Vx_1x_2 = Vx_2x_1$. The general case is

$$J^n(x_1, \dots, x_{n+1}) = J^n(P^n(x_1), \dots, P^n(x_{n+1}))$$

where P^n is a permutation of the symmetric $(n+1)$-group of the variables $x_1, x_2 \dots, x_{n+1}$. The generation of $(n+1)$-iterates from n-iterates is the same as in 2.1 but relabeling of the variables has to be made via following recursion in order to get the k-th line of the corresponding tableau A_{n+1}:

$$P_k^{n+1}(x_\nu) = \begin{cases} P^n(x_\nu) & \text{if} \quad 1 \le P^n(x_\nu) \le k \\ P^n(x_{\nu+1}) & \text{if} \quad k+1 \le P^n(x_{\nu+1}) \le n \\ x_{k+1} & \text{if} \quad \nu = n+1 \end{cases}$$

If $P^n(x_\nu) = x_\nu$ we get the re-labeling of the variables used in 2.1.

As no single permutation can be excluded the total number of possible n -iterates is now $(n+1)!S_n$ and the relevant A_n tableau has $n!S_{n-1}$ lines and n columns.

Whether an asymptotic formula holds as in 2.6 is an open question of great interest in view of the importance of commutativity and its higher analogues as well as other identities such as the Jacobi identity with two operations.

4.4. Examples. Suppose a structure S formally defined as follows: If $x \in S$ and $y \in S$ then $Axy \in S$ where A is a fixed symbol. Putting $Vxy = Axy$ we get the basic system of iterates with V replaced by A. If $L(x)$ is the number of A's figuring in x we obviously have

$$L(Vxy) = L(x) + L(y) + 1$$

Setting both sides equal to n we get successively:

$$L(Vxy) = L(x) + L(y) + 1 = n$$

$$\sum_{\{z|L(z)=n\}} R\left[Vxy = z\right] = \sum_{\kappa+\lambda+1=n} R\left[L(x) = \kappa\right] R\left[L(y) = \lambda\right]$$

where $R\left[E(x, y, ...) = a\right]$ denotes generally the number of solutions $[x_i, y_i...]$ of any equation $E(x, y, ...) = a$ with a fixed a. Since in the present case $R\left[Vxy = z\right] = 1$ - an iterate J can be formally expressed in only one way as $V(J_1, J_2)$- we get

$$\sum_{\{z|L(z)=n\}} 1 = \sum_{\kappa+\lambda+1=n} R\left[L(x) = \kappa\right] R\left[L(y) = \lambda\right],$$

The left hand side being $R\left[L(z) = n\right]$, i.e. the $n-$th Catalan number S_n, we get

$$S_n = \sum_{\kappa+\lambda+1=n} S_\kappa S_\lambda$$

which is the classical recursion for these numbers.

Above reasoning and results can be widely generalized and in our view may be useful in investigating sequences over an alphabet generated by a law. Let us consider for example a structure defined by a binary law Vxy and an application $L(x)$ onto the natural numbers subject to following homomorphism

$$L(V(xy) = P(L(x), L(y)),$$

where $P(s, t)$ is a polynomial in s and t over the ring of integers satisfying a suitable positivity condition. P need not be symmetric in x, y or have only positive coefficients. But e.g. let us take $P(s, t) = s^2 + t^2 + 2$. A structure S obeying above homomorphism can be constructed as follows:

1. The elements $x, y, \ldots \in S$ are sequences of a's and A's. a is the generator, A a fixed symbol.
2. $L(x)$ is the number of A's figuring in x.
3. If $x \in S$ and $y \in S$, $L(x) = s, L(y) = t$, then $V(x, y) = A\underbrace{x \cdots x}_{stimes}A\underbrace{y \cdots y}_{ttimes} \in S$

It is clear that $L(Vxy) = P(L(x), L(y)) = L(x)^2 + L(y)^2 + 2$ as required.

Permutation of the A's in the string of symbols x and y give rise to different structures satisfying homomorphisms with the same polynomial law $P(s, t)$. In the classical case $Vxy = Axy$ generates parenthesizing. But $\overline{V}xy = xAy$ does not.

4.5. "Catalan" numbers relative to homomorphisms. The method used to obtain the recursion formula for the Catalan numbers can be applied to the general case, provided every element a of the structure can be uniquely expressed as $V(x_1x_2)$. In other words the equation $V(u, v) = a$ should have only one solution $[x_1, x_2] : V(x_1x_2) = a$.

Starting with a homomorphism L and a polynomial P, and proceeding as before we set

$$L(Vxy) = P(L(x), L(y)) = n$$

and get the "counting" equation

$$\sum_{\{z,|L(z)=n\}} R\,[Vxy = z] = \sum_{\{s,t|P(s,t)=n\}} R\,[L(x) = s]\,R\,[L(y) = t]$$

where $R[\ldots = \ldots]$ is the number of solutions of the equation in parenthesis and $\{\ldots|\ldots\}$ the all - quantifier. Since we assumed that $R[Vxy = z] = 1$ the left side reduces to

$$\sum_{\{z|L(z)=n\}} 1 = R[L(z) = n]$$

which we denote by S_n^L. We suggest to call these numbers the "Catalan" numbers of the structure relative to the homomorphism (L, P). The recurrence relation now becomes

$$S_n^L = \sum_{P(s,t)=n} S_s^L S_t^L,$$

where the summation extends over the solutions ≥ 0 of the Diophantine equation $P(s,t) = n$. If $P(s,t) = s + t + 1$ we get the ordinary Catalan numbers.

By way of generators $a, b, \cdots$ and fixed parameters (symbols) $A, B, ...$ we may construct algebras whose elements are words formed with the generators and the parameters. $L(x)$ may count the number of A's in x, or the number of B's, or for that matter the number of any configuration of A's and B's, such as for example AB. The composition law Vxy can be set in such a way that given a polynomial $P(s,t)$, the homomorphism $L(Vxy) = P(Lx, Ly)$ is fulfilled, as done in 4.4. In the general case $R[Vxy = z]$ for $L(z) = n$ is a function of both z and n not always equal to 1, depending on the underlying structure. Recursion formulas and generating functions, in the form of power or Dirichlet series for the "Catalan" numbers $S_n^L = R[L(x) = n]$ are not always deducible, but in certain cases can be found. If e.g. $P(s,t) = s + t + k$ and $R[Vxy = z] = n + l$, with fixed integers k and l, and, assuming such a structure can be constructed as outlined, the counting equation leads to following differential equation for the generating function $G(t) = \sum S_n^L t^n$

$$tG'(t) + lG(t) - t^k G(t)^2 = \sum_{n=0}^{k-1} (n + l) S_n^L t^n.$$

4.6. Sets of identities defining an algebra with a binary operation. Suppose an n-algebra is defined by classes h_n of semantically equal n-iterates each class containing λ_i different iterates and each of the S_n iterates figuring in one class only. Then

$$\lambda_1 + \lambda_2 + \cdots + \lambda_{h_n} = S_n.$$

Suppose further the operation Vx_1x_2 satisfies the unicity condition for the solution $Vxa = b$ and form the S_{n-1} n-iterates $V(J_1^{n-1}, x), ..., V(J_{s_{n-1}}^{n-1}, x)$. These iterates are semantically different as otherwise there would be at least a pair $J_i^{n-1} = J_j^{n-1}$ and the algebra would be of lower order. Hence

$$h_n \geq S_{n-1} = \frac{n+2}{4n+2} S_n.$$

Substituting for S_n its above value we successively get

$$h_n \min(\lambda_i) \;\leq\; \lambda_1 + \lambda_2 + ... + \lambda_{h_n} \;\leq\; \frac{4n+2}{n+2} h_n$$

$$\min(\lambda_i) \leq \frac{4n+2}{n+2} < 4.$$

Therefore $\min(\lambda_i) \leq 3$, i.e. the set of classes defining the algebra must contain at least one class containing max. 3 iterates, if unicity conditions apply for the binary operation.

4.7. Multi-variable homomorphism. Homomorphisms $L(x)$ are 1-variable functions. We were unable to find any reference in the literature to a generalization to more variables. In our view a correct extension $L^n(x_1, \ldots, x_n)$ to several variables of this fundamental concept would be as follows:

$$L^n(V^m(x_1 y_1 \ldots w_1), \ldots, (V^m(x_n y_n \ldots w_n)) = W^m(L^n(x_1 x_2 \ldots x_n), \ldots, L^n(w_1 w_2 \ldots w_n)),$$

where V^m and W^m are m-ry operations, the upper indices indicating the number of variables of the operations involved. This definition is purely formal, without regard to the domains of validity of the operations, which have to be prescribed in a concrete application in order to make it an identity.

For $n = 1$, $m = 2$ we get the usual homo (h)-morphism $L(Vxy) = W(Lx, Ly)$ for binary structures. There is however a significant distinction between $n = 1$ and $n \geq 2$, if $V = W$ so that all functions operate in the same domain. If $n = 1$ the series $L(x), L(L(x)), ...L^n(x) = L(L^{n-1}(x)), ...$ i.e. the group of auto (a)-morphisms never leads outside the set of 1-variable a-morphisms. However, this is not true for $n \geq 2$ where from the existence of one n-variable a-morphism we can deduce new a-morphisms of any order. If e.g. $L(x_1 x_2)$ is a 2-variable a-morphism then $\overline{L}(x_2 x_2 x_3) = L(L(x_1 x_2), x_3)$ is a 3-variable a-morphism. The general result establishing the existence of a tower of a-morphisms is a consequence of the definition. In the case $V = W$ the definition is reciprocal in the sense that if L is a n-variable a-morphism then V is a m-variable a-morphism of L.

The search of h-morphisms resp. a-morphisms with many variables of a structure and the investigation of the corresponding groups by way of their composition (iteration) is of importance as shown by the classical case in the Galois Theory of equations over a field. There are analogous theorems if $n \geq 2$, $m \geq 2$:

1) Repeated application of the definition with $V = W$ leads to the fact that L is also an a-morphism of any iterate J of V which in short we write $L(J) = J(L)$.

2) In an equation between iterates of V, which may also involve constants $a, b, c, l \ldots$ as coefficients, if $r_1, r_2, \ldots$ are roots of the equation then $L(r_1, r_2, \ldots)$ are roots of the same equation whose corresponding coefficients are $L(a, a, \ldots), L(b, b, \ldots) \ldots$.

3) If $L(a, a, \ldots) = a$ (idempotent) then $L(r_1, r_2, \ldots)$ is also a root of the same equation.

4) Considering the roots of an equation as functions of its coefficients it can be shown that L is an a-morphism of these functions and vice-versa, if unicity conditions are fulfilled.

Above statements are only indicatively outlined but may convey an idea of what can be done with multi-variable homomorphisms which of course are identities with several operations as described in 4.1. We think they can prove useful for finite structures. We glimpse an eventual connection with n-Categories (see Baez, Dolan, [1998]).

4.8. Relations between Catalan numbers. We give hereunder without proof some relations for the classical Catalan numbers $S_n = \frac{1}{n+1}\binom{2n}{n}$, $n = 1, 2, ...$ which we haven't seen in the literature we are acquainted with:

1.
$$\sum_{i_1+\cdots+i_{\lambda+1}=n-\lambda} S_{i_1} S_{i_2} \ldots S_{i_{\lambda+1}} = S_n - \binom{\lambda+1}{1} S_{n-1} + \cdots + (-1)^k \binom{\lambda-k}{k} S_{n-k}$$

where $k = \left[\frac{\lambda}{2}\right]$, the greatest integer $\leq \frac{\lambda}{2}$ and $1 \leq \lambda \leq n$.

For $\lambda = 1$ this reduces to $\sum_{i_1+.i_2=n-1} S_{i_1} S_{i_2} = S_n$.

2.
$$S_n = c_{\lambda,1} \sum_{i_1+i_2=n-\lambda} S_{i_1} S_{i_2} + \cdots + c_{\lambda,\lambda} \sum_{i_1+\cdots+i_{\lambda+1}=n-\lambda} S_{i_1} S_{i_2} \ldots S_{i_{\lambda+1}}$$

where $c_{\lambda,i}$ are the numbers of the "Pascal" triangle defined in 2.2.

For $\lambda = 1$ this again reduces to the definition of the Catalan numbers.

3.
$$\lim_{n\to\infty} \frac{1}{S_n} \sum_{i_1+\cdots+i_{\lambda+1}=n-\lambda} S_{i_1} S_{i_2} \ldots S_{i_{\lambda+1}} = \frac{\lambda+1}{2^\lambda} \quad , \qquad \lambda \geq 1.$$

4. An iterate of Vxx of order n is a string of $2n+1$ symbols each of which is either a V or an x. It can therefore be written as

$$a_1 b_1 \cdots a_k b_k \qquad \text{with integers } a_i, b_i \geq 1,$$

thus uniquely assigning an integer $1 \leq k < n$ to each iterate.

For example $J_{11}^{(5} = VVVxxVxVxxx$ can be written as $a_1b_1a_2b_2a_3b_3 = 321113$ with $k = 3$.

An expression $a_1 b_1 \ldots a_k b_k$ represents an n-iterate of a binary operation, is well formed so to say, iff k and the a_i, b_i satisfy one of following conditions:

$$\text{i) } k = 1, \qquad a_1 = n, \quad b_1 = n+1$$

ii) $k \geq 2$,

$$a_1 + a_2 + \cdots + a_k = n$$
$$b_1 + b_2 + \cdots + b_k = n + 1$$
$$a_1 \geq b_1$$
$$a_1 + a_2 \geq b_1 + b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_1 + a_2 + \cdots + a_{k-1} \geq b_1 + b_2 + \cdots + b_{k-1}$$

Hence $S_n - 1$ is the number of integer solutions ≥ 1 of above Diophantine system, which simultaneously gives an algorithm to decide in k steps whether any word in V's and x's is an n-iterate of the binary operation Vxx the number of which is S_n whose order is $\sim \frac{1}{\sqrt{\pi}} n^{-3/2} 2^{2n}$.

4.9. Skein polynomials of iterates of a binary operation. The build-up of an iterate J can be characterized by a polynomial $P(J; a, b)$ of 2 variables in the following way. E.g. let us take again $J_{11}^{(5}$ and put indexes in the V's and x's

$$V_1 V_2 V_3 x_1 x_2 \ V_4 x_3 V_5 x_4 x_5 \ x_6.$$

The relation of a variable to the operations is not always the same. For example, x_2 is within the "Wirkungsbereich" (WB) of V_1, V_2 and V_3 but outside of that of V_4 and V_5. Similarly x_6 is within V_1 but outside of V_2, V_3, V_4, V_5. This can easily been seen if the parentheses are put in place. A further distinction to be made is whether the variable is in the first place or the second place of an operation. In above example x_3 is in the first place of V_1, in the second place of V_2 and in the first place of V_4.

We substantiate these remarks by associating to each variable a monomial $a^s b^t$ with a, b in a commutative ring R. We could also assign monomials $a^s b^t c^k$, c standing for being outside of the WB but this doesn't make any essential difference. The exponents s and t are the numbers of the first resp. second place occurrences of the variable. We then sum these monomials to form a polynomial corresponding to the iterate in question. Setting 1 for the polynomial of x we get by this procedure following table for the iterates J_i^n up to $n = 3$, listed in the ordering defined in 2.1 (Exhibit 2):

Iterate J	Polynomial $P(J)$
x	1
Vxx	$a+b$
$VVxxx$	a^2+ab+b
$VxVxx$	$a+ab+b^2$
$VVVxxxx$	$a^3+a^2b+ab+b$
$VVxxVxx$	$a^2+2ab+b^2$
$VVxVxxx$	$a^2+a^2b+ab^2+b$
$VxVVxxx$	$a+a^2b+ab^2+b^2$
$VxVxVxx$	$a+ab+ab^2+b^3$
$\cdots$	$\cdots$

It is easily seen that the formation rule of $P(J)$ obeys the homomorphism

$$P(V(J,J')) = aP(J) + bP(J').$$

If therefore R is the ring of integers the arguments of 4.3 can be applied verbatim. Writing $P(J_i^n;a,b)$ for the polynomial of J_i^n we have $P(J_i^n;1,1) = n+1$ and $P(J_i^n;a,b) = 1$ if $a+b=1$. Because of this, simpler polynomials $Q(J)$ can be defined by setting

$$Q(J_i^n;a,b) = \frac{P(J_i^n;a,b)-1}{a+b-1}$$

with

$$Q(V(J,J^{'})) = aQ(J) + bQ(J^{'}) + 1$$

The function

$$G_P(t) = \sum_J t^{P(J)} = t^1 + t^{a+b} + t^{a^2+ab+b} + t^{a+ab+b^2} + \cdots$$

satisfies the functional equation

$$G_P(t^a)G_P(t^b) = G_P(t) - t.$$

The function

$$G_Q(t) = \sum_J t^{Q(J)} = 1 + t + t^{a+1} + t^{b+1} + \cdots$$

satisfies the functional equation

$$tG_Q(t^a)G_Q(t^b) = G_Q(t) - 1,$$

which reduces to the Catalan equation for $a=b=1$.

Collecting terms in $G_P(t)$ for which $P(J) = P(J^{'})$ we may write

$$G_P(t) = \sum_J N_P\, t^{P(J)}$$

where N_P is the number of solutions J for which $P(J) = P_i$, where P_i is a specific polynomial of the series $P_1, P_2, \cdots$ listed above. The N_P satisfy the equation:

$$N_P = \sum_{\{\kappa,\lambda \mid aP_\kappa + bP_\lambda = P\}} N_{P_\kappa} N_{P_\lambda}.$$

It was originally thought that there was a (1, 1) correspondence between the J's and their polynomials $P(J)$ and our efforts to find the essential algebras $J_i^n = J_j^n$ consisted in equating the corresponding polynomials $P(J_i^n; a, b) = P(J_j^n; a, b)$ and look for distinctive properties of the resulting algebraic manifolds. The correspondence however is not (1, 1) which would mean that all $N_i = 1$. Already for $n = 4$,

$$P(J_4^4) = P(J_7^4) = a^2 + ab + a^2b + ab^2 + b^2 \quad (= P')$$

so that $N_{P'} = 2$.

Our attention has been drawn to the fact that the polynomials we constructed to reflect the way the iterates of a binary operation are built, bear a similarity with the polynomials of knot theory.

4.10. Which algebras are essential? A conjecture. The theorem of 3.2 says that the only essential algebras of order 3, subject to unicity conditions, are the following two:

$$V(V(Vx_1x_2, x_3),\, x_4) = V(x_1,\, V(Vx_2x_3, x_4)) \qquad (1^{(3} = 4^{(3})$$

$$V(V(x_1,\, Vx_2x_3), x_4) = V(x_1,\, V(x_2,\, Vx_3x_4)) \qquad (3^{(3} = 5^{(3}).$$

We notice that as to their form these can be written respectively as

$$V(A, x) = V(x, A)$$

$$V(B, x) = V(x, B),$$

where A and B are the two iterates of order 2, J_1^2 and J_2^2. On the other hand it is seen that the pairs (1, 4) and (3, 5) are the columns of tableau B_3. Is this a lead sign that even in the general case the essential algebras are given by the columns of tableaux B_n ? So that for $n = 4$ the essential algebras would be $J_1^4 = J_9^4,\ J_3^4 = J_{10}^4,\ J_6^4 = J_{12}^4,\ J_8^4 = J_{13}^4,\ J_{11}^4 = J_{14}^4$, and for any n they are given by the S_{n-1} pairs

$$V(J_i^{n-1},\, x\,) = V(x,\, J_i^{n-1}\,) \qquad i = 1, ...S_{n-1}.$$

We suspect this to be true. Use of computer should help at least for low n's.

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Exhibit 1

Ordering and labels of 4-iterates generated from ordering of 3-iterates.

1 $VVVxxxx$	2 $VVxxVxx$	3 $VVxVxxx$	4 $VxVVxxx$	5 $VxVxVxx$
1 $VVVVxxxxx$	2 $VVVxxxVxx$	3 $VVVxxVxxx$	4 $VVxxVVxxx$	5 $VVxxVxVxx$
6 $VVVxVxxxx$	7 $VVxVxxVxx$	8 $VVxVVxxxx$	9 $VxVVVxxxx$	10 $VxVVxxVxx$
3 $VVVxxVxxx$	4 $VVxxVVxxx$	11 $VVxVxVxxx$	12 $VxVVxVxxx$	13 $VxVxVVxxx$
2 $VVVxxxVxx$	5 $VVxxVxVxx$	7 $VVxVxxVxx$	10 $VxVVxxVxx$	14 $VxVxVxVxx$

1 $VVVVxxxxx$	3 $VVVxxVxxx$	6 $VVVxVxxxx$	8 $VVxVVxxxx$	11 $VVxVxVxxx$
9 $VxVVVxxxx$	10 $VxVVxxVxx$	12 $VxVVxVxxxx$	13 $VxVxVVxxx$	14 $VxVxVxVxx$

Tableau A_4

1	**2**	**3**	**4**	**5**
6	**7**	**8**	**9**	**10**
3	4	**11**	**12**	**13**
2	5	7	10	**14**

Tableau B_4

1	3	6	8	11
9	10	12	13	14

Exhibit 2

Iterates J_i^n $(n \leq 4)$		**Labels** $i^{(n}$
J_1^0	x	$1^{(0}$
J_1^1	Vxx	$1^{(1}$
J_1^2	$VVxxx$	$1^{(2}$
J_2^2	$VxVxx$	$2^{(2}$
J_1^3	$VVVxxxx$	$1^{(3}$
J_2^3	$VVxxVxx$	$2^{(3}$
J_3^3	$VVxVxxx$	$3^{(3}$
J_4^3	$VxVVxxx$	$4^{(3}$
J_5^3	$VxVxVxx$	$5^{(3}$
J_1^4	$VVVVxxxxx$	$1^{(4}$
J_2^4	$VVVxxxVxx$	$2^{(4}$
J_3^4	$VVVxxVxxx$	$3^{(4}$
J_4^4	$VVxxVVxxx$	$4^{(4}$
J_5^4	$VVxxVxVxx$	$5^{(4}$
J_6^4	$VVVxVxxxx$	$6^{(4}$
J_7^4	$VVxVxxVxx$	$7^{(4}$
J_8^4	$VVxVVxxxx$	$8^{(4}$
J_9^4	$VxVVVxxxx$	$9^{(4}$
J_{10}^4	$VxVVxxVxx$	$10^{(4}$
J_{11}^4	$VVxVxVxxx$	$11^{(4}$
J_{12}^4	$VxVVxVxxx$	$12^{(4}$
J_{13}^4	$VxVxVVxxx$	$13^{(4}$
J_{14}^4	$VxVxVxVxx$	$14^{(4}$
J_1^5	$VVVVVxxxxxx$	$1^{(5}$
J_2^5	$VVVVxxxxVxx$	$2^{(5}$
…	…	…
J_8^5	$VVVxxVVxxxx$	$8^{(5}$
…	…	…
J_{11}^5	$VVVxxVxVxxx$	$11^{(5}$
…	…	…
J_{42}^5	$VxVxVxVxVxx$	$42^{(5}$
…	…	…

Exhibit 3

Tableaux A_n **and** B_n $(n \le 5)$

A_1

1

B_1

1

A_2

1
2

B_2

1
2

A_3

1	**2**
3	**4**
2	**5**

B_3

1	3
4	5

A_4

1	**2**	**3**	**4**	**5**
6	**7**	**8**	**9**	**10**
3	4	**11**	**12**	**13**
2	5	7	10	**14**

B_4

1	3	6	8	11
9	10	12	13	14

A_5

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
15	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**
6	7	8	9	10	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**
3	4	11	12	13	17	18	25	26	27	**38**	**39**	**40**	**41**
2	5	7	10	14	16	19	21	24	28	30	33	37	**42**

B_5

1	3	6	8	11	15	17	20	22	25	29	31	34	38
23	24	26	27	28	32	33	35	36	37	39	40	41	42

Tableaux $A_n \oplus B_n$ $(n \le 5)$

$A_1 \oplus B_1$

1
1

$A_2 \oplus B_2$

1
2
1
2

$A_3 \oplus B_3$

1	2
3	4
2	5
1	3
4	5

$A_4 \oplus B_4$

1	2	3	4	5
6	7	8	9	10
3	4	11	12	13
2	5	7	10	14
1	3	6	8	11
9	10	12	13	14

$A_5 \oplus B_5$

1	2	3	4	5	6	7	8	9	10	11	12	13	14
15	16	17	18	19	20	21	22	23	24	25	26	27	28
6	7	8	9	10	29	30	31	32	33	34	35	36	37
3	4	11	12	13	17	18	25	26	27	38	39	40	41
2	5	7	10	14	16	19	21	24	28	30	33	37	42
1	3	6	8	11	15	17	20	22	25	29	31	34	38
23	24	26	27	28	32	33	35	36	37	39	40	41	42

Exhibit 4

"Pascal" triangle for the numbers $c_{n,j}$

$n\backslash j$	1	2	3	4	5	6	7	8	9	10	...
1	1										
2	1	1									
3	2	2	1								
4	5	5	3	1							
5	14	14	9	4	1						
6	42	42	28	14	5	1					
7	132	132	90	48	20	6	1				
8	429	429	297	165	75	27	7	1			
9	1430	1430	1001	572	275	110	35	8	1		
10	4862	4862	3432	2002	1001	429	154	44	9	1	
...	...	...	...	...	...	...	...	...	...	...	...

$$c_{n,j} = c_{n-1,j-1} + c_{n-1,j} + c_{n-1,j+1} + \cdots + c_{n-1,n-1}$$

$$\sum_{j=1}^{n} c_{n,,j} = S_n$$

Exhibit 5

Incidence matrix for $n = 4$ relative to tableau A_4 for all possible S_4^2 ($= 14^2$) identities $J_i^{(4} = J_j^{(4}$ and calculation of I_4

($J_i^{(4}$ denoted by i. Blank spaces denote 0's)

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad \text{identity reducible} \\ 0 & \text{if} \quad \text{identity irreducible} \end{cases}$$

$$M(i) = \text{Multiplicity of } J_i$$

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	$\sum_i 1$	$M(i)$
1	1	1	1	1	1										5	1
2	1	1	1	1	1		1			1				1	8	2
3	1	1	1	1	1						1	1	1		8	2
4	1	1	1	1	1						1	1	1		8	2
5	1	1	1	1	1		1			1				1	8	2
6						1	1	1	1	1					5	1
7		1			1	1	1	1	1	1				1	8	2
8						1	1	1	1	1					5	1
9						1	1	1	1	1					5	1
10		1			1	1	1	1	1	1				1	8	2
11			1	1							1	1	1		5	1
12			1	1							1	1	1		5	1
13			1	1							1	1	1		5	1
14		1			1		1			1				1	5	1
														$I_4 =$	88	

Number of J_i^n with multiplicity k : $\quad T_{n,k} = 2^{n-2k+1}\binom{n-1}{2k-2}S_{k-1}$

For $n = 4$, $\quad k = 1$: $\quad T_{4,1} = 8$

For $n = 4$, $\quad k = 2$: $\quad T_{4,2} = 6$

$$I_n = \sum_{k=1}^{\left[\frac{n+1}{2}\right]} (-1)^{k-1} \binom{n-k+1}{k} S_{n-k}^2$$

$$I_4 = \sum_{k=1}^{2} (-1)^{k-1} \binom{5-k}{k} S_{4-k}^2 = \binom{4}{1} S_3^2 - \binom{3}{2} S_2^2 = 88$$

$$(S_2 = 2, \quad S_3 = 5)$$

Exhibit 6

1. Semantical equality of two iterates in same column connected by $\updownarrow$ implies semantical equality of two iterates of order 2 if V satisfies unicity conditions.

A_3

1	2
$\updownarrow$	
3	4
	$\updownarrow$
2	5

A_4

1	2	3	4	5
$\updownarrow$	$\updownarrow$			
6	7	8	9	10
		$\updownarrow$	$\updownarrow$	
3	4	11	12	13
				$\updownarrow$
2	5	7	10	14

A_5

1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$									
15	16	17	18	19	20	21	22	23	224	25	26	27	28
					$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$				
6	7	8	9	10	29	30	31	32	33	34	35	36	37
		$\updownarrow$	$\updownarrow$							$\updownarrow$	$\updownarrow$	$\updownarrow$	
3	4	11	12	13	17	18	25	26	27	38	39	40	41
	$\updownarrow$			$\updownarrow$		$\updownarrow$			$\updownarrow$				$\updownarrow$
2	5	7	10	14	16	19	21	24	28	30	33	37	42

2. Semantic equality of two iterates in same column or different columns having a common element connected by $\updownarrow$ implies semantic equality of two iterates of order 3 if V satisfies unicity conditions.

A_4

1	2	3	4	5
$\updownarrow$		$\updownarrow$		
6	7	8	9	10
$\updownarrow$		$\updownarrow$	$\updownarrow$	$\updownarrow$
3	4	11	12	13
			$\updownarrow$	$\updownarrow$
2	5	7	10	14

A_5

1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\updownarrow$	$\updownarrow$				$\updownarrow$	$\updownarrow$							
15	16	17	18	19	20	21	22	23	24	25	26	27	28
$\updownarrow$	$\updownarrow$				$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$		$\updownarrow$	$\updownarrow$		
6	7	8	9	10	29	30	31	32	33	34	35	36	37
			$\updownarrow$	$\updownarrow$			$\updownarrow$	$\updownarrow$		$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
3	4	11	12	13	17	18	25	26	27	38	39	40	41
			$\updownarrow$	$\updownarrow$								$\updownarrow$	$\updownarrow$
2	5	7	10	14	16	19	21	24	28	30	33	37	42