# A general formula in Additive Number Theory


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**Abstract**

Classical Additive Number Theory in $\mathbb{Z}$ investigates the existence of a finite integer $\theta$ such that for a given infinite sequence of increasing non-negative integers $a_\nu = f(\nu)$, $\nu = 1, 2, \ldots$ the Diophantine equation

$$f(x_1) + f(x_2) + \ldots + f(x_\theta) = n$$

has solutions $x_i \geq 1$ for any $n \geq 0$, and determines, if possible, their number, asymptotically or otherwise (Goldbach, Waring, Hilbert, Hardy-Littlewood, Vinogradov, Erdös, ...). Our approach is different.

We denote by $A(n, N, \theta) = A(n.\theta)$ the number of solutions in integers $x_i \geq 0$ of the Diophantine system:

$$a_1x_1 + a_2x_2 + \ldots + a_Nx_N = n \tag{1}$$

$$x_1 + x_2 + \ldots + x_N = \theta \ . \tag{1'}$$

$A(n.\theta)$ expresses in how many ways $n$ is a sum of $\theta$ integers taken from the set $\{a_1, \ldots, a_N\}$.

The generating function for the $A(n, \theta)$, $n = 0, 1, \ldots,$ $\theta = 0, 1, \ldots,$ is:

$$\sum_{\substack{n=0\ldots\infty \\ \theta=0\ldots\infty}} A(n, \theta)\ x^n y^\theta = \prod_{\nu=1}^{N} \frac{1}{1 - x^{a_\nu} y} \quad ,$$

as seen by expanding all the $(1 - x^{a_\nu} y)^{-1}$ formally in power series and, after multiplication, collecting terms with equal exponents in $x$ and $y$, respectively.

Considering $x$ and $y$ as complex variables we have by Cauchy's theorem for the coefficients of power series of several complex variables:

$$A(n, \theta) = \frac{1}{(2\pi i)^2} \int_{c_1} \int_{c_2} \frac{1}{x^{n+1} y^{\theta+1}} \prod_{\nu=1}^{N} \frac{1}{1 - x^{a_\nu} y}\, dxdy \ , \tag{2}$$

where the integrals are taken over the circles $|x| = c_1 < 1$, $|y| = c_2 < 1$, respectively.

The partial fraction expansion with regard to $y$ of $\prod_{\nu=1}^{N}(1 - x^{a_\nu} y)^{-1}$ gives

$$\prod_{\nu=1}^{N} \frac{1}{1 - x^{a_\nu} y} = \sum_{\nu=1}^{N} \frac{1}{y^{N-1} L'(x^{a_\nu})(1 - x^{a_\nu} y)} \ ,$$

where $L'(t)$ is the derivative of $L(t) = \prod_{\nu=1}^{N}(t - x^{a_\nu})$.

Inserting in (2) and expanding $(1 - x^{a_\nu}y)^{-1}$ in power series of $y$ within the circle $c_2$ we get successively:

$$A(n,\theta) = \frac{1}{(2\pi i)^2}\int_{c_1}\int_{c_2} \frac{1}{x^{n+1}y^{\theta+1}} \left\{\sum_{\nu=1}^{N} \frac{1}{y^{N-1}L'(x^{a_\nu})(1 - x^{a_\nu}y)}\right\}dxdy$$

$$= \frac{1}{2\pi i}\int_{c_1} \frac{1}{x^{n+1}} \left\{\frac{1}{2\pi i}\int_{c_2} \frac{1}{y^{\theta+N}} \sum_{\nu=1}^{N}\frac{1}{L'(x^{a_\nu})(1 - x^{a_\nu}y)}\, dy\right\}dx$$

$$= \frac{1}{2\pi i}\int_{c_1} \frac{1}{x^{n+1}} \left\{\sum_{\nu=1}^{N} \frac{1}{2\pi i}\int_{c_2} \frac{1}{y^{\theta+N}L'(x^{a_\nu})}(1 + x^{a_\nu}y + \ldots\ )dy\right\}dx \quad .$$

Integrating over $|y| = c_2$ we obtain by Cauchy's theorem:

$$A(n,\theta) = \frac{1}{2\pi i}\int_{c_1} \frac{1}{x^{n+1}}\left\{\sum_{\nu=1}^{N} \frac{x^{a_\nu(\theta+N-1)}}{L'(x^{a_\nu})}\right\}dx$$

$$= \sum_{\nu=1}^{N} \frac{1}{2\pi i}\int_{c_1} \frac{1}{x^{n+1-a_\nu(\theta+N-1)}}\frac{1}{L'(x^{a_\nu})}\, dx. \qquad (3)$$

In order to expand $1/L'(x^{a_\nu})$ in a power series of $x$ we transform $L'(x^{a_\nu})$ as follows:

$$L'(x^{a_\nu}) = (x^{a_\nu} - x_1^a)\ldots(x^{a_\nu} - x^{a_{\nu-1}})(x^{a_\nu} - x^{a_{\nu+1}})\ldots(x^{a_\nu} - x^{a_N})$$

$$= x^{a_1+\ldots+a_{\nu-1}}(x^{a_\nu-a_1} - 1)\ldots(x^{a_\nu-a_{\nu-1}} - 1)x^{a_\nu(N-\nu)}(1 - x^{a_{\nu+1}-a_\nu})\ldots(1 - x^{a_N-a_\nu})$$

$$= (-1)^{\nu-1}x^{a_1+\ldots+a_{\nu-1}+a_\nu(N-\nu)}(1 - x^{a_\nu-a_1})\ldots(1 - x^{a_\nu-a_{\nu-1}})(1 - x^{a_{\nu+1}-a_\nu})\ldots(1 - x^{a_N-a_\nu}).$$

Inserting in (3) for $L'(x^{a_\nu})$ their above expressions and after calculations in the exponents, we obtain:

$$A(n,\theta) = \sum_{\nu=1}^{N} \frac{(-1)^{\nu-1}}{2\pi i}\int_{c_1} \frac{1}{x^{n+1+a_1+\ldots+a_\nu-(\theta+\nu)a_\nu}}P_\nu(x)dx \qquad (4)$$

where

$$P_\nu(x) = \frac{1}{(1 - x^{a_\nu-a_1})\ldots(1 - x^{a_\nu-a_{\nu-1}})(1 - x^{a_{\nu+1}-a_\nu})\ldots(1 - x^{a_N-a_\nu})} \quad .$$

Since all exponents $a_\nu - a_\mu$ are positive, we can expand for $|x| \le c_1$ the factors $P_\nu(x)$ in power series of $x$:

$$P_\nu(x) = \sum_{\lambda=0}^{\infty}B_\nu(\lambda)x^\lambda$$

where $B_\nu(\lambda)$ are respectively the number of non-negative integer solutions of the linear Diophantine equations

$$(a_\nu - a_1)x_1 + \ldots + (a_\nu - a_{\nu-1})x_{\nu-1} + (a_{\nu+1} - a_\nu)x_\nu + \ldots + (a_N - a_{N-1})x_{N-1} = \lambda \ .$$

Substituting in (4) the $P_\nu(x)$ by their respective power series and using again Caushy's theorem we finally arrive at

$$A(n,\theta) = \sum_{\nu=1}^{N} (-1)^{\nu-1} B_\nu(s_\nu) \ , \tag{5}$$

with $s_\nu = n + \sum_{i=1}^{\nu} a_i - (\nu+\theta)a_\nu$ and $B_\nu(s_\nu)$ respectively, the number of solutions of each of the following linear Diophantine equations

$$(a_\nu - a_1)x_1 + \ldots + (a_\nu - a_{\nu-1})x_{\nu-1} + (a_{\nu+1} - a_\nu)x_\nu + \ldots + (a_N - a_{\nu-1})x_{N-1} = s_\nu \tag{6}$$

$$\nu = 1, \ldots, N \ .$$

This formula reduces the investigation of the number of solutions of the initial system to that of the number of solutions of $N$ linear Diophantine equations involving the difference sets (positive) $\{a_\nu - a_\mu\}$ and the numbers $s_\nu$.

Geometrically speaking this means that the number of *Gitterpunkte* in the intersection of the two $N$-dimensional planes (1) and (1') in the positive quadrant $x_i \geq 0$ is equal to the alternate sum of the number of *Gitterpunkte* of the $(N-1)$-dimensional planes (6) in the same quadrant.

As standard examples, theorems and conjectures we may cite $a_\nu = (\nu-1)^2$, $\theta = 4$ (Lagrange), $a_\nu = (\nu-1)^k$ (Waring), $a_\nu = \nu$-th prime, $\theta = 2$, $n$ even (Goldbach), $a_\nu = \nu^p$, $\theta = 2$, $n = n_1^p$, $p \geq 3$ (Fermat).

Obviously in order to attack the problem for a given sequence we have to take into account that $N$ is linked to $n$ by a function $N(n)$ (ex.g. for Lagrange $N(n) = [n^{\frac{1}{2}}]$). This complicates the matter but still an ad hoc suggestion would be to approximate the $B_\nu(s_\nu)$ as follows

$$B_\nu(s_\nu) \sim \frac{s_\nu^{N-2}}{(N-2)!(a_\nu - a_1)\cdots(a_N - a_\nu)} \ ,$$

which is valid for $n \to \infty$ but fixed $N$ (Polya-Szegö, Aufgaben und Lehrsätze aus der Analysis I; loosing no generality the differences $a_\nu - a_\mu$ can be assumed free of common divisors $> 1$).

Summing over $\nu$ (we write $N$ for short of $N(n)$) and reverting again to the polynomials $L(t)$, written now as $L_N(t)$, we obtain from (5)

$$\frac{1}{(N-2)!} \sum_{\nu=1}^{N} \frac{s_\nu^{N-2}}{L'_N(a_\nu)} \ , \tag{7}$$

as a plausible heuristic estimate of $A(n,\theta)$ for $n \to \infty$.

The behaviour of the expressions involving $n$ and $\theta$:

$\dfrac{n + \sum_{i=1}^{\nu} a_i - (\nu+\theta)a_\nu}{|a_\nu - a_\mu|}$ The cutting points of the planes (6) with the coordinate axes,

$\dfrac{n + \sum_{i=1}^{\nu} a_i - (\nu+\theta)a_\nu}{\sqrt{\sum_{\nu=1}^{N}(a_\nu - a_\mu)^2}}$ The distances of the planes (6) from the origin, would play, we believe, a decisive role in any such attempt.

As to its form the sum (7) bears a striking resemblance to the sums

$$\sum_{\nu=1}^{N} \frac{a_\nu^t}{L'_N(a_\nu)}$$

encountered in Lagrange interpolation with $a_\nu$ replaced by $s_\nu$ in the numerator. As known these expressions considered as functions of the exponent $t$ are equal to:

$$\begin{cases} \dfrac{(-1)^{N-1}}{a_1 \cdots a_N} \displaystyle\sum_{\sum i = -t-1} \dfrac{1}{a_1^{i_1} \cdots a_N^{i_N}} & \text{for} \quad t \leq -1 \\ 0 & \text{for} \quad 0 \leq t \leq N-2 \\ 1 & \text{for} \quad t = N-1 \\ \displaystyle\sum_{\sum i = t-N+1} a_1^{i_1} \cdots a_N^{i_N} & \text{for} \quad N \leq t \end{cases}$$

Above facts may prove, eventually, useful in further developments.