Variance Maintained by Stochastic Forcing of Non-Normal Dynamical Systems Associated with Linearly Stable Shear Flows

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The level of variance maintained in a stochastically forced asymptotically stable linear dynamical system with a non-normal dynamical operator cannot be fully characterized by the decay rate of its normal modes, unlike normal dynamical systems. The nonorthogonality of modes may lead to transient growth which supports variance far in excess of that anticipated from the decay rate given by the eigenvalues of the operator. As an example, the variance maintained by stochastic forcing in a canonical shear flow is found to increase with a power of the Reynolds number between 1.5 and 3. This great amplification of variance suggests a fundamentally linear mechanism underlying shear flow turbulence.

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Consider the stochastically forced linear autonomous dynamical system

$$\frac{d}{dt}x_i=\mathcal{A}_{ij}x_j+\mathcal{G}_{ij}\epsilon_j\,, \tag{1}$$

in which x is a complex vector of generalized coordinates and ϵ a random forcing taken to be a δ-correlated Gaussian white-noise process with zero mean so that

$$\begin{aligned}&\langle\epsilon_i\rangle=0\,,\\&\langle\epsilon_i(t)\epsilon_j(t')\rangle=\xi\delta_{ij}\delta(t-t')\,,\end{aligned} \tag{2}$$

where $\langle\,\rangle$ denotes ensemble averaging, and the asterisk denotes complex conjugation. The random forcing excites with equal probability and independently each forcing function, specified by the columns of the matrix $\mathcal{G}_{ij}$. We are interested in the variance $\langle E^\infty\rangle=\lim_{t\to\infty}\langle z_i^*(t)\times x_i(t)\rangle$ which exists for asymptotically stable systems in which all the eigenvalues of the linear dynamical operator $\mathcal{A}$ have negative real parts.

Relatively more consideration has been given in the past to the stochastic dynamics of linear operators that are normal (i.e., $\mathcal{A}^\dagger\mathcal{A}=\mathcal{A}\mathcal{A}^\dagger$, where $\dagger$ denotes the Hermitian transpose) while in this Letter we concentrate on stochastic excitation of non-normal operators such as govern the dynamics of perturbations in fluid shear flow and for which $\mathcal{A}^\dagger\mathcal{A}\neq\mathcal{A}\mathcal{A}^\dagger$. In the case of a normal $\mathcal{A}$ the motion can be resolved into the N distinct orthogonal normal modes of $\mathcal{A}$ and each mode analyzed in isolation with the total variance found as the sum of contributions from each of the individual normal modes (Wang and Uhlehnbeck [1]). Furthermore, the variance contributed by each mode is inversely proportional to the damping rate of that mode so that the stationary state variance supported is (see Ref. [1])

$$\langle E^t_{\rm normal}\rangle=\sum_{i=1}^{N}-\frac{\xi}{2\sigma_i}\,, \tag{3}$$

where $\sigma_i<0$ is the real part of the ith distinct eigenvalue of the normal operator $\mathcal{A}$. In normal dynamical systems the forcing is the only energy source for the normal modes and the energy associated with the stochastically maintained variance is accumulated from this forcing, resulting in high variance if damping is small.

Consider now a fluid with a background flow field having nonvanishing rate of strain but with sufficient dissipation so that all sufficiently small perturbations impressed on the flow eventually decay. Linearization of the dynamical system about the background flow results in a non-normal linear dynamical operator and associated set of modes that individually decay but that are not mutually orthogonal. This mathematical property of nonorthogonality of modes is indicative of an important physical property: The lack of mode orthogonality corresponds to the potential for extraction of energy from the background flow field by a subspace of perturbations leading to transient growth despite the absence of exponential instability, a result known since the pioneering work of Orr [2] (see Refs. [3–14]). However, it should be noted that every asymptotically stable non-normal operator does not necessarily support transient growth. The condition for transient growth for an operator $\mathcal{A}$ is most transparently derived from consideration of the equation for the evolution of the perturbation magnitude in the L_2 norm, which in the absence of forcing is

$$\frac{d}{dt}x_i^*x_i=x_i^*(\mathcal{A}_{ij}^\dagger+\mathcal{A}_{ij})x_j\,. \tag{4}$$

It is clear from (4) that it is necessary and sufficient for transient growth that the largest eigenvalue of the operator $\frac{1}{2}(\mathcal{A}^\dagger+\mathcal{A})$ be positive, which is equivalent to the requirement that $\sup\mathrm{Re}[\mathcal{F}(\mathcal{A})]$ be positive, where $\mathcal{F}(\mathcal{A})$ is the numerical range of $\mathcal{A}$, i.e., the set of the values $x^\dagger\mathcal{A}x$ for $||x||=1$ (see Ref. [12]). For a normal operator $\mathcal{F}(\mathcal{A})$ is the convex hull of the spectrum set of $\mathcal{A}$, denot-

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ed by $\Lambda(\mathcal{A})$, so that the condition for transient growth can be identified with the occurrence of a positive real part of $\Lambda(\mathcal{A})$. The condition for growth in the case of a non-normal operator depends on the positiveness of a (necessarily real) eigenvalue of $\frac{1}{2}(\mathcal{A}^{\dagger}+\mathcal{A})$ and is not equivalent to the requirement that $\mathcal{A}$ be asymptotically unstable.

Recently a more complete understanding of the role of 3D time dependent perturbations in transferring energy from the mean flow to the perturbation field has emerged and the universality of this process in shear flows has been recognized (see Ref. [11]). The original 2D growth mechanism of Orr (in which transient upshear tilting perturbations grow by inducing down gradient Reynolds stresses) and the streamwise roll mechanism of Landahl (in which spanwise variations of cross-stream velocity alternatively lift and depress material parcels in the sheared background flow to produce perturbation velocity fields dominated by streamwise streaks; see Ref. [13]) have been found to occur in a characteristic combination which are accompanied by rapid transfer of mean flow energy to the perturbations and produce the ubiquitously observed coherent structures (see Refs. [10,14]).

Tapping the mean flow energy in this way can lead to levels of perturbation variance orders of magnitude larger than would have been supported by an otherwise equivalent damped normal system. The energy balance in such a non-normal system is between the stochastic driving together with the induced extraction of energy from the background flow, on the one hand, and the dissipation, on the other hand. Under stochastic forcing the variance in a non-normal system may be maintained primarily by the stochastically induced transfer of background flow energy to the perturbation field. While the potential for transient growth from specific initial conditions has been amply demonstrated in previous work (see Refs. [2-9, 11-14]), it is not immediate that the existence of a subspace of growing perturbations is sufficient to produce variance greatly enhanced in comparison with that maintained under stochastic excitation in any equivalently damped normal system, such as that associated with an unsheared fluid. For example, in 2D unbounded constant shear flow with Rayleigh damping the maintained variance at any shear is identical to that obtained in the absence of shear (see Ref. [15]), despite the existence of a subspace of transiently growing perturbations that increases with shear (see Ref. [16]). One purpose of this Letter is to examine whether transient growth of a subset of perturbations can result in enhanced variance in a stochastically forced flow.

The linearized 3D Navier-Stokes equations governing evolution of disturbances in steady mean flow with streamwise (x) velocity, U, varying only in the cross-stream y direction (see Ref. [7]) are

$$\frac{\partial}{\partial t}\begin{bmatrix}\hat{v}\\ \hat{\omega}\end{bmatrix}=\begin{bmatrix}\mathcal{L} & 0\\ \mathcal{C} & \mathcal{S}\end{bmatrix}\begin{bmatrix}\hat{v}\\ \hat{\omega}\end{bmatrix}, \tag{5}$$

in which

$$\mathcal{L}=\Delta^{-1}(-i\alpha U\Delta+i\alpha U_{yy}+\Delta\Delta/R), \tag{6a}$$

$$\mathcal{S}=-i\alpha U+\Delta/R, \tag{6b}$$

$$\mathcal{C}=-i\beta U_y, \tag{6c}$$

where $K^2\equiv\alpha^2+\beta^2$, $\Delta\equiv d^2/dy^2-K^2$, and $R\equiv U_0L/\nu$ is the Reynolds number, based on the maximum velocity, U_0, occurring in a channel of half-width L in a fluid with kinematic viscosity ν. We consider the evolution of single Fourier components of the cross-stream velocity $v=\hat{v}\times\exp(i\alpha x+i\beta z)$ and the cross-stream component of perturbation vorticity $\omega=\hat{\omega}\exp(i\alpha x+i\beta z)$. No slip boundary conditions are imposed at $y=\pm 1$ which require $v=\partial v/\partial y=\omega=0$ at $y=\pm 1$. Plane Poiseuille flow with $U=1=y^2$ is chosen for the examples.

Consider now the discrete equivalent of (5). The state vector for an N level discretization is $\psi=[\hat{v}_1\cdots\hat{v}_N\times\hat{\omega}_1\cdots\hat{\omega}_N]^T$, and we denote with $\mathcal{T}$ the discretized form of

$$\begin{bmatrix}\mathcal{L} & 0\\ \mathcal{C} & \mathcal{S}\end{bmatrix},$$

by which the continuous dynamical system (5) is approximated as a finite dynamical system.

To determine the evolution of perturbation energy $E(\alpha,\beta)$, we introduce the energy metric $\mathcal{M}$ given by

$$\mathcal{M}=\frac{1}{8K^2}\begin{bmatrix}-\Delta & 0\\ 0 & I\end{bmatrix}, \tag{7}$$

where I is the identity matrix. The perturbation energy is then $E(\alpha,\beta)=\psi^{\dagger}\mathcal{M}\psi$ (see Ref. [7]). The dynamical equation (5) can then be transformed into variables of the generalized velocities $x=\mathcal{M}^{1/2}\psi$ so that the operator appearing in (1) becomes

$$\mathcal{A}=\mathcal{M}^{1/2}\mathcal{T}\mathcal{M}^{-1/2}. \tag{8}$$

In the energy metric the variance corresponds to the ensemble average energy $\langle E^{\infty}\rangle$. For plane Poiseuille flow $\mathcal{A}$ is asymptotically stable for $R<5772.22$ (Orszag [17]) and we will limit our investigations to $R\leq 5000$, so that the existence of stationary statistics is immediate.

The asymptotic ensemble energy $\langle E^{\infty}\rangle$ is obtained with the aid of the Fourier transform pair (for an alternative procedure see Ref. [18]):

$$x(t)=\int_{-\infty}^{\infty}\hat{x}(\omega)e^{i\omega t}d\omega, \tag{9a}$$

$$\hat{x}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}x(t)e^{-i\omega t}dt. \tag{9b}$$

The Fourier transform of (1) is

$$\hat{x}(\omega)=\mathcal{R}(\omega)\mathcal{G}\hat{\epsilon}(\omega), \tag{10}$$

with resolvent

$$\mathcal{R}(\omega)=(i\omega\mathcal{I}-\mathcal{A})^{-1}. \tag{11}$$

The ensemble average energy is

$$\langle E^{\infty}\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{R}_{ij}(\omega')\mathcal{G}_{jk}\langle \hat{\epsilon}_k(\omega)\hat{\epsilon}_m^*(\omega')\rangle \times \mathcal{R}_{il}^*(\omega')\mathcal{G}_{lm}^* e^{i(\omega-\omega')t}\, d\omega\, d\omega' . \qquad (12)$$

With white-noise excitation, $\langle \hat{\epsilon}_k(\omega)\hat{\epsilon}_m^*(\omega')\rangle = (\dot{\xi}/2\pi)\delta_{km} \times \delta(\omega-\omega')$ and for unitary forcing distributions $\mathcal{G}\mathcal{G}^\dagger = \mathcal{I}$ we have

$$\langle E^{\infty}\rangle = \frac{\dot{\xi}}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, d\omega , \qquad (13)$$

where the frequency response is given by

$$F(\omega) = \mathrm{Tr}[\mathcal{R}^\dagger(\omega)\mathcal{R}(\omega)] . \qquad (14)$$

Note that the energy response function $F(\omega)$ cannot be simply characterized as a summation of individual contributions from the poles of the resolvent as would be the case if the operator were normal. Recent results (see Ref. [19]) underscore this point by noting that for a general linear operator the norm of the resolvent is bounded by

$$\frac{1}{\mathrm{dist}(i\omega, \Lambda(\mathcal{A}))} \leq \|\mathcal{R}(\omega)\| \leq \frac{1}{\mathrm{dist}(i\omega, \mathcal{F}(\mathcal{A}))} , \qquad (15)$$

where the right inequality is valid for $i\omega \notin \overline{\mathcal{F}(\mathcal{A})}$ and the left inequality for $i\omega \notin \Lambda(\mathcal{A})$ and dist denotes the distance function of a point from a set [recall that $\Lambda(\mathcal{A})$ is the spectrum set and $\mathcal{F}(\mathcal{A})$ is the numerical range of $\mathcal{A}$]. For a normal operator $\mathcal{F}(\mathcal{A})$ becomes the convex hull of $\Lambda(\mathcal{A})$ and, consequently, the inequalities in (15) become equalities, signifying that the system response at a given frequency ω is characterized solely by the proximity of $i\omega$ to the spectrum of $\mathcal{A}$. However, for a non-normal operator $\mathrm{dist}(i\omega, \mathcal{F}(\mathcal{A}))$ may be much less than $\mathrm{dist}(i\omega, \Lambda(\mathcal{A}))$ rendering the familiar estimate of the response by the distance from the contour $i\omega$ to the poles inadequate.

The increase of the variance of the stationary state as a function of Reynolds number, R, is shown for spanwise wave number $\beta = 1$ and unit forcing in Fig. 1. This wave number was chosen for concreteness and is not exceptional. In order to demonstrate clearly the role of non-normality in increasing the response of the system we have included in the same figure the variance that would have resulted if the system were interpreted as an equivalent normal system by applying Eq. (3). Note the dramatic increase of variance with Reynolds number. As $R \to 0$ the dynamical operator approaches the (normal) diffusion operator leading to a linear dependence of variance on R, as is the case for all normal systems. For higher R the variance of the streamwise roll components, $\alpha = 0$, grows as R^3. This is due to $O(R)$ transient amplitude growth over the time $O(R)$ during which input energy accumulates before it dissipates (see Refs. [7,9]). For oblique waves and for Reynolds numbers $800 < R < 5000$ the variance grows approximately as $R^{3/2}$. These results hold for a wide range of spanwise and streamwise wave numbers.

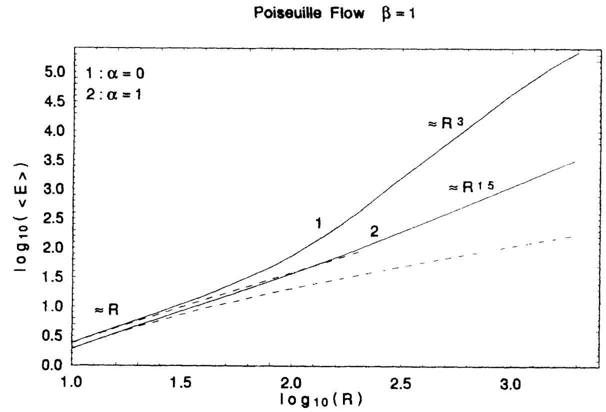

FIG. 1. Variation of ensemble average energy as a function of Reynolds number R for plane Poiseuille flow. The full line indicates the variance of the non-normal operator. The dashed line represents the variance that would result if the operator were interpreted as a normal operator. Both cases are for spanwise wave number $\beta = 1$. Curve 1 is for streamwise wave number $\alpha = 0$, and curve 2 for $\alpha = 1$.

The spectral response function $F(\omega)$ for the case of streamwise rolls ($\alpha = 0$) is plotted for $R = 500$ and 2000 in Fig. 2 and for comparison the variance supported by an equivalent normal system is also shown. Over the primary passband of the response the variance supported by the non-normal operator is substantially larger than that supported by the equivalent normal system. Maximum response is found in the neighborhood of zero frequency which would characterize low-pass disturbances such as those arising from boundary roughness. The normal and non-normal responses converge at high frequencies as the

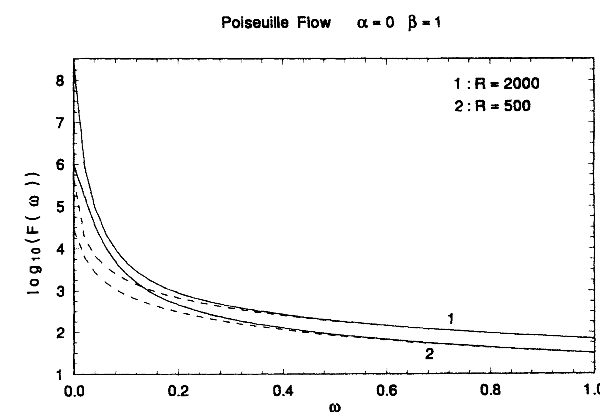

FIG. 2. The frequency response function $F(\omega)$ in the energy norm for plane Poiseuille flow with streamwise wave number $\alpha = 0$ and spanwise wave number $\beta = 1$. Curve 1 corresponds to Reynolds number $R = 2000$; curve 2 to $R = 500$. The dashed curves show the frequency response that would result if the operator were interpreted as a normal operator.

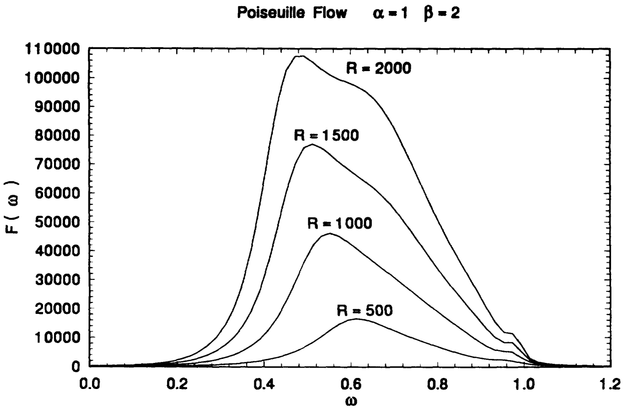

FIG. 3. The frequency response function $F(\omega)$ in the energy norm for plane Poiseuille flow with streamwise wave number $\alpha=1$ and spanwise wave number $\beta=2$ for various Reynolds numbers R.

dynamical operator becomes diagonal and consequently normal [cf. Eq. (11)].

The typical growth of variance as a function of R for an oblique plane wave perturbation is shown in Fig. 3. The main contribution to the variance arises from frequencies resulting in phase velocities lying within the flow, i.e., $0 \leq \omega/\alpha \leq 1$. Note that the maximum response for higher R shifts to lower frequencies corresponding to disturbances with phase speeds nearer the boundary flow speed.

We have demonstrated that the maintained variance in shear flow may greatly exceed the variance anticipated from the decay rate of the modes of the dynamical operator. Transient growth, which is possible because of the nonorthogonality of the modes of a non-normal operator, leads to enhanced levels of variance arising from transfer of the energy of the mean flow to the perturbations. This mechanism may provide the amplification of free stream disturbances necessary to support turbulence for subcritical Reynolds numbers. For example, we calculate using the above methods that a background noise level of 1% rms in the velocity field sustains variance of typical turbulent intensity (10% rms) at $R=1000$ for Couette flow, and $R=2000$ for Poiseuille flow. The centrality of the linear growth mechanism retained in this analysis was realized by Joseph [20] and recently underscored by Henningson and Reddy [21], who point out that in forced shear flow the nonlinear terms make no contribution to the transfer between mean flow energy and perturbation energy. Taken together these results suggest that a linear first order mean field approximation for turbulence in shear flows can be constructed in which the nonlinear effects are parametrized. These effects are identified as nonlinear spectral transfer, here parametrized as stochastic forcing, and diffusive dissipation and disruption of perturbations, which may be parametrized using an eddy viscosity. Some numerical results supporting this interpretation have been obtained (see Refs. [4,22]).

While we have chosen an example from the realm of hydrodynamics, these results are also pertinent to other physical systems governed by asymptotically stable non-normal dynamical operators. We have shown in this work that when a non-normal system supports transient growth the stochastically maintained variance can greatly exceed the variance anticipated from the decay rate of the modes of the dynamical operator.

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