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GRAVITATIONAL TIDES IN THE OUTER PLANETS. I. IMPLICATIONS OF CLASSICAL TIDAL THEORY

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ABSTRACT

Classical tidal theory is applied to the gravitational excitation of the atmospheres of the gaseous planets. The only departure made from classical theory is the retention of the effects of nonhdrostaticity which are important in the deeper atmosphere or wherever one expects extremely small static stability. The meridional structure of the tidal response is shown to depend only on the ratio of the period of gravitational forcing to the period of rotation of the planet. Forcing by the low-inclination orbits of the satellites of Jupiter, Saturn, and Uranus excites primarily symmetric Hough modes. Consideration of the vertical structure equation shows that although the gravitational tidal forcing is proportional to the first symmetric spherical harmonic with zonal wavenumber 2, the tidal response will be concentrated in higher order meridional structures confined equatorward of 50° N on Jupiter, 76° N on Saturn, and 45° N on Uranus. The meridional structure of these modes resembles the visible banding on these planets. The excitation of the tides depends on the distribution of static stability in the interior. Estimates are made showing that observation of the tidal response of the planets at the visible cloud level may be within reach of current observational capability. Detection of this signal is shown to provide information about the thermodynamic structure of the interior.

A primary purpose of the present paper, in addition to the above, is the presentation of computational results concerning the eigenvalues and eigenfunctions relevant to gravitational tides in the outer planets.

Subject headings: planets and satellites: general

1. INTRODUCTION

It has been suggested that observations of oscillations in the gaseous planets can provide additional constraints on their interior structure and advance understanding of their dynamic behavior. For instance, Vorontsov (1984a, b; Vorontsov & Zharkov 1981; Vorontsov, Gudkova, & Zharkov 1989; Vorontsov, Zharkov, & Lubimov 1976) investigated the free oscillations of Jupiter and Saturn and stressed their use for determining the interior structure of these planets. In another context Flasar & Gierasch (1986) detected mesoscale gravity wave activity in *Voyager* images and argued that observations (e.g., by a descending probe) of the wind shear below the visible clouds could provide information on the distribution of static stability as implied by Lindzen & Tung's (1976) theory of banded convection on Earth. Also, Bercovici & Schubert (1987) suggested that observation of acoustic oscillations, trapped below Jupiter's tropopause, could provide information on Jupiter's thermal structure and the structure of winds.

The first observations on the oscillations of Jupiter were reported by Deming et al. (1989). Using a 20 element linear array in the 8–13 μm band, they analyzed brightness fluctuations of the Jovian disk at approximately the 0.5 bar level. They investigated only the power spectra of high zonal wavenumbers excluding zonal wavenumber 2 which corresponds to the lunar forcing (D. Deming, 1992, private communication). Although Deming et al. (1989) could resolve temperature fluctuations of amplitude as low as 0.07 K, they failed to detect a signal at the expected acoustic frequencies; instead they detected oscillations at the frequency which corresponds to the rotation of the planet. Subsequently Schmider, Mosser, & Fossat (1991), using Doppler shift measurements of the solar sodium D lines reflected by Jupiter, presented preliminary evidence of coherent velocity oscillations of 8 ± 4 m s^{-1} with periods between 10 and 20 minutes, at approximately the 3 bar level which were interpreted by Mosser et al. (1991) as acoustic oscillations. The acoustic modes are thought to be excited by turbulent convection (Goldreich & Kumar 1988). The details of the excitation are uncertain, and calculations of the amplitude of the acoustic oscillations on the outer planets differ (cf. Deming et al. 1989; Marley 1991). They are expected to be significantly weaker than their solar counterparts because the heat output from the Sun is six orders of magnitudes greater than the 14 W m^{-2} that are radiated away from Jupiter, and consequently the observer of the outer planets has to face the problem of faint oscillation signals.

Observations are most likely to succeeed in detecting a signal if directed toward well-defined sources of excitation in the outer planets. It is fortuitous that the outer planets do possess such a source: the gravitational tidal potential imposed by their revolving satellites. For example, in the case of Jupiter the gravitational tidal potential is dominated by Io, and this potential is 116 times larger than the lunar tidal potential on Earth (Lindzen 1991; Lindzen & Ioannou 1991). The relative disadvantage of the less energetic excitation of the Jovian acoustic modes compared to solar acoustic modes may be offset by the advantage gained through the coherent tidal excitation.

It is the purpose of this paper to specify the gravitational tidal forcing of the outer planets and investigate the characteristics of the tidal response in the visible atmospheric envelope of these planets. Subsequent papers will use these results to deal with interior tides, tidal dissipation, and tidal acceleration of the mean flow.

Section 2 of this paper reviews classical tidal theory and its extension to the outer planets. Section 3 discusses the tidal geopotential due to satellites of the outer planets. Section 4 discusses the solution of Laplace's tidal equation for Hough modes appropriate to tides on the outer planets and provides tabulations of the eigenvalues and eigenfunctions necessary for subsequent studies. Section 5 describes the vertical structure of tidal modes, noting that only relatively high-order meridional mode numbers can propagate vertically with growing amplitude. Section 6 discusses expected magnitudes for tides in the visible atmosphere of Jupiter, and § 7 provides some additional discussion.

2. FORMULATION OF THE THEORY OF GRAVITATIONAL TIDES AS IT APPLIES TO THE ATMOSPHERES OF THE OUTER PLANETS

Our treatment and notation follows that presented in Chapman & Lindzen (1970), but extended to include nonhydrostatic effects. Consider the response of a planetary atmosphere to tidal forcing with gravitational potential Ω. The planet is assumed to be a sphere of radius, a, rotating with angular velocity, ω, and with surface gravitational acceleration, g. The tidal response of the atmosphere will be assumed to be a perturbation on a basic atmospheric state of the planet prescribed by the distribution of the mean pressure p_0 and mean density ρ_0 which are assumed to be in hydrostatic balance and to satisfy the perfect gas law. Denote with θ the colatitude measured from the axis of rotation, ϕ the azimuthal angle, and z the radial distance measuring the radial location in the atmospheric envelope of the planet, and assumed small compared to the planetary radius, a (see Fig. 1). The tidal velocity fields—u, northerly meridional velocity, v, westerly zonal velocity, and w, radial velocity—tidal pressure, p, and density, ρ, satisfy the following inviscid equations in a frame rotating with the planet:

$$\frac{\partial u}{\partial t} - 2\omega \cos\theta v = -\frac{1}{a}\frac{\partial P}{\partial \theta}, \tag{2.1}$$

$$\frac{\partial v}{\partial t} + 2\omega \cos\theta u = -\frac{1}{a \sin\theta}\frac{\partial P}{\partial \phi}, \tag{2.2}$$

$$\rho_0 \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} - \rho_0 \frac{\partial \Omega}{\partial z} - g\rho, \tag{2.3}$$

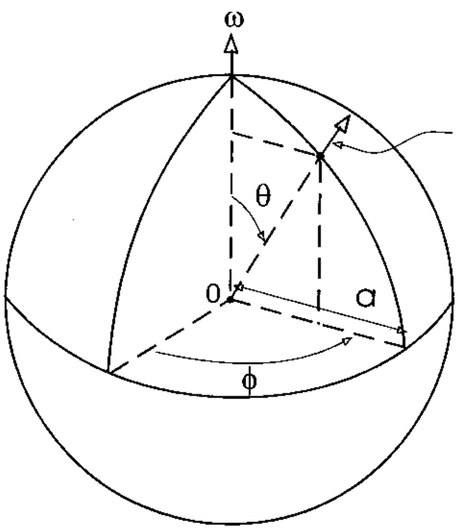

FIG. 1.—Coordinates used in tidal theory

with

$$P = \frac{p}{\rho_0} + \Omega, \tag{2.4}$$

the reduced pressure. The continuity equation is

$$\frac{\partial \rho}{\partial t} + w\frac{d\rho_0}{dz} + \rho_0 \chi = 0, \tag{2.5}$$

where χ is the velocity divergence:

$$\chi = \frac{\partial w}{\partial z} + \frac{1}{a \sin\theta}\frac{\partial v}{\partial \phi} + \frac{1}{a \sin\theta}\frac{\partial(u \sin\theta)}{\partial \theta}. \tag{2.6}$$

The linearized thermodynamic equation for adiabatic motion of a perfect gas is

$$\frac{\partial p}{\partial t} = c^2 \frac{\partial \rho}{\partial t} - \rho_0 \frac{c^2}{g} N^2 w, \tag{2.7}$$

with c, the speed of sound, given by

$$c^2 = \gamma \frac{p_0}{\rho_0}, \tag{2.8}$$

where γ is the ratio of specific heat at constant pressure to that at constant volume, and N^2, the Brunt-Väisälä frequency, is defined by

$$N^2 = \frac{g}{H}\left(\kappa + \frac{dH}{dz}\right), \tag{2.9}$$

where $\kappa = (\gamma - 1)/\gamma$, $H = RT_0/g$ is the scale height, R is the gas constant per unit mass, and T_0 the mean temperature. Finally, the tidally induced temperature field is determined from the linearized perfect gas law:

$$\frac{T}{T_0} = \frac{p}{p_0} - \frac{\rho}{\rho_0}. \tag{2.10}$$

Note we have retained in the momentum equations only the coriolis deflection due to the radial component of rotation which is an appropriate approximation for large-scale motions in stable thin atmospheric spherical shells (Eckart 1960; Phillips 1966). We have also neglected the mean motion of the background state because it is expected to be negligible compared to the phase speed of the migrating tidal forcing (Lindzen 1991). Under these simplifications the inviscid and adiabatic equations prove to be separable.

Assume that the tidal fields on the planet have reached a periodic state under the periodic tidal forcing. The dependence of the perturbed quantities on time and azimuthal angle will be then the same as that of the forcing and of the form $e^{i\sigma t + is\phi}$. We continue to denote the perturbed quantities with the same symbols as previously, it being understood that the quantities refer to the (σ, s) mode. The complete fields are obtained by multiplication by this exponential term. Physically relevant quantities correspond, as usual, to the real part of the resulting fields. We proceed as in Chapman & Lindzen (1970), solving equations (2.1)–(2.2) for the (u, v) component of velocity to obtain

$$u = \frac{i\sigma}{4a\omega^2(f^2 - \cos^2\theta)}\left(\frac{\partial}{\partial \theta} + \frac{s \cot\theta}{f}\right)P, \tag{2.11}$$

$$v = -\frac{\sigma}{4a\omega^2(f^2 - \cos^2\theta)}\left(\frac{\cos\theta}{f}\frac{\partial}{\partial \theta} + \frac{s}{\sin\theta}\right)P, \tag{2.12}$$

with $f = \sigma/(2\omega)$, which is one-half the ratio of the rotation period of the planet, $2\pi/\omega$, to the period of the externally imposed forcing, $2\pi/\sigma$.

The divergence, χ, is expressed by

$$\chi = \frac{\partial w}{\partial z} + \frac{i\sigma}{4\omega^2 a^2} F(P) , \tag{2.13}$$

where F is the Laplace tidal operator,

$$F \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{f^2 - \cos^2\theta}\frac{\partial}{\partial\theta}\right) - \frac{1}{f^2 - \cos^2\theta}\left(\frac{s}{f}\frac{f^2 + \cos^2\theta}{f^2 - \cos^2\theta} + \frac{s^2}{\sin^2\theta}\right). \tag{2.14}$$

The continuity equation (2.13) is separable. If the separation constant is denoted by $\epsilon_n^{\sigma,s}$, the meridional structure is determined by the eigenproblem:

$$F(\Theta_n^{\sigma,s}) = -\frac{1}{\epsilon_n^{\sigma,s}}\Theta_n^{\sigma,s} . \tag{2.15}$$

Boundedness of the solutions of equation (2.15) at the poles determines the eigenvalues $\epsilon_n^{\sigma,s}$ and the eigenfunctions (Hough functions) $\Theta_n^{\sigma,s}(\theta)$ (Hough 1897, 1898). The Hough functions, $\Theta_n^{\sigma,s}(\theta)$, form a complete set in terms of which the tidal fields can be expanded. It is convenient to define the associated functions

$$U_n^{\sigma,s}(\theta) = \frac{1}{f^2 - \cos^2\theta}\left(\frac{\partial}{\partial\theta} + \frac{s\cot\theta}{f}\right)\Theta_n^{\sigma,s} \tag{2.16}$$

for the expansion of the meridional velocity, and

$$V_n^{\sigma,s}(\theta) = \frac{1}{f^2 - \cos^2\theta}\left(\frac{\cos\theta}{f}\frac{\partial}{\partial\theta} + \frac{s}{\sin\theta}\right)\Theta_n^{\sigma,s} \tag{2.17}$$

for the expansion of the zonal velocity. It is, also, customary to express the separation constant in terms of the equivalent depth $h_n^{\sigma,s}$ which is defined as

$$\epsilon_n^{\sigma,s} \equiv \frac{g h_n^{\sigma,s}}{4a^2\omega^2} . \tag{2.18}$$

It is also practical to use as an independent variable the divergence, χ, which can be expanded in Hough modes as

$$\chi = \sum_1^\infty \chi_n(z)\Theta_n(\theta)e^{i\sigma t + is\phi} , \tag{2.19a}$$

the superscripts σ and s being understood. Similarly, we expand the lunar forcing

$$\Omega = \sum_1^\infty \Omega_n(z)\Theta_n(\theta)e^{i\sigma t + is\phi} . \tag{2.19b}$$

The equation for the vertical structure of the nth Hough mode is given by

$$\frac{d^2\chi_n}{dx^2} - \frac{d\chi_n}{dx} + \left[\frac{N^2H^2}{gh_n} + \frac{\sigma^2 H}{\gamma g}\left(1 - \frac{\gamma H}{h_n}\right)\right]\chi_n = \frac{i\sigma H\Omega_n}{\gamma g}\left(\frac{2}{a^2} - \frac{\sigma^2}{gh_n}\right), \tag{2.20}$$

with the vertical coordinate z replaced by the log-pressure coordinate x (Holton 1992):

$$x \equiv \int_0^z \frac{dz}{H} . \tag{2.21}$$

In the atmospheric envelope the forcing terms on the right-hand side of (2.20) can be neglected (cf. Chapman & Lindzen 1970). In terms of

$$y_n = \chi_n e^{-x/2} , \tag{2.22}$$

the vertical structure equation (2.20) assumes the canonical form

$$\frac{d^2 y_n}{dx^2} + \left[\frac{N^2H^2}{gh_n} + \frac{\sigma^2 H}{\gamma g}\left(1 - \frac{\gamma H}{h_n}\right) - \frac{1}{4}\right]y_n = 0 . \tag{2.23}$$

The σ^2 term arises from inclusion of the acceleration term in the radial momentum equation. It can be shown that the radial acceleration term can be neglected when $N^2 \gg \sigma^2$ (Lindzen 1990), as is the case in the visible atmospheric envelopes of the planets, in which case equation (2.23) reduces to the familiar classical hydrostatic tidal equation (Chapman & Lindzen 1970)

$$\frac{d^2 y_n}{dx^2} + \left(\frac{N^2H^2}{gh_n} - \frac{1}{4}\right)y_n = 0 . \tag{2.24}$$

Below the visible part of the atmospheres of the outer planets the distribution of the static stability is unknown. It is usually assumed that the static stability is sharply reduced in this region, although locally stable regions are sometimes assumed to exist (Conrath & Gierasch 1984; Geirasch & Conrath 1987; Achterberg & Ingersoll 1990). In either case below the visible clouds the effects of nonhydrostaticity have to be taken into account, and we have to revert to the nonhydrostatic vertical structure equation (2.23). Here we will be concerned with the response of the visible part of the atmospheric envelope for which the hydrostatic equation (2.24) is adequate (it should be noted that N^2 can be larger than σ^2 even when N^2 is observationally indistinguishable from zero (Lindzen 1990)). Assuming hydrostaticity the tidal fields are given by

$$w_n = \frac{i\sigma}{g}\Omega_n + \gamma h_n e^{x/2}\left[\frac{dy_n}{dx} + \left(\frac{H}{h_n} - \frac{1}{2}\right)y_n\right], \tag{2.25}$$

$$P_n = -\frac{i\gamma g h_n}{\sigma} e^{x/2}\left(\frac{dy_n}{dx} - \frac{y_n}{2}\right), \tag{2.26}$$

$$v_n = iu_n = \frac{i\gamma g h_n}{4a\omega^2} e^{x/2}\left(\frac{dy_n}{dx} - \frac{y_n}{2}\right), \tag{2.27}$$

$$\frac{T_n}{T_0} = \frac{\Omega_n}{gH^2}\frac{dH}{dx} + \frac{i\gamma h_n}{\sigma H}e^{x/2} \times \left[\frac{\kappa H}{h_n}y_n + \frac{1}{H}\frac{dH}{dx}\left(\frac{d}{dx} + \frac{H}{h_n} - \frac{1}{2}\right)y_n\right]. \tag{2.28}$$

In terrestrial tidal theory the lunar forcing is introduced as a part of the boundary condition at the solid surface (Chapman & Lindzen 1970). The outer planets are gaseous spheres with no solid surface, and the lunar forcing is distributed over the interior of the planet. The tidal fields in the atmospheric envelope can be determined after the response of the interior of the planet has been solved (Ioannou & Lindzen 1993). For the moment we will assume that the interior is a source of tidal

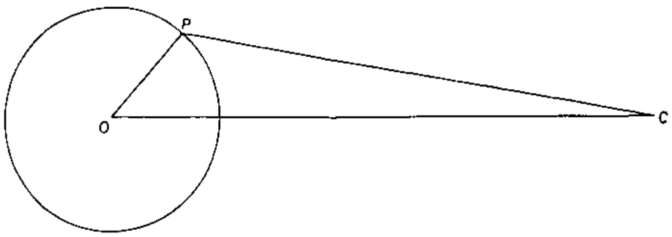

FIG. 2.—Geometry for calculation of the tidal potential; O is the center of the planet, C the center of the disturbing body. $D = OC$; $a = OP$; $\Theta = P\hat{O}C$.

activity, and we will limit our investigation to the effect of the atmospheric envelope on the excited tides.

As an upper boundary condition it is sufficient to assume that y_n is bounded as $x \to \infty$. However, when the upper layers of the atmosphere are isothermal, equation (2.24) may possess bounded wave solutions of the form $e^{\pm i\lambda x}$, with

$$\lambda = \left(\frac{\kappa H}{h_n} - \frac{1}{4}\right)^{1/2} . \quad (2.29)$$

The solution $e^{i\lambda x}$ is associated with an upgoing wave, while $e^{-i\lambda x}$ with a downgoing wave. If we impose a radiation condition, the latter must be rejected, and the solution is uniquely determined.

3. LUNAR FORCING

The gravitational tidal forcing on a planet of radius, a, due to a satellite of mass M_s at a distance D from the center of the planet is given by

$$\Omega = -\frac{3}{2}\frac{GM_s a^2}{D^3}\left(\frac{1}{3} - \cos^2 \Theta\right) , \quad (3.1)$$

where G is the universal gravitational constant and $\Theta = P\hat{O}C$, shown in Figure 2 (Lamb 1932; Chapman & Lindzen 1970).

Although the outer planets have many satellites, it is fortuitous that the tidal potential is dominated on each of the planets by the contribution of a single satellite (Lindzen 1991; Lindzen & Ioannou 1991). In Table 1 we tabulate general parameters pertaining to the outer planets, with Earth listed to provide a comparison, and in Table 2 a list of the major satellites of the outer planets with the corresponding tidal potential at the surface of the planet. Note that the lunar potential of Io on Jupiter is two orders of magnitude larger than that of the Moon on Earth, and the tidal potential due to Europa and Ganymede is significant.

To determine the meridional structure of the Hough modes we must specify the frequency of the forcing, σ, and the zonal wavenumber, s. The principal tidal component for low-inclination orbits such as those of Io, Titan, and Ariel is the semidiurnal tide with frequency $\sigma = 2\omega_L$ where, ω_L, is the relative rate of rotation of the satellite and zonal wavenumber $s = 2$. The relative rate of rotation of the satellite, as seen from an observer rotating with the planet, is determined by

$$\omega_L \pm \omega_M = \omega + \omega_Y , \quad (3.2)$$

where ω_M is the rotation rate of the satellite around the planet (the minus sign is for a retrograde orbit as in the case of Neptune's Triton) and ω_Y the rotation rate of the planet around the Sun, which will be disregarded as small. The resulting f, which determines the Hough functions, for the outer planets is given in Table 3.

Specifically, Io revolves around Jupiter in an orbit of maximum inclination of 0°04. The dominant component of tidal forcing on the surface of Jupiter is symmetric with respect to the equator with zonal wavenumber $s = 2$, and $f = 0.766$. The meridional dependence of the tidal potential according to equation (3.1) takes the form $\Omega = -314.336\, P_{2,2}(\theta)\, m^2\, s^{-2}$, where θ is the colatitude and $P_{2,2}$ the normalized associated Legendre polynmial, which is related to the associated Legendre polynomials by $P_{n,s}/P_n^s = (2n + 1)^{1/2}/2$, $s > 0$. For Titan, with a maximum inclination of 0°33, the tidal potential ($s = 2$ and $f = 0.9721$) on the surface of Saturn is $\Omega = -13.8857\, P_{2,2}(\theta)\, m^2\, s^{-2}$. For Ariel, orbiting around Uranus with maximum inclination of 0°0034, $s = 2$, and $f = 0.71495$, the tidal potential is $\Omega = -7.30499\, P_{2,2}(\theta)\, m^2\, s^{-2}$. For comparison, the primary lunar forcing on Earth (the maximum inclination of the orbit of the moon is 20°) is semidiurnal ($s = 2$ and $f = 0.98277$) and $\Omega = -2.55216\, P_{2,2}(\theta)\, m^2\, s^{-2}$.

4. THE MERIDIONAL STRUCTURE OF THE TIDAL RESPONSE

To determine the meridional structure of the tidal response we must find the eigenvalues $\epsilon_n^{\sigma,s}$ and the eigenfunctions, $\Theta_n^{\sigma,s}$, of the Laplace tidal equation (2.15). Note that the eigenfunctions and eigenvalues depend only on the value of the zonal wavenumber s and the parameter $f(=\sigma/[2\omega])$ which are determined by the forcing. The meridional structure is independent of the absolute rotation rate, mass, radius, and thermodynamic state of the planet. These properties are important for the specification of the vertical structure.

In the case of a nonrotating planet ($\omega = 0$) the eigenfunctions of equation (2.15) are the spherical harmonics, $P_n^s(\theta)e^{is\phi}$, with $P_n^s(\theta)$ the associated Legendre polynomials

TABLE 1

PROPERTIES OF THE JOVIAN PLANETS

Parameter	Jupiter	Saturn	Uranus	Neptune	Earth
Equatorial radius (10^3 km)	71.4	60.3	25.6	24.8	6.371
Rotation period (hr)	9.92	10.66	17.24	16.11	24
Equatorial gravity (ms^{-2})	22.9	9.1	8.8	11.1	9.8
Emission temperature (K)	124	95	59	59	255
Emission pressure (bar)	0.4	0.3	0.4	0.5	0.4
Scale height (km)	20	39	25	20	7.8
Sound speed	810	705	560	560	410
Linear rotation speed (m s^{-1})	12562	9873	2592	2687	463

NOTES.—After Ingersoll 1990. For reference purposes the properties of Earth are included. Also the linear speed of rotation at the equator (rotation frequency × radius) has been included.

TABLE 2

RELEVANT PROPERTIES OF EARTH SATELLITES AND JOVIAN PLANETS

Planet	Satellite	$\frac{D}{a}$	D (10^3 km)	M_S (10^{20} kg)	Potential ($m^2\ s^{-2}$)	Ratio to Earth
Earth	Moon	60.3	384.4	734.9	3.51	1
Jupiter	Io	5.905	421.6	894	405.81	116
	Europa	9.396	670.9	480	54.08	15.4
	Ganymede	14.99	1070	1482	41.13	11.17
	Callisto	26.37	1883	1076.6	5.48	1.56
Saturn	Rhea	8.736	527.04	24.9	4.13	1.18
	Titan	20.25	1221.85	1346	17.93	5.11
	Iapetus	59.03	3561.3	18.8	0.01	0.003
Uranus	Miranda	4.95	129.8	0.71	1.49	0.425
	Ariel	7.3	191.2	14.4	9.50	2.69
	Umbriel	10.15	266	11.8	2.87	0.82
	Titania	16.64	435.8	34.3	1.90	0.54
	Oberon	22.4	582.6	28.7	0.66	0.19
Neptune	Triton	14	354.3	1300	124.92	35.6

NOTES.—After Lindzen 1991. Note that for Jupiter the tidal potential of Io dominates, for Saturn that of Titan, and for Uranus that of Ariel.

($n - s = 0, 1, 2, \ldots$). The corresponding eigenvalues are independent of the zonal wavenumber, s, and are given by

$$\epsilon_n^{\sigma,s} = \frac{f^2}{n(n+1)} . \quad (4.1)$$

Note, also, that the meridional structure is independent of the forcing frequency σ. Hough (1898) obtained equation (4.1) and its modification for slow rotation. The inclusion of rapid rotation drastically modifies equation (4.1), and the eigenfunctions deviate from the associated Legendre polynomials, making the perturbation techniques employed in the astrophysical literature for the analysis of nonradial oscillations of slowly rotating stellar bodies inadequate (*cf.* Unno et al. 1979). To determine the eigenfunctions of the Laplace tidal equation (2.15) in the presence of rotation, Hough (1897, 1898) expanded the eigenfunctions in normalized associated Legendre polynomials:

$$\Theta_n^{\sigma,s} = \sum_{m=s}^{\infty} C_{n,m}(\sigma) P_{m,s} , \quad (4.2)$$

reducing equation (2.15) to a third-order linear recursion relation in the coefficients of the expansion, C_m (Chapman & Lindzen 1970). The coefficients are determined by solving the infinite set of linear equations in C_m. Solutions of this linear set of equations exist only when the determinant of the coefficients of C_m is zero. This condition determines the discrete eigenvalues $\epsilon_n^{\sigma,s}$, which in turn determine the expansion of the eigenfunction in terms of associated Legendre polynomials. In practice the infinite determinant is truncated, and convergence is assumed when the eigenvalue does not change by more than a desired accuracy when higher order truncations are considered; once the eigenvalue $\epsilon_n^{\sigma,s}$ has been determined, C_m can be calculated. In order to achieve adequate accuracy for the eigenvalues $\epsilon_n^{\sigma,s}$ for the case of gravitational forcing of the outer planets we had to proceed to orders of truncation as high as 100 terms. Consideration of these high-order truncations led us to the realization that the representation (4.2) is an asymptotic expansion for the Hough functions which should be truncated to N terms, where N is the optimal truncation. However accurately the eigenvalues are determined by truncation of the infinite determinant, the coefficients C_m will eventually diverge.

TABLE 3

VALUE OF NONDIMENSIONAL PARAMETER f

Planet	Satellite	f	Critical Latitude
Jupiter	Io	0.766	49°996
	Europa	0.8836	62.08
	Ganymede	0.9422	70.4
Saturn	Titan	0.9721	76.4
	Rhea	0.9017	64.4
Uranus	Ariel	0.71495	45.64
Neptune	Triton	1.12	...

NOTES.—Value of the nondimensional parameter $f(=\sigma/2\omega)$ and the critical latitude for some of the satellites of the outer planets.

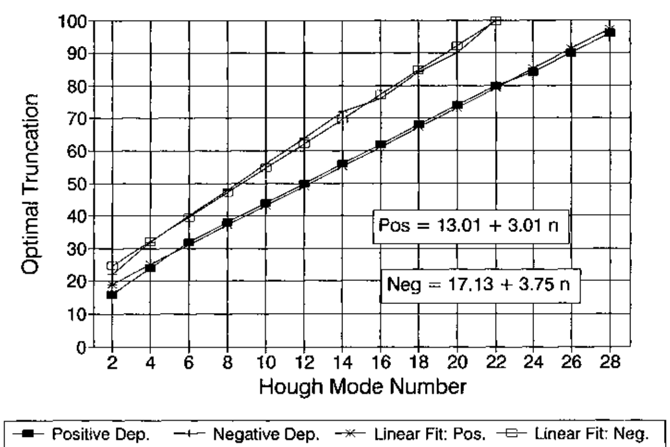

FIG. 3.—The optimal truncation, N, for the asymptotic representation (4.2) of the symmetric Hough modes in terms of normalized associated Legendre polynomials. The top curve is for the Hough modes with negative equivalent depth. The bottom is for the Hough modes with positive equivalent depth. The lines are the result of a linear fit and show that N is a linear function of the order n of the Hough modes.

The optimal truncation N depends on the degree of accuracy of the eigenvalue and the order of the Hough mode, n. In Figure 3 we plot the optimal truncation N for various eigenvalues for the lunar forcing of Io on Jupiter. It is worth noting that Kelvin (1875) anticipated this behavior while discussing the objections of Airy to the representation of the solutions of the Laplace tidal equation in the form of equation (4.2).

The eigenvalues of the Laplace tidal equation (2.15) with their associated equivalent heights are listed in Table 4 for the semidiurnal forcing of Jupiter by Io, of Saturn by Titan in Table 5, and of Uranus by Ariel in Table 6. The eigenvalues $\epsilon_n^{\sigma,s}$ for Saturn are very close to those for semidiurnal forcing of Earth. Also, the eigenvalues for Jupiter ($f = 0.766$) and Uranus ($f = 0.715$) are close. The difference in the value of the equivalent heights in the outer planets is due to the large value of $4\omega^2 a^2/g$, which is approximately 25,250, 42,845 and 3050 km, respectively, for Jupiter, Saturn, and Uranus (for Earth this factor is approximately 81 km). In Table 7 we list the expansion coefficients relating the relevant Hough functions and normalized associated Legendre polynomials for the tidal forcing of Jupiter by Io. Similarly, Table 8 lists the respective expansion coefficients for tidal forcing of Saturn by Titan. Note the existence of negative eigenvalues leading to negative equivalent depths. In general, negative eigenvalues occur when $f < 1$; that is, when a critical latitude exists for which the period of the forcing is equal to the local inertial period. Poleward of the critical latitude the Coriolis forces dominate inertia, while equatorward of this latitude inertia dominates the Coriolis forces. The Hough functions with negative equivalent depths are concentrated between the critical latitude and the pole, oscillating poleward of the critical latitude, while those with positive equivalent depth are concentrated at low latitudes, oscillating between the critical latitudes. This behavior is apparent in Figures 4, 5, and 6 where the structure of some of the Hough modes and their associated U, V functions (cf. eqs. [2.16] and [2.17]) are shown as a function of latitude for the case of Jupiter.

Rotation, by leading to a deviation of Hough functions from spherical harmonics, also leads to the projection of gravitational tidal forcing on a variety of Hough modes. This is accentuated when critical latitudes exist. This is especially important on Jupiter and Uranus, where the existence of critical latitudes at 50° and 45°, for forcing by Io and Ariel, respectively, leads to projection of the tidal potential on a multitude of Hough modes. For example, for the case of the tide on Jupiter forced by Io we derive the following expansion for the tidal potential at the surface of the planet in units of $m^2\ s^{-2}$:

$$\begin{aligned}\Omega = & -275.6\Theta_2 - 77.17\Theta_4 - 39.31\Theta_6 - 24.72\Theta_8 \\ & - 17.35\Theta_{10} - 13.03\Theta_{12} - 10.24\Theta_{14} - 8.33\Theta_{16} \\ & - 6.94\Theta_{18} - 5.9\Theta_{20} - 5.1\Theta_{22} - 4.46\Theta_{24} - 3.95\Theta_{26} \\ & - 3.53\Theta_{28} - \cdots - 104.92\Theta_{-2} - 41.51\Theta_{-4} \\ & - 23.13\Theta_{-6} - 15.15\Theta_{-8} - 10.88\Theta_{-10} - 8.29\Theta_{-12} \\ & - 6.59\Theta_{-14} - 5.4\Theta_{-16} - 4.53\Theta_{-18} - 3.87\Theta_{-20} - \cdots . \end{aligned} \tag{4.3}$$

Note the absence of strong dominance of a single mode. This gives the opportunity for selective amplification of various

TABLE 4

SYMMETRIC EIGENVALUES AND EQUIVALENT DEPTHS FOR JUPITER

Hough Mode	ϵ_n	h_n (km)	Hough Mode	ϵ_n	h_n (km)
2	3.758487E − 02	948.99	−2	−1.080836E − 02	−272.48
4	8.098624E − 03	203.48	−4	−2.922360E − 03	−73.77
6	3.365069E − 03	84.96	−6	−1.320263E − 03	−23.32
8	1.824786E − 03	46.07	−8	−7.467236E − 04	−18.85
10	1.141528E − 03	28.52	−10	−4.792306E − 04	−12.10
12	7.805261E − 04	19.71	−12	−3.333514E − 04	−8.41
14	5.670424E − 04	14.32	−14	−2.451456E − 06	−6.19
16	4.304666E − 04	10.87	−16	−1.878714E − 04	−4.74
18	3.378652E − 04	8.53	−18	−1.484120E − 04	−3.74
20	2.722101E − 04	6.87	−20	−1.203718E − 04	−3.03
22	2.239800E − 04	5.65	−22	−9.939643E − 05	−2.51
24	1.875149E − 04	4.73	−24	−8.353678E − 05	−2.11
26	1.592784E − 04	4.02	−26	−7.118338E − 05	−1.80
28	1.369693E − 04	3.45	−28	−6.138224E − 05	−1.55
30	1.190381E − 04	3.01	−30	−5.348961E − 05	−1.35
32	1.044106E − 04	2.64	−32	−4.697422E − 05	−1.19
34	9.232199E − 05	2.33	−34	−4.122795E − 05	−1.04
36	8.221701E − 05	2.08	−36	−3.520887E − 05	−0.89
38	7.368374E − 05	1.86	−38	−2.840234E − 05	−0.72
40	6.639003E − 05	1.68	−40	−2.092765E − 05	−0.53
42	5.987355E − 05	1.51	−42	−1.291258E − 05	−0.33
44	5.330689E − 05	1.35	−44	−4.534090E − 06	−0.11
46	4.611540E − 05	1.16			
48	3.830297E − 05	0.96			
50	3.004025E − 05	0.76			
52	2.148877E − 05	0.54			
54	1.279005E − 05	0.32			
56	4.074059E − 06	0.10			

NOTES.—The first symmetric eigenvalues ϵ_n of Laplace tidal equation with $s = 2, f = 0.766$, and the corresponding equivalent depths h_n for Jupiter. On the left we list the eigenvalues with positive equivalent depth, and on the right the negative equivalent depths.

TABLE 5

Symmetric Eigenvalues and Equivalent Depths for Saturn

Hough Mode	ϵ_n	h_n (km)	Hough Mode	ϵ_n	h_n (km)
2.........	8.246436E − 02	3533.13	−2.........	−1.8937E − 04	−8.11
4.........	2.171901E − 02	930.54	−4.........	−4.781813E − 05	−2.05
6.........	9.736592E − 03	417.16	−6.........	−2.129341E − 05	−0.91
8.........	5.479009E − 03	234.75	−8.........	−1.199814E − 05	−0.51
10.........	3.500136E − 03	149.96	−10.........	−7.660111E − 06	−0.33
12.........	2.424672E − 03	103.88	−12.........	−4.933109E − 06	−0.21
14.........	1.776840E − 03	76.13	−14.........	−1.815862E − 06	−0.08
16.........	1.357047E − 03	58.14			
18.........	1.069761E − 03	45.83			
20.........	8.646747E − 04	37.05			
22.........	7.132173E − 04	30.56			
24.........	5.982345E − 04	25.63			
26.........	5.089092E − 04	21.80			
28.........	4.381512E − 04	18.77			
30.........	3.811580E − 04	16.33			
32.........	3.345832E − 04	14.34			
34.........	2.960372E − 04	12.68			
36.........	2.637781E − 04	11.30			
38.........	2.365113E − 04	9.70			
40.........	2.132581E − 04	9.13			
42.........	1.932692E − 04	8.28			
44.........	1.759614E − 04	7.53			
46.........	1.608765E − 04	6.89			
48.........	1.476494E − 04	6.33			
50.........	1.359881E − 04	5.83			
52.........	1.256541E − 04	5.38			
54.........	1.164539E − 04	4.98			
56.........	1.082217E − 04	4.64			

Notes.—The first symmetric eigenvalues ϵ_n of Lapalce tidal equation with $s = 2, f = 0.9721$, and the corresponding equivalent depths h_n for Saturn. On the left we list the eigenvalues with positive equivalent depth and on the right the negative equivalent depths.

TABLE 6

Symmetric Eigenvalues and Equivalent Depths for Uranus

Hough Mode	ϵ_n	h_n (km)	Hough Mode	ϵ_n	h_n (km)
2.........	2.934927E − 02	89.61	−2.........	−1.522564E − 02	−46.48
4.........	6.049680E − 03	18.47	−4.........	−4.175062E − 03	−12.75
6.........	2.492817E − 03	7.61	−6.........	−1.893966E − 03	−5.78
8.........	1.348076E − 03	4.11	−8.........	−1.072236E − 03	−3.27
10.........	8.422856E − 04	2.57	−10.........	−6.882067E − 04	−2.11
12.........	5.755384E − 04	1.75	−12.........	−4.788286E − 04	−1.47
14.........	4.179604E − 04	1.27	−14.........	−3.522671E − 04	−1.08
16.........	3.172138E − 04	0.97	−16.........	−2.698075E − 04	−0.82
18.........	2.489334E − 04	0.76	−18.........	−2.133274E − 04	−0.65
20.........	2.005358E − 04	0.61	−20.........	−1.729891E − 04	−0.53
22.........	.6149908E − 04	0.50	−22.........	−1.429447E − 04	−0.44
24.........	1.381201E − 04	0.42	−24.........	−1.201385E − 04	−0.37
26.........	1.173157E − 04	0.35	−26.........	−1.024834E − 04	−0.31
28.........	1.008800E − 04	0.31	−28.........	−8.824701E − 05	−0.27
30.........	8.767105E − 05	0.27	−30.........	−7.687176E − 05	−0.23
32.........	7.689566E − 05	0.23	−32.........	−6.757466E − 05	−0.21
34.........	6.799098E − 05	0.21	−34.........	−5.986866E − 05	−0.18
36.........	6.054052E − 05	0.18	−36.........	−5.332619E − 05	−0.16
38.........	5.411733E − 05	0.16	−38.........	−4.725262E − 05	−0.14
40.........	4.792647E − 05	0.15	−40.........	−4.075966E − 05	−0.12
42.........	4.116779E − 05	0.13	−42.........	−3.350233E − 05	−0.10
44.........	3.370184E − 05	0.10	−44.........	−2.565042E − 05	−0.08
46.........	2.568625E − 05	0.08	−46.........	−1.737237E − 05	−0.05
48.........	1.728836E − 05	0.05	−48.........	−8.802626E − 06	−0.03
50.........	8.653784E − 06	0.03	−50.........	−8.509288E − 08	−0.0003

Notes.—The first symmetric eigenvalues ϵ_n of Laplace tidal equation with $s = 2, f = 0.71495$, and the corresponding equivalent depths h_n for Uranus. On the left we list the eigenvalues with positive equivalent depth, and on the right the negative equivalent depths.

TABLE 7

EXPANSION COEFFICIENTS

NORMALIZED LEGENDRE FUNCTION EXPANSIONS IN TERMS OF THE ASSOCIATED HOUGH FUNCTIONS FOR THE PLANET JUPITER AND THE MOON IO

$f = 0.765999972$ Azimuthal number $s = 2$

Epsilon	3.7585E-02	8.0986E-03	3.3651E-03	1.8248E-03	1.1415E-03	7.8053E-04	5.6704E-04	4.3047E-04	3.3787E-04	2.7221E-04	2.2398E-04	1.8752E-04	1.5928E-04
	H(2,2)	H(2,4)	H(2,6)	H(2,8)	H(2,10)	H(2,12)	H(2,14)	H(2,16)	H(2,18)	H(2,20)	H(2,22)	H(2,24)	H(2,26)
P(2,2)	0.876772	0.245504	0.125055	0.078630	0.055187	0.041448	0.032593	0.026496	0.022088	0.018779	0.016222	0.014196	0.012559
P(2,4)	-0.468962	0.280715	0.176684	0.117985	0.084955	0.064657	0.051240	0.041860	0.035012	0.029838	0.025819	0.022624	0.020036
P(2,6)	0.105746	-0.724141	-0.285559	-0.157067	-0.102640	-0.073960	-0.056676	-0.045294	-0.037314	-0.031456	-0.027000	-0.023514	-0.020725
P(2,8)	-0.012726	0.536310	-0.218474	-0.213512	-0.169908	-0.134819	-0.109170	-0.090323	-0.076162	-0.065266	-0.056697	-0.049826	-0.044224
P(2,10)	0.000949	-0.214015	0.634406	0.305691	0.165305	0.101490	0.068561	0.049678	0.037917	0.030100	0.024632	0.020644	0.017636
P(2,12)	-0.000048	0.054911	-0.565097	0.136607	0.213548	0.195168	0.166062	0.140323	0.119519	0.102949	0.089679	0.078932	0.070117
P(2,14)	0.000002	-0.009917	0.298599	-0.549332	-0.321969	-0.174436	-0.097975	-0.057392	-0.034728	-0.021441	-0.013323	-0.008192	-0.004860
P(2,16)	0.000000	0.001335	-0.109384	0.573864	-0.051606	-0.193339	-0.203080	-0.183197	-0.159521	-0.138234	-0.120364	-0.105619	-0.093455
P(2,18)	0.000000	-0.000139	0.029924	-0.366537	0.462629	0.333314	0.188263	0.099619	0.049176	0.020442	0.003857	-0.005803	-0.011419
P(2,20)		0.000012	-0.006397	0.168114	-0.565022	-0.029849	0.160910	0.199871	0.191338	0.170833	0.149510	0.130512	0.114394
P(2,22)		-0.000001	0.001103	-0.059434	0.419315	-0.373132	-0.337011	-0.204936	-0.107234	-0.046452	-0.010243	0.011018	0.023336
P(2,24)		0.000000	-0.000157	0.016879	-0.226895	0.539996	0.104157	-0.120438	-0.188602	-0.193486	-0.177331	-0.156308	-0.136088
P(2,26)			0.000019	-0.003959	0.096326	-0.457001	0.281762	0.331324	0.221808	0.119645	0.049321	0.005030	-0.021808
P(2,28)			-0.000002	0.000783	-0.033413	0.282659	-0.500338	-0.168842	0.074724	0.170786	0.191060	0.180765	0.160661
P(2,30)			0.000000	-0.000133	0.009723	-0.138429	0.479389	-0.190392	-0.315542	-0.236491	-0.135202	-0.057045	-0.004659
P(2,32)			0.000000	0.000019	-0.002419	0.055980	-0.332842	0.447919	0.222069	-0.026103	-0.147333	-0.184605	-0.181929
P(2,34)				-0.000003	0.000522	-0.019192	0.183545	-0.486436	0.101318	0.289784	0.247049	0.152238	0.068644
P(2,36)				0.000000	-0.000099	0.005682	-0.084090	0.375258	-0.384945	-0.262521	-0.023252	0.119017	0.174259
P(2,40)				0.000000	0.000017	-0.001473	0.032890	-0.229452	0.478448	-0.016963	-0.254826	-0.252021	-0.169189
P(2,42)					-0.000002	0.000338	-0.011187	0.116882	-0.408098	0.313907	0.289395	0.071266	-0.086676
P(2,44)						-0.000069	0.003354	-0.051053	0.273953	-0.456155	-0.060324	0.211964	0.250401
P(2,46)						0.000013	-0.000896	0.019488	-0.153190	0.429966	-0.237509	-0.302402	-0.115975
P(2,48)						-0.000002	0.000215	-0.006591	0.073621	-0.314936	0.420729	0.128407	-0.162894
P(2,50)						0.000000	-0.000047	0.001996	-0.031023	0.191610	-0.439917	0.158574	0.301771
P(2,52)						0.000000	0.000009	-0.000546	0.011625	-0.100246	0.350445	-0.373764	-0.185481
P(2,54)						0.000000	-0.000002	0.000136	-0.003915	0.046075	-0.230570	0.437485	-0.079941
P(2,56)							0.000000	-0.000031	0.001195	-0.018880	0.130301	-0.378750	0.317223
P(2,58)							0.000000	0.000006	-0.000333	0.006973	-0.064740	0.268403	-0.422695
P(2,60)								-0.000001	0.000085	-0.002340	0.028724	-0.162916	0.398417
P(2,62)								0.000000	-0.000020	0.000719	-0.011510	0.086910	-0.303419
P(2,64)								0.000000	0.000004	-0.000203	0.004201	-0.041435	0.197008
P(2,66)								0.000000	-0.000001	0.000053	-0.001406	0.017865	-0.112261
P(2,68)									0.000000	-0.000013	0.000434	-0.007028	0.057171
P(2,70)									0.000000	0.000003	-0.000124	0.002541	-0.026351
P(2,72)										-0.000001	0.000033	-0.000849	0.011096
P(2,74)										0.000000	-0.000008	0.000263	-0.004299
P(2,76)										0.000000	0.000002	-0.000076	0.001541
P(2,78)											0.000000	0.000021	-0.000514
P(2,80)											0.000000	-0.000005	0.000160

TABLE 7—*Continued*

Epsilon	1.3697E-04	-1.0792E-02	-2.9216E-03	-1.3197E-03	-7.4668E-04	-4.7917E-04	-3.3325E-04	-2.4505E-04	-1.8773E-04	-1.4839E-04	-1.2023E-04	
	H(2,28)	H(2,58)	H(2,60)	H(2,62)	H(2,64)	H(2,66)	H(2,68)	H(2,70)	H(2,72)	H(2,74)	H(2,76)	SUM OF SQUARES
P(2,2)	0.011215	0.333782	0.132046	0.073571	0.048188	0.034616	0.026389	0.020969	0.017177	0.014404	0.012303	0.998510
P(2,4)	0.017905	0.740539	0.233812	0.123563	0.079360	0.056481	0.042839	0.033934	0.027742	0.023230	0.019823	0.996166
P(2,6)	-0.018451	0.541500	-0.119240	-0.096265	-0.069612	-0.052149	-0.040636	-0.032709	-0.027017	-0.022782	-0.019537	0.996103
P(2,8)	-0.039589	0.210372	-0.554651	-0.293623	-0.184975	-0.129962	-0.097775	-0.077042	-0.062757	-0.052417	-0.044646	0.980900
P(2,10)	0.015303	0.051359	-0.626423	-0.107208	-0.014578	0.007604	0.013161	0.013947	0.013267	0.012179	0.011044	0.998059
P(2,12)	0.062798	0.008637	-0.415349	0.310534	0.258463	0.194736	0.150354	0.119879	0.098256	0.082360	0.070305	0.952090
P(2,14)	-0.002650	0.001062	-0.191004	0.569856	0.263106	0.134501	0.077746	0.049491	0.033912	0.024597	0.018655	0.999084
P(2,16)	-0.083356	0.000100	-0.065658	0.537915	-0.048614	-0.141990	-0.142772	-0.126914	-0.110053	-0.095362	-0.083154	0.922813
P(2,18)	-0.014616	0.000007	-0.017641	0.353065	-0.398375	-0.309441	-0.212320	-0.149996	-0.110850	-0.085345	-0.067985	0.980158
P(2,20)	0.100925	0.000000	-0.003820	0.177314	-0.544840	-0.167666	-0.011535	0.044346	0.062639	0.066538	0.064844	0.913810
P(2,22)	0.030279	0.000000	-0.000682	0.071499	-0.475440	0.163827	0.246515	0.219150	0.180146	0.147134	0.121550	0.925173
P(2,24)	-0.118348		-0.000102	0.023851	-0.313218	0.435429	0.288072	0.148118	0.067633	0.024066	0.000513	0.928478
P(2,26)	-0.037595		-0.000013	0.006721	-0.166381	0.513106	0.067374	-0.108152	-0.153748	-0.154917	-0.142732	0.850422
P(2,28)	0.139245		-0.000001	0.001625	-0.073845	0.428898	-0.241943	-0.292668	-0.223164	-0.155145	-0.105584	0.927611
P(2,30)	0.028296		0.000000	0.000341	-0.028014	0.284401	-0.447953	-0.233721	-0.051624	0.040640	0.080938	0.810994
P(2,32)	-0.163716		0.000000	0.000063	-0.009231	0.157195	-0.482252	0.020011	0.196896	0.215803	0.188850	0.859004
P(2,34)	0.008623		0.000000	0.000010	-0.002675	0.074494	-0.392506	0.293693	0.296745	0.177152	0.079823	0.825599
P(2,36)	0.181030			0.000001	-0.000688	0.030831	-0.262042	0.448120	0.168578	-0.044851	-0.134417	0.761793
P(2,40)	-0.083086				-0.000158	0.011293	-0.149250	0.454025	-0.091418	-0.251498	-0.229105	0.799550
P(2,42)	-0.160041				-0.000033	0.003697	-0.074232	0.362988	-0.327494	-0.274889	-0.106328	0.732228
P(2,44)	0.184634				-0.000006	0.001091	-0.032746	0.243890	-0.441872	-0.103456	0.126124	0.713520
P(2,46)	0.051290				-0.000001	0.000292	-0.012957	0.142242	-0.428623	0.148233	0.277391	0.675694
P(2,48)	-0.241627					0.000071	-0.004639	0.073449	-0.338368	0.349075	0.238732	0.681823
P(2,50)	0.155585					0.000016	-0.001513	0.034035	-0.228679	0.432278	0.043118	0.613805
P(2,52)	0.109595					0.000003	-0.000452	0.014293	-0.135972	0.405815	-0.192866	0.539686
P(2,54)	-0.288227						-0.000124	0.005481	-0.072362	0.317385	-0.362246	0.573381
P(2,56)	0.230150						-0.000032	0.001931	-0.034879	0.215636	-0.420997	0.539345
P(2,58)	0.004364						-0.000007	0.000628	-0.015361	0.130301	-0.385283	0.420628
P(2,60)	-0.253373						-0.000002	0.000190	-0.006223	0.071093	-0.299192	0.344915
P(2,62)	0.396055						0.000000	0.000053	-0.002332	0.035397	-0.204250	0.299585
P(2,64)	-0.408358						0.000000	0.000014	-0.000812	0.016211	-0.125126	0.223223
P(2,66)	0.333988						0.000000	0.000003	-0.000264	0.006871	-0.069719	0.129380
P(2,68)	-0.231332							0.000001	-0.000080	0.002709	-0.035673	0.058113
P(2,70)	0.140259							0.000000	-0.000023	0.000997	-0.016882	0.020660
P(2,72)	-0.075948								-0.000006	0.000344	-0.007432	0.005947
P(2,74)	0.037228								-0.000001	0.000112	-0.003057	0.001414
P(2,76)	-0.016682									0.000034	-0.001179	0.000282
P(2,78)	0.006886									0.000009	-0.000428	0.000048
P(2,80)	-0.002633										-0.000147	0.000007

NOTES.—Expansion coefficients relating normalized Hough functions Θ, denoted as $H(S, N)$, where N is the order, and normalized associated Legendre functions denoted as $P(S, N)$. Symmetric gravitational modes with zonal number $s = 2$, and $f = 0.766$. Also shown are the corresponding eigenvalues ϵ_n. The last column shows the sum of the squares of the coefficients of the expansion of each Legendre polynomial in terms of the Hough functions. If the representation was exact this sum would be 1.

TABLE 8

EXPANSION COEFFICIENTS

NORMALIZED LEGENDRE FUNCTION EXPANSIONS IN TERMS OF THE ASSOCIATED HOUGH FUNCTIONS FOR THE PLANET SATURN AND THE MOON TITAN
$f = 0.9721$ Azimuthal number $s = 2$

Epsilon	8.2464383E-02	2.1719032E-02	9.7365936E-03	5.4790100E-03	3.5001379E-03	2.4246766E-03	1.7768442E-03	1.3570484E-03	1.0697693E-03	8.6467490E-04	-1.8923005E-04	
	H(2,2)	H(2,4)	H(2,6)	H(2,8)	H(2,10)	H(2,12)	H(2,14)	H(2,16)	H(2,18)	H(2,20)	- H(2,88)	SUM OF SQUARES
P(2,2)	0.9644875069	0.2253837392	0.1029129901	0.0603250661	0.0403824066	0.0293466456	0.0225440658	0.0180213038	0.0148410115	0.0125061507	0.0174892791	0.999271679
P(2,4)	-0.2625871094	0.7638474847	0.4231087183	0.2634965978	0.1812047257	0.1335871555	0.1035018134	0.0831933979	0.068768901	0.0581044384	0.0823673655	0.9840705323
P(2,6)	0.0284475365	-0.5812941011	0.3749036384	0.385370189	0.3133518241	0.2503361232	0.2029686804	0.1678340372	0.1413863128	0.1210599445	0.1832536026	0.9262245869
P(2,8)	-0.0016104474	0.1648472187	-0.7192910828	-0.0501357537	0.1624314787	0.2122642208	0.2112188455	0.1952833608	0.1762856443	0.1580937694	0.2941077167	0.8438291485
P(2,10)	0.0000559224	-0.0255977238	0.3754836339	-0.6220488803	-0.3441611062	-0.113471928	0.0105133482	0.0712649247	0.0989816908	0.1097720528	0.3845883503	0.8348554587
P(2,12)	-0.0000013128	0.0025451781	-0.1053379829	0.566836028	-0.3278903486	-0.4046625208	-0.2952915507	-0.1839474228	-0.1019231837	-0.0464779539	0.4325601177	0.9243600329
P(2,14)	0.0000000227	-0.0001770381	0.0191144339	-0.2500913133	0.6348011382	0.0359984158	-0.2425148779	-0.2967222586	-0.2676548873	-0.2178449507	0.4308684406	0.918780168
P(2,16)		0.0000091278	-0.0024617696	0.0701016339	-0.4247741553	0.5212526046	0.3114537393	0.0273105353	-0.1339167947	-0.2001856838	0.3868847339	0.7624947705
P(2,18)		-0.0000003629	0.0002383489	-0.0139429758	0.1708337548	-0.5570795055	0.2515977561	0.3868289289	0.24873214	0.0892657473	0.3168208433	0.7228657596
P(2,20)		-0.000000005	-0.0000180515	0.0020948343	-0.0480197519	0.3134623657	-0.571071438	-0.0699459827	0.2520289991	0.3071073984	0.2385479231	0.6463226498
P(2,22)			0.0000011012	-0.0002477447	0.0101612141	-0.1189956502	0.4578327073	-0.4308746246	-0.3085630272	0.0005886049	0.1661351449	0.532339381
P(2,24)			-0.0000000529	0.0000237562	-0.0016940518	0.0335837332	-0.2304947222	0.5415777022	-0.1692448128	-0.3666942176	0.1075113302	0.522232104
P(2,26)				-0.000001889	0.0002296077	-0.0074352928	0.0841561024	-0.3640622082	0.5086795671	0.1185161466	0.0648817674	0.4166895001
P(2,28)				0.0000001258	-0.0000258944	0.0013371801	-0.023854491	0.1696340688	-0.477130098	0.3429950001	0.036622325	0.375986438
P(2,30)				0.00000001	0.000002474	-0.0002002674	0.0054681994	-0.0602367227	0.2843422223	-0.51615957	0.0193818176	0.3513052601
P(2,32)					-0.0000002026	0.0000254533	-0.0010421915	0.0171476993	-0.1251636871	0.4027694806	0.0096379639	0.1782772238
P(2,34)					0.0000000068	-0.0000027863	0.000168542	-0.0040413513	0.0435354423	-0.2198120267	0.0045114845	0.0502493762
P(2,36)						0.0000002638	-0.0000234932	0.0008065933	-0.0124438609	0.0926409377	0.0019911752	0.0087418089
P(2,40)						0.0000000032	0.000002858	-0.0001386649	0.0030001783	-0.031715085	0.0008298479	0.0010155556
P(2,42)							-0.000000304	0.000020809	-0.0006217543	0.0090997464	0.0003270204	0.0000832993
P(2,44)							0.0000000015	-0.0000027551	0.0001123683	-0.0022360458	0.0001220063	0.0000050274
P(2,46)								0.0000003213	-0.0000179149	0.0004781643	0.0000431446	0.0000002308
P(2,48)								-0.0000000015	0.0000025429	-0.0000901012	0.0000144767	0.0000000083
P(2,50)									-0.00000032	0.0000151113	0.0000046121	0.0000000002
P(2,52)									0.0000000022	-0.0000022745	0.0000013902	7.1062953E-12
P(2,54)										0.0000003093	0.0000003742	2.3568280E-13
P(2,56)										-0.0000000369	0.0000000027	1.3692000E-15
P(2,58)										-0.0000000076		5.7149757E-17
P(2,60)												
P(2,62)												
P(2,64)												
P(2,66)												
P(2,68)												
P(2,70)												
P(2,72)												
P(2,74)												
P(2,76)												
P(2,78)												
P(2,80)												

NOTES.—Expansion coefficients relating normalized Hough functions Θ, denoted as $H(S, N)$, where N is the order, and normalized associated Legendre functions denoted as $P(S, N)$. Symmetric gravitational modes with zonal number $s = 2$, and $f = 0.9721$. Also shown are the corresponding eigenvalues ϵ_n. The last column shows the sum of the squares of the coefficients of the expansion of each Legendre polynomial in terms of the Hough functions.

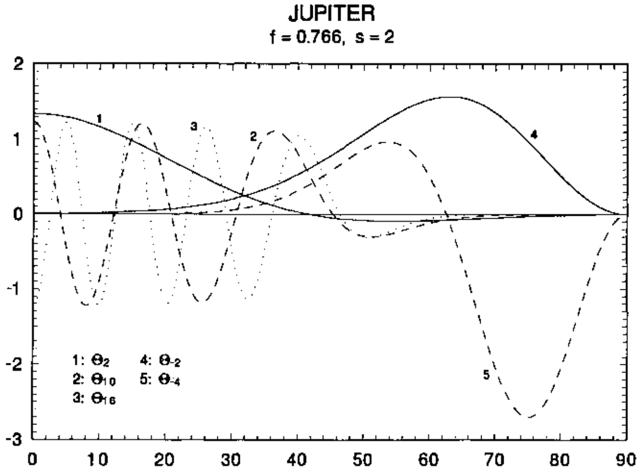

FIG. 4.—Some of the Hough functions for $s = 2$ and $f = 0.766$. Curves numbered 1, 2, and 3 correspond to the second, tenth, and sixteenth symmetric mode of positive equivalent depth. Curves 4 and 5 correspond to the first two symmetric modes with negative equivalent depth. The critical latitude is 50°; note that the modes with positive equivalent depth decay poleward of the critical latitude, while the modes with negative equivalent depth decay equatorward of this latitude.

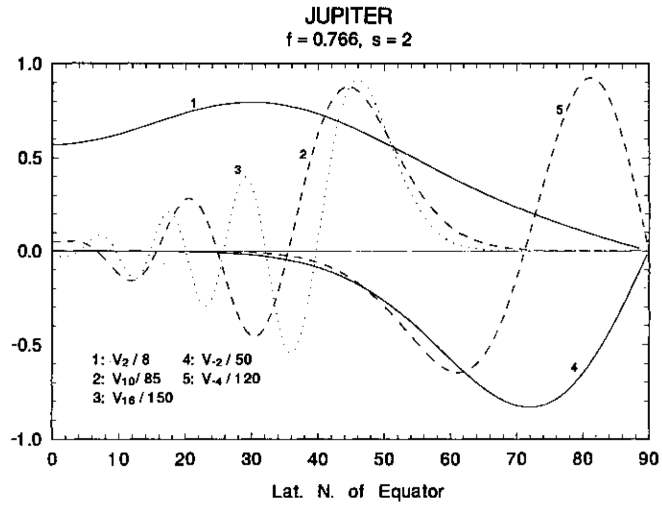

FIG. 6.—Some of the associated V for $s = 2$ and $f = 0.766$. These functions are the expansion basis of the zonal velocity. Curves numbered 1, 2, and 3 correspond to the second, tenth, and sixteenth symmetric mode of positive equivalent depth. Curves 4 and 5 correspond to the first two symmetric modes with negative equivalent depth. The functions have been divided by the amounts shown.

Hough modes in the atmospheric envelope of the planet. For Saturn, the tidal potential due to Titan gives a critical latitude of 76°, which leads to a small contribution by the Hough modes with negative equivalent depth. For Neptune, the retrograde orbit of Triton leads to absence of eigenvalues associated with negative equivalent depths, but further conclusions should be reserved in this case because the highly inclined orbit of Triton requires a separate treatment.

5. THE VERTICAL STRUCTURE OF THE TIDAL RESPONSE

While the Laplace tidal equation determines the meridional structure of the tidal response, the vertical structure equation

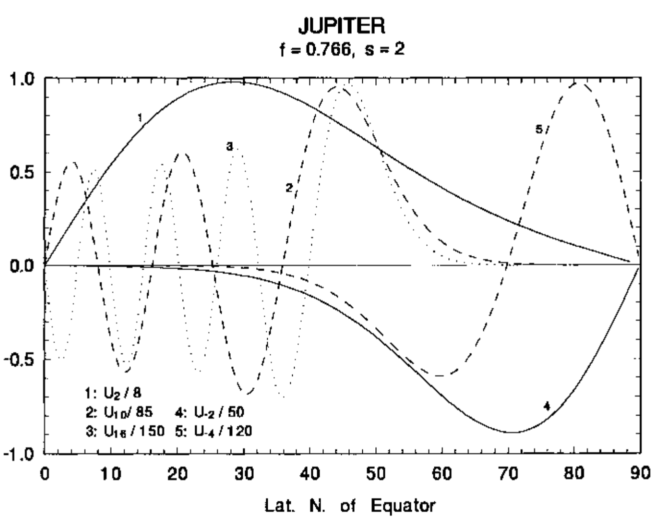

FIG. 5.—Some of the associated U for $s = 2$ and $f = 0.766$. These functions are the expansion basis of the northerly velocity. Curves numbered 1, 2, and 3 correspond to the second, tenth, and sixteenth symmetric mode of positive equivalent depth. Curves 4 and 5 correspond to the first two symmetric modes with negative equivalent depth. The functions have been divided by the amounts shown.

depends on the properties of the planet and on its thermodynamic state. Consider the hydrostatic version of the vertical structure equation (2.24). Assuming slow variation of the coefficients of equation (2.24) the tidal fields are either exponentially trapped or vertically propagating. When conditions for propagation are satisfied, the amplitude of the tidal fields will grow with height as $\rho_0^{-1/2}$ (cf. eq. [2.22]). The nth Hough mode propagates in the vertical whenever

$$0 < h_n^{\sigma,s} < \frac{4N^2H^2}{g} . \tag{5.1}$$

Note that modes with negative equivalent depth are trapped. The first Hough modes with large equivalent depths may also violate equation (5.1) and be trapped. The propagating modes, which correspond to sufficiently small positive equivalent depths, will be selectively amplified due to their exponential growth with height (see Fig. 7). Although the tidal forcing is proportional to the first symmetric spherical harmonic, the response of the atmosphere will be dominated at sufficiently great heights by the propagating modes, in proportion to their degree of excitation in the interior of the planet. The large value of the factor $4\omega^2a^2/g$, as we have seen in the previous section, leads to large equivalent depths and trapping of the first Hough modes. The response of the atmospheres is expected to be at higher Hough modes. The outer planets should be contrasted to Earth for which, for an isothermal state, all semidiurnal symmetric Hough modes propagate. It is thus important to delineate which Hough modes propagate vertically in the observable outer atmospheric envelope and determine their general structure.

According to equation (2.24) the local vertical wavelength λ_z is approximately

$$\lambda_z = \frac{2\pi H}{[(N^2H^2)/(gh_n^{\sigma,s}) - (1/4)]^{1/2}} . \tag{5.2}$$

Tables 9–12 list the first Hough modes that propagate in the atmospheric envelopes of the outer planets; the terrestrial case

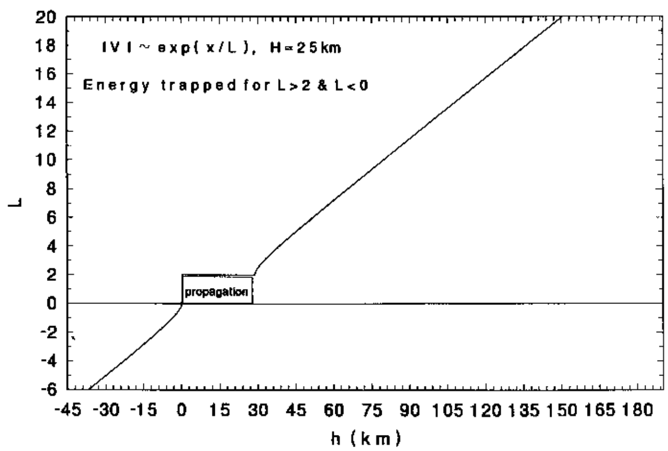

FIG. 7.—Energy trapping measured by L as a function of equivalent depth. For this graph we have assumed an isothermal atmosphere with scale height $H = 25$ km ($T = 167$ K), simulating conditions on Jupiter. Vertical distance is measured by x which in dimensional variables corresponds to a height Hx. For $0 < h < 29$ km energy propagates, and velocity amplitudes grow with $L = 2$. For $h > 29$ km energy is trapped, but there is a growth of the velocity amplitudes. This growth diminishes as h increases. For $h < 0$ energy is trapped, and the velocity amplitudes decay (after Lindzen 1967).

is included for reference (for the purpose of these tables we have assumed that the atmospheres are isothermal). Note that on Jupiter modes must be of order $n \geq 10$ to propagate vertically; on Saturn they must be even higher ($n \geq 20$) to propagate vertically. Uranus is the exception; vertical propagation is allowed for $n \geq 4$ (the same is true for Neptune). It is interesting to note that the bands on Jupiter and Saturn correspond to the structure of the first propagating Hough modes on the corresponding planets (Lindzen 1991).

We have so far assumed the tidal response to be linear and disregarded the effects of mechanical or thermal dissipation. At great heights the exponential growth of the vertically propagating tidal modes, which leads to violation of the linearity assumption, is retarded by enhanced dissipation because molecular viscosity and thermal conductivity are inversely proportional to the ambient density. When dissipation is insufficient to balance the growth of the amplitude of the tides, the large tidal temperature fluctuations create convectively unstable regions which cause breaking of the tides (Lindzen 1981). A heuristic criterion for the importance of dissipative

TABLE 9

SYMMETRIC HOUGH MODES FOR EARTH

Hough Mode	h_n (km)	λ_{hor} (km)	λ_z (km)	τ_{res}
2.........	7.85	12403	293	6 minutes
4.........	2.11	6440	54	1.5 hr
6.........	0.96	4338	34	2.6 hr
8.........	0.54	3253	25	3.7 hr
10.........	0.35	2619	20	4.7 hr
12.........	0.24	2169	16	

NOTES.—The first vertically propagating symmetric Hough modes for lunar semidiurnal forcing on the Earth with $f = 0.966$, $s = 2$, and $\tau_{tide}/(2\pi) = 1.97$ hr. Here h_n is the equivalent height of the mode, λ_{hor} the total horizontal wavenumber of the equivalent gravity wave, λ_z the vertical wavelength, and τ_{res} the time it takes for a wave to traverse a region of depth equal to the scale height $H = 7.6$ km. The atmosphere is taken to be isothermal. For reference, the period of the lunar forcing $\tau_{tide} = 12.42$ hr.

TABLE 10

SYMMETRIC HOUGH MODES FOR JUPITER

Hough Mode	h_n (km)	λ_{hor} (km)	λ_z (km)	τ_{res}
10.........	28.8	19780	504	12 minutes
12.........	19.71	16363	310	26 minutes
14.........	14.32	13947	235	39 minutes
16.........	10.87	12152	192	51 minutes
18.........	8.53	10765	164	1 hr
20.........	6.87	9660	143	1.2 hr
22.........	5.65	8761	127	1.4 hr
24.........	4.73	8061	115	1.6 hr

NOTES.—The first vertically propagating symmetric Hough modes at the visible atmospheric envelope of Jupiter for semidiurnal forcing by Io, with $f = 0.766$, $s = 2$, and $\tau_{tide}/(2\pi) = 1$ hr. The atmosphere is taken isothermal with $N = 2 \times 10^{-2}$ s^{-1}. Here h_n is the equivalent height of the mode, λ_{hor} the total horizontal wavenumber of the equivalent gravity wave, λ_z the vertical wavelength, and τ_{res} the time it takes for a wave to traverse a region of depth equal to the scale height $H = 25$ km. For reference, the period of the lunar forcing $\tau_{tide} = 6.5$ hr.

TABLE 11

SYMMETRIC HOUGH MODES FOR SATURN

Hough Mode	h_n (km)	λ_{hor} (km)	λ_z (km)	τ_{res}
20.........	37.05	11461	1088	5 minutes
22.........	30.56	10409	724	9 minutes
24.........	25.63	9532	570	15 minutes
26.........	21.80	8792	480	19 minutes
28.........	18.77	8157	418	23 minutes
30.........	16.33	7609	373	27 minutes

NOTES.—The first vertically propagating symmetric Hough modes at the visible atmospheric envelope of Saturn for semidiurnal forcing by Titan, with $f = 0.972$, $s = 2$, and $\tau_{tide}/(2\pi) = 52.4$ minutes. The atmosphere is taken isothermal with $N = 8 \times 10^{-3}$ s^{-1}. Here h_n is the equivalent height of the mode, λ_{hor} the total horizontal wavenumber of the equivalent gravity wave, λ_z the vertical wavelength, and τ_{res} the time it takes for a wave to traverse a region of depth equal to the scale height $H = 39$ km. For reference, the period of the lunar forcing $\tau_{tide} = 5.5$ hr.

TABLE 12

SYMMETRIC HOUGH MODES FOR URANUS

Hough Mode	h_n (km)	λ_{hor} (km)	λ_z (km)	τ_{res}
4.........	18.47	17498	424	25 minutes
6.........	7.61	11232	190	81 minutes
8.........	4.11	8255	129	129 minutes
10.........	2.57	6527	99	2.9 hr
12.........	1.75	5386	80	3.6 hr
14.........	1.27	4588	68	4.3 hr
16.........	0.97	4011	58	5 hr
18.........	0.76	3549	51	5.7 hr

NOTES.—The first vertically propagating symmetric Hough modes at the visible atmospheric envelope of Uranus for semidiurnal forcing by Ariel, with $f = 0.715$, $s = 2$, and $\tau_{tide}/(2\pi) = 115$ minutes. The atmosphere is taken isothermal with $N = 10^{-2}$ s^{-1}. Here h_n is the equivalent height of the mode, λ_{hor} the total horizontal wavenumber of the equivalent gravity wave, λ_z the vertical wavelength, and τ_{res} the time it takes for a wave to traverse a region of depth equal to the scale height $H = 25$ km. For reference, the period of the lunar forcing $\tau_{tide} = 12.1$ hr.

processes is given by comparing the associated dissipation time scale, τ_{diss}, to the time scale associated with the period of the tidal fields $1/\sigma$ and the time it takes a group of waves to traverse one scale height, τ_{res}. Dissipation processes are important when $\tau_{diss} < \min(1/\sigma, \tau_{res})$. If the source of dissipation is due to molecular viscosity, with kinematic viscosity ν, an estimate of the dissipation time scale is given by $\tau_{diss} \approx \rho_0 \lambda_z^2/\nu$, which increases quadratically with the vertical wavelength.

The calculation of τ_{res} requires an estimate of the vertical group velocity for tidal modes. This is most easily done by considering internal gravity waves in the neighborhood of the equator whose frequencies, zonal wavelengths, and vertical wavelengths are identical to those of the tidal modes (Lindzen 1970). If we write the zonal wavenumber as k, and the meridional wavenumber as m, then we have the dispersion relation

$$gh_n = \frac{\sigma^2}{\alpha^2}, \tag{5.3}$$

where $\alpha = (k^2 + m^2)^{1/2}$, is the total horizontal wavenumber, and $k = s/a$, with, a, the radius of the planet (Lindzen 1970); m is adjusted in order to make h_n correspond to the value calculated for a Hough mode. For such gravity waves the vertical group velocity is given by

$$C_g = \frac{N\lambda_z^2}{2\pi\lambda_{hor}}, \tag{5.4}$$

where $\lambda_{hor} = 2\pi/\alpha$, and $\tau_{res} \approx H/C_g$. The residence time, τ_{res}, is inversely proportional to the vertical wavelength. Consequently the higher Hough modes which have smaller vertical wavelengths will be preferentially damped. Moreover, either breaking of the tides or limitation of tidal amplitude by dissipation implies exchange of tidal wave momentum with the mean flow. This momentum exchange will be investigated in a following paper. In Tables 9–12 we list the total horizontal wavelengths and residence times corresponding to the first propagating Hough modes.

6. EXCITATION OF ATMOSPHERIC TIDES

Excitation of the tides in the planetary interior is formulated and investigated in an accompanying paper (Ioannou & Lindzen 1993). However, it is useful to review some of these results in order to estimate magnitudes that should be realizable in the visible atmosphere of Jupiter.

At the deeper levels of the atmospheric layer the static stability is usually considered to be neutral, so that according to equation (5.1) all modes are trapped. However, the existence of very small stability, in inverse proportion to the local temperature, allows propagation of the high-order Hough modes. Note that in these regions the nonhydrostatic equation (2.23) must be considered.

If the interior excitation depends only on the external forcing, the dominant mode would be the first vertically propagating mode. However, the interior is capable of selectively amplifying higher modes because of resonances which may exist when stratification is present (Ioannou & Lindzen 1993). The excitation of the atmosphere is proportional to radial displacement, ξ_f, which is the difference between the displacement of a material surface of the atmosphere under the action of the tidal forcing and the displacement of an equipotential surface: $\xi_{equilib} = \Omega/g$. For example, in the absence of any restoring force and for sufficiently long period forcing $\xi_f = 0$ because the surface will assume the figure of the equipotential to good accuracy, leading to insignificant atmospheric response. When the interior is neutral, the weak restoring forces associated with compression and rotation lead typically to $\xi_f \approx 0.1\ \xi_{equilib}$ (Ioannou & Lindzen 1993), and to small tidal fields in the atmosphere. The presence of some static stability dramatically enhances ξ_f so that it exceeds in magnitude the equilibrium displacement $\xi_{equilib}$. Specification of the effective forcing ξ_f allows estimation of the order of magnitude of the various tidal fields. As in the previous discussion we consider only tidal fields associated with vertically propagating modes. Assuming for simplicity an isothermal state we obtain from equations (2.25)–(2.28)

$$T_n \approx \frac{\xi_f}{h_n} T_0 e^{x/2}, \tag{6.1}$$

$$w_n \approx \sigma \xi_f e^{x/2}, \tag{6.2}$$

$$u_n \approx \frac{g}{a\sigma} \xi_f e^{x/2} U_n, \tag{6.3}$$

where x is the distance measured in scale heights above the interior level with positive Brunt-Väisälä frequency. U_n is the normalization factor of the associated Hough function (see Figs. 6, 7). For example, for the first vertically propagating mode on Jupiter and with the choice $x = 8$ and $\xi_f = 1$ m ($\xi_{equilib} = 0.7$ m), we get $T \approx 0.3$ K, $w \approx 0.01$ m s^{-1}, and $u \approx v \approx 2$ m s^{-1}. Such magnitudes are well within the sensitivity of present observations by Deming et al. (1989). Detection of such tides will provide useful constraints on models of the interior structure of Jupiter.

7. DISCUSSION

We have applied classical tidal theory to the atmospheres of the outer planets. The only departure made from classical theory was retention of the effects of nonhydrostaticity which are important in the deeper atmosphere. The meridional structure of the tidal response was shown to depend only on the ratio of the period of gravitational forcing to the period of rotation of the planet. Forcing by low-inclination orbits of the satellites of Jupiter, Saturn, and Uranus excite primarily symmetric Hough modes with respect to the equator of the planet. For each forcing period σ, there exists a critical latitude when $\sigma < 2\omega$. The meridional modes split into two classes: those with positive equivalent depths oscillate equatorward of the critical latitude, and those with negative equivalent depth poleward of the critical latitudes. Consideration of the vertical structure equation shows that negative equivalent depth modes are trapped in the stable atmosphere, and only higher positive equivalent depths can propagate vertically in the atmospheric envelope. The vertically propagating modes increase in amplitude in inverse proportion to the square root of the ambient density and are expected to dominate the tidal signal sufficiently high in the atmosphere. Remarkably, although the gravitational forcing is proportional to the first symmetric spherical harmonic with zonal wavenumber 2, the tidal response is concentrated in higher order meridional structures confined between the critical latitudes. For Jupiter, the tidal forcing by Io defines a critical latitude at 50°, and the first propagating mode is the tenth Hough mode (see Fig. 5). For Saturn, the tidal forcing by Titan defines a critical latitude at 76°, and the first propagating mode is the twentieth. Uranus's forcing by Ariel associates a critical latitude at 45°, and the first propagating mode is the fourth. As has already been noted the meridional structure of these modes corresponds to the visible bands on the outer planets. In this paper we have estimated tidal damping which leads to acceleration of the mean flow by tides, which could play a role in producing mean banding. A paper exploring this possibility is in preparation.

Finally, using results from an accompanying paper (Ioannou & Lindzen 1993) we have noted that tides should be observable in the visible atmosphere of Jupiter provided that there is no small, nonzero, static stability within the planet.

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