Generalized Stability Theory. Part II: Nonautonomous Operators

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ABSTRACT

An extension of classical stability theory to address the stability of perturbations to time-dependent systems is described. Nonnormality is found to play a central role in determining the stability of systems governed by nonautonomous operators associated with time-dependent systems. This pivotal role of nonnormality provides a conceptual bridge by which the generalized stability theory developed for analysis of autonomous operators can be extended naturally to nonautonomous operators. It has been shown that nonnormality leads to transient growth in autonomous systems, and this result can be extended to show further that time-dependent nonnormality of nonautonomous operators is capable of sustaining this transient growth leading to asymptotic instability. This general destabilizing effect associated with the time dependence of the operator is explored by analyzing parametric instability in periodic and aperiodic time-dependent operators. Simple dynamical systems are used as examples including the parametrically destabilized harmonic oscillator, growth of errors in the Lorenz system, and the asymptotic destabilization of the quasigeostrophic three-layer model by stochastic vacillation of the zonal wind.

1. Introduction

The first part of this study addressed the growth of perturbations to time-independent systems in which there was a source of energy such as the baroclinic–barotropic available energy of atmospheric shear flows. In such time-independent flows availability of background flow energy for perturbation growth can be determined by linearizing the equations of motion about the fixed background flow and searching for growing perturbations. Systematic application of this program in the asymptotic limits of short and long time, as well as in the physically most relevant intermediate times, led to the generalized stability theory of autonomous operators described in Part I. It was found that the stability analysis of time-independent flows understood in this generalized sense provides a comprehensive conceptual framework for analyzing the growth of perturbations on timescales relevant to the rapid deepening phase of cyclogenesis, as well as to the analysis of forecast error growth on the synoptic cyclone timescales of 12 to 48 hours. In addition, stochastic forcing of an autonomous operator modeling the background flow was found to provide a mechanistic theory for the statistically stationary state of the fully turbulent flow. In summary, generalized stability theory applied with an appropriate choice of time-independent flow, which models either a stationary or a mean background state, addresses a wide variety of phenomena related to statistical as well as deterministic flow dynamics (Farrell 1989; Borges and Hartmann 1992; Farrell and Ioannou 1993a,b,c, 1994a,b, 1995; DelSole and Farrell 1995; Buizza and Palmer 1995; Reddy and Henningson 1993; Trefethen et al. 1993). However, there remain a set of issues involving the $t \rightarrow \infty$ growth of errors along time-dependent trajectories and the development of finite perturbations in time-dependent flows for which inclusion of temporal variability in the underlying dynamical system is essential. The advantage of generalized stability theory in addressing this set of problems is persuasive in that modal instability has no counterpart in such nonautonomous systems, while the Lyapunov vectors and their associated optimal excitations in nonautonomous systems are natural extensions of the singular vectors associated with optimal modes and the structures into which they evolve in the autonomous version of generalized stability theory (Lorenz 1965; Lacarra and Talagrand 1988; Farrell 1990; Moore and Farrell 1993; Palmer 1993; Yoden and Nomura 1993).

The results of Lyapunov (1907) and Oseledec (1968) ensure that associated with the asymptotic error growth rate there is a unique Lyapunov vector and a unique optimal excitation for this Lyapunov vector, both of which can be identified at each point of the solution trajectory of a nonlinear system. The form of

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these asymptotic structures and the manner in which the asymptotic regime is approached as $t \to \infty$ are important questions because as the time interval of a forecast initially increases from zero the mean spatial wavenumber of the optimal perturbation increases, suggesting potentially severe resolution requirements on NWP models for forecasting in the intermediate range (Buizza and Palmer 1995; Hartmann et al. 1995). Ultimately, however, the optimal perturbation must approach that of the optimal excitation of the first Lyapunov vector and the spatial scale of the optimal must therefore converge to a statistical limit. It is of interest to determine at each forecast interval the spatial scales that must be included in order to resolve the optimal perturbations over that forecast interval and the limit of these as they converge to the first Lyapunov excitation.

It has not been established that divergence of solution trajectories on a chaotic attractor correctly models error growth in the atmosphere at space scales and timescales appropriate for synoptic forecasting. Growth of errors, even very great growth of errors, can arise in asymptotically stable systems. Such systems when subjected to inevitable external noise sources become unpredictable without the necessary existence of an asymptotic instability especially so when the system is nonnormal and supports large transient amplification of a subspace of perturbations. Even if a positive Lyapunov exponent exists in a time-dependent system, this asymptotic instability may arise from externally forced time dependence of the solution trajectory rather than from intrinsic time dependence associated with the trajectory on its chaotic attractor. Moreover, in cases with asymptotic instability arising from time dependence this asymptotic instability may be very weak and dominant only in the limit of irrelevantly long time as far as practical forecast is concerned. The prediction problem would then turn on analysis of intermediate time optimals and the projection of error sources on these optimals. Regardless of which of these scenarios obtains in the atmosphere at any given time, the appropriate method of analysis is generalized stability theory extended to nonautonomous operators.

It is sometimes assumed that existence of a positive Lyapunov exponent, which is indicative of sensitive dependence on initial conditions in the asymptotic limit $t \to \infty$, arises in fluid systems because of the specific nature of the nonlinearity associated with advection. We demonstrate that the source and particular configuration of time dependence in a nonautonomous operator is to a great extent irrelevant to determining its asymptotic stability, that destabilization by time dependence is a generic property of nonnormal dynamical systems, and that the underlying cause of this universal destabilization of time-dependent systems is the instigation of growth processes associated with nonnormality in such systems.

The existence of a positive Lyapunov exponent and its magnitude in a damped time-dependent system depends on the relative magnitude of the destabilizing effects of nonnormality of the mean operator and both nonnormality and variance of the perturbation operator on the one hand and the stabilizing effect of damping on the other. We show using a three-layer quasigeostrophic baroclinic model with stochastic variation of the zonal wind that there is for physically realistic damping rates a necessary level of variance needed to support a positive Lyapunov exponent and associated asymptotic error growth.

In what follows we review some relevant theoretical results on time-dependent operators and use these results to develop the stability theory of nonautonomous systems, making illustrative applications to model problems.

2. Stability theory for nonautonomous dynamical systems

The linear time-dependent dynamical system

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}(t)\mathbf{u} \tag{1}$$

has the solution

$$\mathbf{u}(t) = \mathbf{\Phi}_{[t,t_0]}\mathbf{u}(t_0). \tag{2}$$

The propagator $\mathbf{\Phi}_{[t,t_0]}$ maps the state of the system at time t_0 to its state at time t. It is given by the following time-ordered exponential (Coddington and Levinson 1955):

$$\mathbf{\Phi}_{[t,t_0]} = \mathbf{I} + \int_{t_0}^{t} \mathbf{A}(s)ds + \int_{t_0}^{t} \mathbf{A}(r)dr \int_{t_0}^{r} \mathbf{A}(s)ds + \cdots, \tag{3}$$

which is equivalent in the limit to the ordered product of infinitesimal propagators

$$\mathbf{\Phi}_{[t,t_0]} = \lim_{\delta t \to 0} \prod_{j=1}^{n} e^{A(t_j)\delta t}, \tag{4}$$

for t_j lying in the mesh $t_0 + (j-1)\delta t < t_j < t_0 + j\delta t$ with $t = t_0 + n\delta t$. The propagator obeys the semigroup property $\mathbf{\Phi}_{[t,s]}\mathbf{\Phi}_{[s,t_0]} = \mathbf{\Phi}_{[t,t_0]}$ and solves

$$\frac{d\mathbf{\Phi}_{[t,t_0]}}{dt} = \mathbf{A}(t)\mathbf{\Phi}_{[t,t_0]} \quad \text{with} \quad \mathbf{\Phi}_{[t_0,t_0]} = \mathbf{I}. \tag{5}$$

Analysis of the stability properties of this dynamical system differs from analysis of the stability of autonomous operators in that system (5) has asymptotic behavior that cannot be determined by examining the behavior of the temporal eigenmodes of $\mathbf{A}(t)$, which are not defined. However, the optimal perturbations retain their meaning and provide the required description of the stability of the system for all time. In fact, it remains possible to uniquely define an asymptotic exponential rate of growth or decay for time-dependent operators, which is the first Lyapunov exponent (Lyapunov 1907), defined as[1]

$$\lambda = \lim_{t\to\infty} \sup \frac{\ln(\|\mathbf{\Phi}(t)\|)}{t}. \quad (6)$$

The first Lyapunov exponent assumes the role played by the spectral abscissa in autonomous systems and reduces to the spectral abscissa (the most rapidly growing mode) in the limit that the system becomes independent of time. It follows that any dynamical system with $\lambda > 0$ is asymptotically unstable.

We have shown that for autonomous systems the optimal initial perturbation that maximizes large time growth is the biorthogonal of the least damped mode. A slight modification of this argument establishes in the asymptotic limit the existence of a unique optimal perturbation producing maximum excitation of the time dependent first Lyapunov vector associated with the limiting operation in (6). The singular value decomposition (SVD) of the propagator at sufficiently large t is given with exponential accuracy by

$$\mathbf{\Phi}_{[t,t_0]} = \mathbf{U}_{[t,t_0]}\mathbf{S}_{[t,t_0]}\mathbf{V}^{\dagger}_{[t,t_0]}, \quad (7)$$

with $\mathbf{S}_{[t,t_0]}$ a null matrix save for its first entry, $\mathbf{S}_{11}$, being nonzero and equal to $e^{\lambda(t-t_0)}$, where λ is the first Lyapunov exponent and the time dependence of the unitary matrices $\mathbf{U}$, $\mathbf{V}$ has been included to indicate that this decomposition depends on both the initial time t_0 and the final time t. We show that optimal excitation (by the first column of $\mathbf{V}$) of the first Lyapunov vector (the first column of $\mathbf{U}$) is asymptotically independent of t and depends only on the initial time t_0 and that the first Lyapunov vector asymptotically becomes independent of the initial time t_0 and depends only on t.

The asymptotic form of $\mathbf{S}_{[t,t_0]}$ in which only the $\mathbf{S}_{11}$ entry survives implies that $\mathbf{\Phi}_{[t,t_0]}$ is a matrix of rank 1, which requires for $s > t$ that the propagator $\mathbf{\Phi}_{[s,t_0]}$ be also of rank 1 (because of the semigroup property $\mathbf{\Phi}_{[s,t_0]} = \mathbf{\Phi}_{[s,t]}\mathbf{\Phi}_{[t,t_0]}$). Consequently, $\mathbf{\Phi}_{[s,t_0]}$ and $\mathbf{\Phi}_{[t,t_0]}$ are of the same rank and share the same null space. The vector that leads to the maximum norm of the propagator at s is the constant vector orthogonal to this unique null space, which depends only on t_0. This proves that at each point on the trajectory a unique vector can be defined that optimally excites the asymptotically maximally growing Lyapunov vector. Perturbations that lie in the subspace perpendicular to the vector that optimally excites the first Lyapunov vector will necessarily grow at a rate bounded by the second Lyapunov exponent. The above argument can then be repeated on this perpendicular subspace to obtain the optimal excitation of the second Lyapunov exponent, and by induction it can be shown that all the columns of $\mathbf{V}$ are associated with Lyapunov exponents and depend asymptotically only on t_0.

The inverse operator $\mathbf{\Phi}_{[t_0,t]} = \mathbf{\Phi}^{(-1)}_{[t,t_0]}$ interchanges the roles of $\mathbf{V}$ and $\mathbf{U}$, and it follows by applying the above inductive argument to the inverse operator that the first Lyapunov vector depends only on t and is independent of the initial time t_0 and that this is true for the whole of $\mathbf{U}$.

In practice, in order to verify convergence to the Lyapunov vector at a given time t, it is necessary to show that integration of the tangent linear system yields a Lyapunov vector that is independent of the initial time in the past at which the integration was started. Conversely, it can be seen from (7) that convergence to the optimal excitation of the Lyapunov vector at t_0 requires forward integration until the determined optimal perturbation becomes independent of time. Although convergence to the value of the Lyapunov exponent is slow in practice, we have found that convergence to the Lyapunov vector and its optimal excitation is usually rapid.

If we assume that the time-dependent operator is comprised of a sum of a time-independent mean operator and a stochastic operator, then an extension of the Lyapunov results of Oseledec (1968) establishes the existence of unique Lyapunov exponents almost surely in probability. All the properties established for deterministic operators carry over to the stochastic case, with the exception that the first Lyapunov vector and its optimal excitation depend on the specific time realization, while the associated first Lyapunov exponent is independent of the particular realization (Arnold and Kliemann 1983).

3. Example: The Lorenz equation

The Lorenz system of differential equations provides a convenient model for demonstrating perturbation dynamics of nonautonomous systems. The standard form of these equations,

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & -x \\ y & 0 & -b \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad (8)$$

with parameters $b = 8/3$, $\sigma = 10$, $r = 28$, produces a time-dependent chaotic trajectory in state variables $[x(t), y(t), z(t)]$. From any starting point the trajectory rapidly converges to the familiar butterfly-shaped strange attractor shown in Fig. 1. At each point along the trajectory there is associated the tangent linear system:

[1] The superior limit has been used, with the notation "lim sup" and defined as the supremum of the limit points, because the usual limit may not exist as, for example, in the case of the one-dimensional propagator $\mathbf{\Phi}_{[t,0]} = e^t \sin t$. The existence of the Lyapunov exponents defined as in (6) is guaranteed for linear differential systems with continuous and bounded coefficients.

$$\frac{d}{dt}\begin{pmatrix}\delta x\\ \delta y\\ \delta z\end{pmatrix} = \begin{pmatrix}-\sigma & \sigma & 0\\ r - z(t) & -1 & -x(t)\\ y(t) & x(t) & -b\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\\ \delta z\end{pmatrix}, \quad (9)$$

where the linearization has been performed about the time-varying solution.

The Lyapunov exponents for these parameter values are approximately $\lambda_1 = 0.88$, $\lambda_2 = 0$, $\lambda_3 = -14.55$. Convergence to these values requires extensive integration of the tangent linear system because the convergence is slow, as can be seen from the example shown in Fig. 2.

The uniqueness of the first Lyapunov vector (the state vector associated with the largest Lyapunov exponent) and of the optimal excitation of the first Lyapunov vector permits identification of these vectors at each point along the trajectory. As was previously remarked, convergence to these vectors is rapid, typically requiring integration over a time interval not longer than ten units. In addition to the Lyapunov vector and the optimal excitation vector, the directions of the numerical abscissa [the eigenvector associated with the largest eigenvalue of $\mathbf{A} + \mathbf{A}^\dagger$ in which $\mathbf{A}$ is the tangent linear operator in (9)] and of the trajectory itself can be distinguished. An interesting additional diagnostic is provided by the eigenvector of $\mathbf{A}$ at each point along the trajectory where the eigenanalysis is performed by fixing the values of the components of $\mathbf{A}$ at each time t. In Figs. 3a,b the relation among these quantities is displayed for a typical segment of the solution trajectory by plotting as a function of time the instantaneous growth rate of a vector lying in the direction of the numerical abscissa (the maximum possible rate of growth), the instantaneous growth rate of a perturbation vector lying in the direction of the solution trajectory, the instantaneous growth rate of the Lyapunov vector, the instantaneous growth rate of the optimal excitation of the Lyapunov vector, and the maximum of the real parts of the eigenvalues of $\mathbf{A}$. While the Lyapunov exponent associated with the direction of the trajectory necessarily vanishes in the mean, this zero-average growth rate results from cancellation of large and rapid local variations in growth rate, which at times nearly equal the maximum possible growth rate given by the numerical abscissa. It is remarkable that the numerical abscissa itself exhibits frequent excursions of negative growth rate, indicating a temporary decrease in all deviations from the solution trajectory (cf. Ahlquist and Sivillo 1995).

The average of the numerical abscissa is ≈ 4.2 (cf. Fig. 4a), which is considerably greater than the first Lyapunov exponent, which is ≈ 0.88 (cf. Fig. 4b). The optimal excitation of the Lyapunov vector has an intermediate average growth rate of 2.7 (cf. Fig. 4c). The statistics of the average growth of perturbations tangent to the trajectory is shown in Fig. 4d.

The Lyapunov vector and the vector in the direction of the trajectory both necessarily lie in the plane of the attractor, while the vector in the direction of the numerical abscissa and the vector of the optimal excitation of the Lyapunov vector are not so constrained. Indeed it can be seen from Figs. 5a and 5b that the angle between the attractor plane and these vectors may be as large as 18°. Nevertheless, the strong convergence to the attractor in the Lorenz system indicated by the

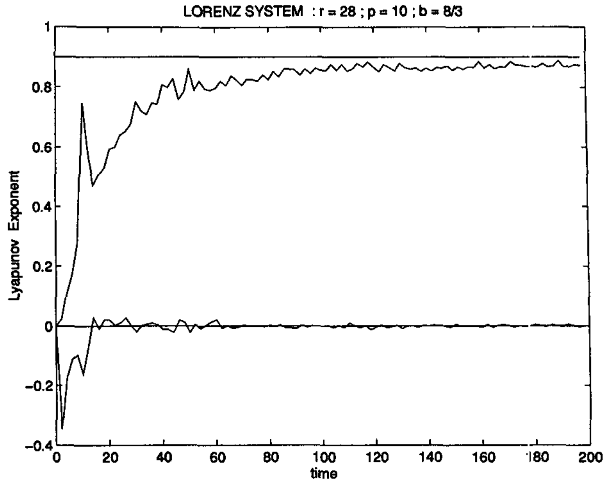

FIG. 2. Convergence of the Lyapunov exponents for the Lorenz system with $b = 8/3$, $\sigma = 10$, and $r = 28$. The first two exponents are shown in the graph. The first Lyapunov exponent has value $\lambda_1 = 0.88$, the second, characterized by perturbations along the trajectory, has the value $\lambda_2 = 0$, and the third, characterized by perturbations off the attractor, has the value $\lambda_3 = -14.55$. Note that the sum of the Lyapunov exponents gives the rate of volume contraction in configuration space, which is $-(\sigma + 1 + b)$.

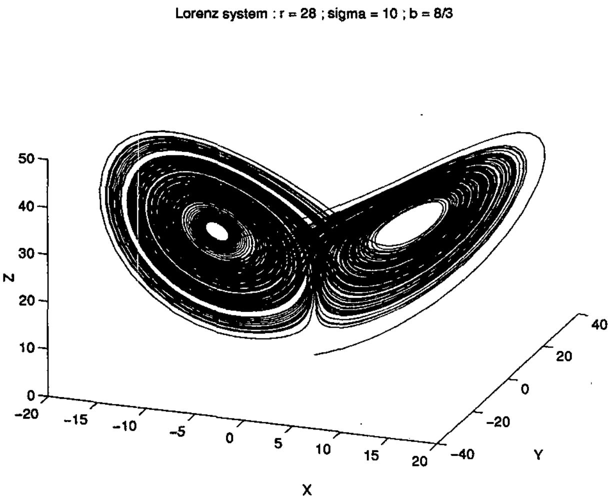

FIG. 1. A trajectory of the Lorenz system with $b = 8/3$, $\sigma = 10$, and $r = 28$. The trajectory rapidly converges to the familiar butterfly-shaped attractor.

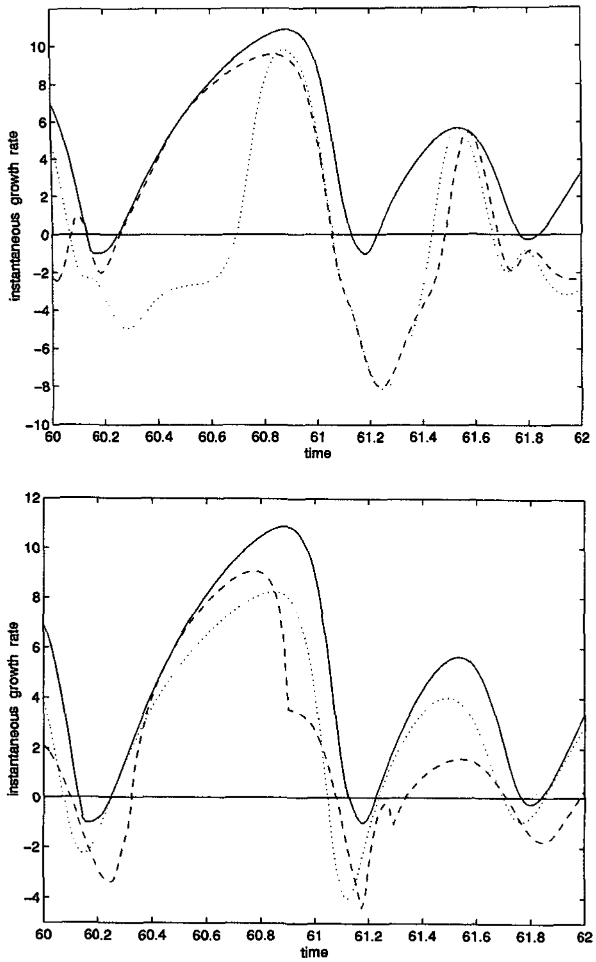

FIG. 3. (a) An interval along the trajectory of the Lorenz system with parameters $b = 8/3$, $\sigma = 10$, and $r = 28$, showing the numerical abscissa (solid), the growth rate of the Lyapunov vector (dashed), and the growth rate of the vector in the direction of the trajectory (dotted). The numerical abscissa bounds all instantaneous growth rates. Remarkably, both the trajectory and the Lyapunov vector episodically grow at rates practically indistinguishable from this theoretical maximum, and all three growth rates are episodically negative, indicating convergence of all perturbation trajectories. (b) An interval along the trajectory of the Lorenz system with parameters $b = 8/3$, $\sigma = 10$, and $r = 28$, showing the numerical abscissa (solid), the maximum growth rate of the eigenvector (dashed), and the growth rate of the vector in the direction of the optimal excitation of the Lyapunov vector (dotted). The numerical abscissa bounds all instantaneous growth rates, but both the maximum growth rate eigenvector and the optimal excitation of the Lyapunov vector episodically grow at rates practically indistinguishable from this theoretical maximum, and all three growth rates are episodically negative.

large negative third Lyapunov exponent ($\lambda_3 = -14.55$) leads to small average deviations of these vectors from the attractor, as can be seen from Figs. 5a and 5b. This example demonstrates that the perturbation of maximum instantaneous growth and the perturbation giving rise to maximum asymptotic growth do not necessarily lie on the attractor; it can also be seen from examples not shown that perturbations producing optimal growth at intermediate times do not in general lie on the attractor. This observation has implications for procedures used in the assimilation of data into forecast models in which the initial conditions are constrained to lie on the attractor. Exogenous forcing (e.g., from latent heat release) may produce perturbations of high growth rate that do not lie on the attractor.

Eigenanalysis of the instantaneous matrix **A** reveals instantaneous eigenvalues with large maximum real part excursions (cf. Fig. 4e). This maximum real part indicates the asymptotic growth that would be attained by perturbations to the system if the system were maintained in the instantaneous state that occurs at time t.

The role played in time-independent systems by the most unstable eigenvector of **A** as the asymptotic response of the system to perturbation is replaced in time-dependent systems by the Lyapunov vector. Just as is the case in time-independent systems, much of the physically relevant perturbation dynamics in time-dependent systems occurs on timescales shorter than would be required for achieving $t \to \infty$ asymptotics, so that the intermediate timescale of generalized stability theory is most often appropriate. An important advantage of generalized stability theory is that transient growth and its $t \to 0$ and $t \to \infty$ asymptotics have obvious generalizations from time-independent to time dependent systems through SVD analysis of the propagator. Modal instability theory has no such extension to time-dependent dynamics because the normal modes do not exist when the system is inhomogeneous in time, except in the special case of simple harmonic time dependence for which Floquet analysis can be applied.

The Lorenz system is useful as a model for perturbation dynamics because its attractor is embedded in three-dimensional space and as a result can be easily visualized. However, the analogy between the Lorenz system and the dynamics of the atmosphere should be pursued with caution. While the mean state of the Lorenz system $(\bar{x}, \bar{y}, \bar{z}) = (0.115, 0.114, 23.7)$ supports exponential instability (with growth rate $\lambda = 2.45$) and instantaneous realizations often are unstable (with mean growth rate $\lambda = 1.6$, cf. Fig. 4e), the atmosphere is not significantly unstable at synoptic or planetary scales and forecast error growth in the limit of long time must be understood to arise from the time dependence of the atmospheric system.

4. Parametric instability is a consequence of the nonnormality of the operator

We obtain a bound on the Lyapunov exponent by considering the evolution of $r(t) = \|\mathbf{u}(t)\|^2$, which is easily seen to obey

$$\frac{dr}{dt} = \mathbf{u}^\dagger(\mathbf{A}(t) + \mathbf{A}^\dagger(t))\mathbf{u}, \qquad (10)$$

implying

$$2\int_0^t \lambda_{\min}(s)ds \leq \ln\left(\frac{r(t)}{r(0)}\right) \leq 2\int_0^t \lambda_{\max}(s)ds, \quad (11)$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues of the Hermitian operator $(\mathbf{A}(t) + \mathbf{A}^{\dagger}(t))/2$. This leads to the following bound for the Lyapunov exponent:

$$\limsup_{t\to\infty} \frac{\int_0^t \lambda_{\min}(s)ds}{t} \leq \lambda \leq \limsup_{t\to\infty} \frac{\int_0^t \lambda_{\max}(s)ds}{t}. \quad (12)$$

For autonomous dynamical systems the right inequality leads to the energy bound of Joseph (1976), which is typically indicative only of perturbation growth for very small times. It is possible for time-dependent operators to sustain the instantaneous growth rate predicted by the numerical abscissa leading in the asymptotic limit to instability, despite having stable eigenvalues at each instant of time. This destabilization is referred to in the literature as parametric instability and is exemplified by the Mathieu equation commonly associated with destabilization of the harmonic oscillator by sinusoidal perturbations of its restoring force. An atmospheric example of harmonic parametric destabilization is the generation of gravity waves by the time-dependent Brunt–Väisälä frequency (Orlanski 1973). Despite its at most neutral stability at each time instant, the time-dependent operator associated with the perturbed harmonic oscillator is unstable for specific intervals of amplitude and frequency of restoring force perturbation. It is immediate from (12) that nonnormality of the evolution operator is a necessary condition for parametric asymptotic instability of such systems with stable eigenvalues since if the operator were at each instant asymptotically stable and normal it would necessarily also have negative Lyapunov exponents.

As an example of the role of nonnormality in parametric instability, consider the harmonic oscillator with time-dependent restoring force:

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2(t) & -2\gamma \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix}. \quad (13)$$

The operator in (13) has commutator

$$\mathbf{A}\mathbf{A}^{\dagger} - \mathbf{A}^{\dagger}\mathbf{A} = \begin{pmatrix} 1-\omega^4(t) & -2\gamma(1+\omega^2(t)) \\ 2\gamma(1+\omega^2(t)) & 1-\omega^4(t) \end{pmatrix}, \quad (14)$$

indicating that $\mathbf{A}$ is nonnormal unless both $\gamma = 0$ and ω constant and equal to unity. If the frequency is constant and $\gamma = 0$, we are at liberty to rescale time by $1/\omega$ to make $\omega = 1$ and render the operator formally normal in the L_2 norm, which for this scaling coincides with the energy norm. This rescaling is equivalent to the coordinate transformation $\tilde{x} = x$, $\tilde{v} = v/\omega$. But such a transformation with constant ω cannot succeed in making the operator uniformly normal when the operator is time dependent, leaving open the possibility of a positive Lyapunov exponent according to (12), given that the maximum eigenvalue of $(\mathbf{A}^{\dagger} + \mathbf{A})/2$ is $\lambda_{\max} = -\gamma + \sqrt{\gamma^2 + (1-\omega^2)^2/4} > 0$. It can be shown that not only does this destabilization occur for time-dependent ω but that it persists even when $\gamma > 0$, in which case the operator is stable at each time instant.

This synergism of nonnormality and time dependence leading to asymptotic instability can be succinctly demonstrated by considering a discontinuous change in ω between ω_1 and ω_2 every T units of time. With $\gamma = 0$, the propagator after a full period takes the form

$$\mathbf{\Phi}_{2T} = \prod_{i=1,2}\left[\mathbf{I}\cos(\omega_i T) + \mathbf{A}_i\frac{\sin(\omega_i T)}{\omega_i}\right], \quad (15)$$

with $\mathbf{I}$ the identity matrix and

$$\mathbf{A}_i = \begin{pmatrix} 0 & 1 \\ -\omega_i^2 & 0 \end{pmatrix}. \quad (16)$$

The first Lyapunov exponent can be readily calculated from

$$\lambda = \lim_{n\to\infty}\frac{\ln(\|\mathbf{\Phi}_{2T}^n\|)}{2nT}, \quad (17)$$

where n is the number of periods, and it can be verified that positive Lyapunov exponents are obtained as shown in Fig. 6. The reason for the instability is that transient growth instigated at the starting time is continued with further growth when the switch to the second operator takes place, forestalling the decay that would occur in the autonomous case. This synergism fails when the switchover takes place near an integer multiple of one-half the natural period of one of the oscillations. It is important to realize that a finite pulsing period T is necessary for destabilization of this oscillator. As $T \to 0$, the Lyapunov exponent vanishes.

It is natural to inquire whether this support of parametric instability by nonnormal systems with periodic parameter modulation leads also to asymptotic instability for parameter modulations of more general form. One extreme limit of parametric modulation is stochastic modulation of the system's parameters, and it is remarkable that under the broadest assumptions stochastic modulation leads to asymptotic instability (Carrier 1970; Has'minskii 1980; Arnold and Kliemann 1983; Arnold et al. 1986; Colonius and Kliemann 1993).

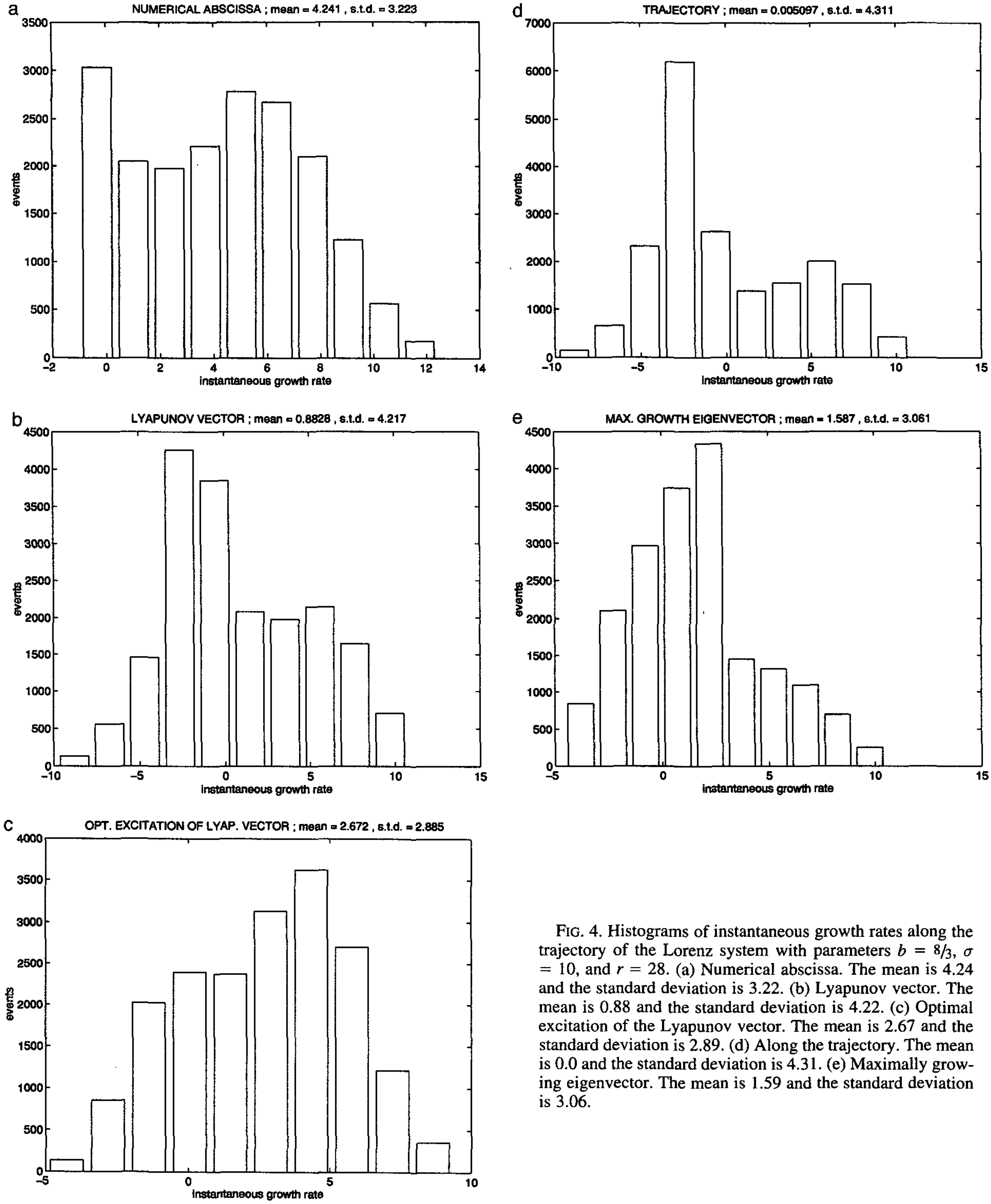

FIG. 4. Histograms of instantaneous growth rates along the trajectory of the Lorenz system with parameters $b = 8/3$, $\sigma = 10$, and $r = 28$. (a) Numerical abscissa. The mean is 4.24 and the standard deviation is 3.22. (b) Lyapunov vector. The mean is 0.88 and the standard deviation is 4.22. (c) Optimal excitation of the Lyapunov vector. The mean is 2.67 and the standard deviation is 2.89. (d) Along the trajectory. The mean is 0.0 and the standard deviation is 4.31. (e) Maximally growing eigenvector. The mean is 1.59 and the standard deviation is 3.06.

This universal instability of nonautonomous dynamical systems can be most easily understood by a modification for our purpose of an example originally due to Zeldovich et al. (1984), which served as an illustration of a theorem on the product of random matrices by Fustenberg (1963). We consider a dynamical system governed by a nonnormal 2×2 matrix constructed to have zero trace and imaginary eigenvalues $\lambda = (i, -i)$ but with column eigenvector matrix:

$$\mathbf{E}_j = \mathbf{R}_{\phi_j}\begin{pmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{pmatrix}, \tag{18a}$$

with

$$\mathbf{R}_{\phi_j} = \begin{pmatrix} \cos\phi_j & -\sin\phi_j \\ \sin\phi_j & \cos\phi_j \end{pmatrix}, \tag{18b}$$

corresponding to rotation[2] by an angle ϕ_j. The eigenvectors themselves subtend an angle θ so that these eigenvectors are nonorthogonal for $\theta \neq \pi/2$. A dynamical matrix with these properties is easily constructed:

$$\mathbf{A}_j = \mathbf{E}_j\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\mathbf{E}_j^{-1} = \mathbf{R}_{\phi_j}\begin{pmatrix} i & -2i\cot\theta \\ 0 & -i \end{pmatrix}\mathbf{R}_{\phi_j}^{\dagger}. \tag{19}$$

It is immediate that $\mathbf{A}_j$ is neutrally stable for any orientation ϕ_j and any nonnormality θ. However, if ϕ_j is a random variable uniformly distributed on $[0, \pi/2]$ and sampled at an interval of time δt, then the resulting dynamical system will be shown below to be asymptotically unstable if it is nonnormal ($\theta \neq \pi/2$), despite the stability of $\mathbf{A}_j$ at each t_j.

The propagator of the system at time $t_n = n\delta t$ is

$$\mathbf{\Phi}_{[t_n,0]} = \prod_{j=1}^{n} \mathbf{T}_j, \tag{20a}$$

which results from application of n random $\mathbf{T}_j$, each of the form

$$\mathbf{T}_j = e^{\mathbf{A}_j\delta t} = \mathbf{R}_{\phi_j}\begin{pmatrix} e^{i\delta t} & -\cot\theta(e^{i\delta t} - e^{-i\delta t}) \\ 0 & e^{-i\delta t} \end{pmatrix}\mathbf{R}_{\phi_j}^{\dagger}. \tag{20b}$$

Each $\mathbf{T}_j$ has $\det(\mathbf{T}_j) = 1$ and the SVD decomposition of $\mathbf{T}_j = \mathbf{U\Sigma V}^\dagger$ has singular values $(\sigma, 1/\sigma)$ with $\sigma \approx 1 + 2\cot\theta\delta t$ for small δt. If the unitary column vectors of $\mathbf{V}$, $\mathbf{v}_{1,2}$ are used as a basis at time t_j, the system advances this basis over the succeeding time interval δt according to

$$\mathbf{T}_j\mathbf{v}_1 = \sigma\mathbf{u}_1$$

$$\mathbf{T}_j\mathbf{v}_2 = \frac{1}{\sigma}\mathbf{u}_2, \tag{21}$$

with $\mathbf{u}_{1,2}$ the orthogonal column vectors of $\mathbf{U}$. Therefore, a normalized state vector at time t_j, denoted by $\mathbf{x}_j$ and at an angle ϕ_j to the first singular vector of $\mathbf{T}_j$,

$$\mathbf{x}_j = \cos\phi_j\mathbf{v}_1 + \sin\phi_j\mathbf{v}_2, \tag{22}$$

[2] Rotation matrices are unitary, that is, $\mathbf{R}_{\phi_j}\mathbf{R}_{\phi_j}^{\dagger} = \mathbf{I}$, with dagger denoting the hermitian transpose.

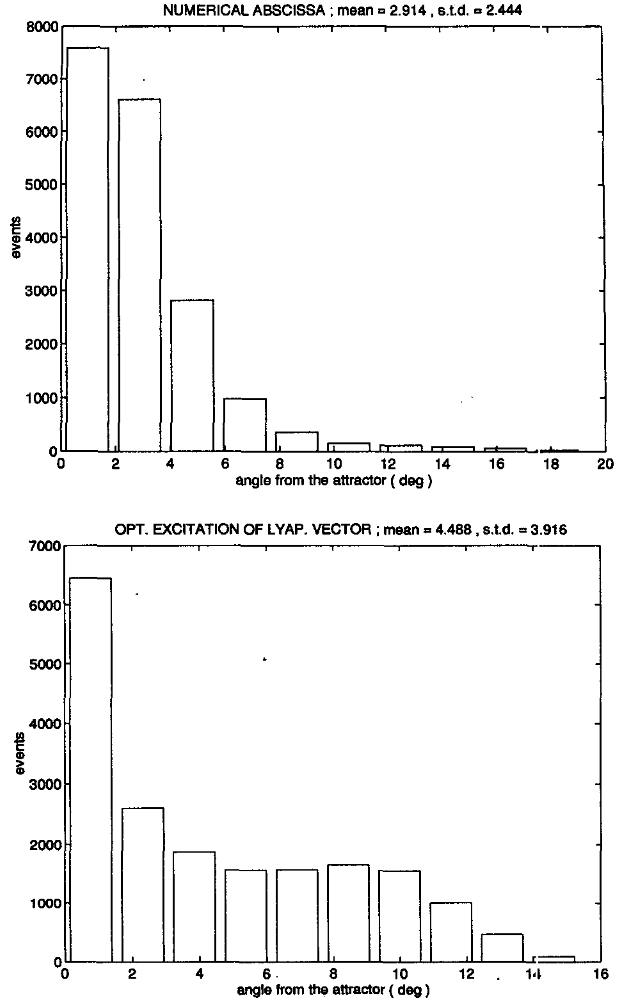

FIG. 5. For the Lorenz system with $b = 8/3$, $\sigma = 10$, and $r = 28$ the (a) histogram of the instantaneous angle between the local plane of the attractor and the eigenvector associated with the numerical abscissa. The mean is 2.91 degrees, and the standard deviation is 2.44 degrees. (b) Histogram of the instantaneous angle between the local plane of the attractor and the vector associated with optimal excitation of the Lyapunov vector. The mean is 4.49 degrees, and the standard deviation is 3.92 degrees.

is advanced over the succeeding interval δt to $\mathbf{X}_{j+1} = \mathbf{T}_j\mathbf{x}_j$, with incremental magnification

$$G_j = \frac{\|\mathbf{X}_{j+1}\|}{\|\mathbf{x}_j\|} = \left(\sigma^2\cos^2\phi_j + \frac{\sin^2\phi_j}{\sigma^2}\right)^{1/2}, \tag{23}$$

with $G_j > 1$ for

$$|\cos\phi_j| > \frac{1}{\sqrt{1+\sigma^2}}. \tag{24}$$

Because our nonnormal system is chosen to have $\sigma > 1$, (24) is satisfied with probability greater than 0.5 for uniformly distributed ϕ_j, and we conclude that the state vector grows more often than it decays under transfor-

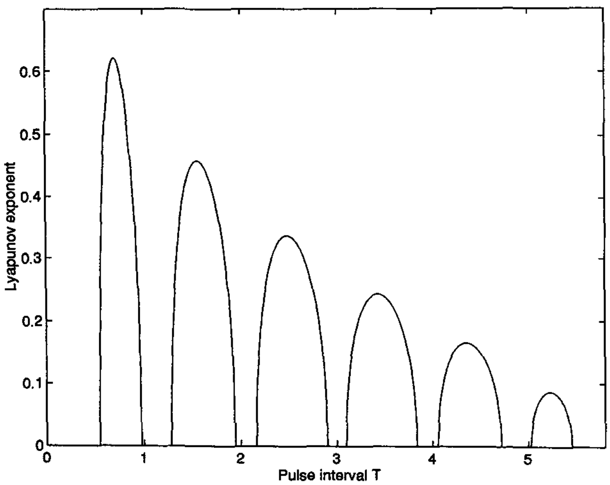

FIG. 6. Lyapunov exponent for the switched oscillator example as a function of the period of the switch T nondimensionalized by π/ω_2 for $\omega_1 = 0.5$ and $\omega_2 = 3$. Although instantaneously neutral at all times, the oscillator is destabilized except in the neighborhood of integer values of the switching period.

mation by randomly oriented $\mathbf{T}_j$s. But in order to establish exponential growth in the mean, we must calculate the expectation of the incremental magnifications over all possible angles ϕ_j for each θ. This is because the Lyapunov exponent can be found as the asymptotic rate of expansion of an initial unit vector $\mathbf{x}_0$ in terms of its normalized vectors $\mathbf{x}_j$ after j mappings according to

$$\lambda = \lim_{n\to\infty} \frac{\ln(\|\mathbf{\Phi}_{[t_n,0]}\mathbf{x}_0\|)}{n\,\delta t}$$

$$= \lim_{n\to\infty} \frac{\ln(\|\mathbf{T}_n\mathbf{x}_{n-1}\| \cdots \|\mathbf{T}_1\mathbf{x}_0\|)}{n\,\delta t} = \lim_{n\to\infty} \frac{\sum_{j=1}^{n} \ln G_j}{n\,\delta t}. \quad (25)$$

From which λ is seen to be the mean of $\ln(G_j)/\delta t$ denoted by $\langle \ln(G_j)\rangle/\delta t$, where the average is taken over all uniformly distributed angles ϕ_j. The resulting Lyapunov exponent as a function of θ is shown in Fig. 7. It is clear from this figure that the composite action of random matrices with unit determinant is to lead to exponential expansion of an initial unit vector.

In the limit $\delta t \to 0$, the Lyapunov exponent calculated according to (25) vanishes, as was also the case for the deterministic pulsed harmonic oscillator. This result can be seen from a Taylor expansion of the propagator (20b) to be a consequence of the fact that by construction trace $(\mathbf{A}_j) = 0$. The Lyapunov exponent as $\delta t \to 0$ is given for any unit vector $\mathbf{x}$ by

$$\lambda = \frac{\langle \ln\|(\mathbf{I} + \mathbf{A}_j\delta t)\mathbf{x}\|\rangle}{\delta t}$$

$$= \left\langle \mathbf{x}^\dagger \frac{\mathbf{A}_j^\dagger + \mathbf{A}_j}{2} \mathbf{x} \right\rangle + O(\delta t), \quad (26)$$

where $\mathbf{I}$ is the identity. Because trace $(\mathbf{A}_j) = 0$ the expected value: $\langle \mathbf{x}^\dagger(\mathbf{A}_j^\dagger + \mathbf{A}_j)\mathbf{x}\rangle$ vanishes and the contribution to the Lyapunov exponent comes only from the $O(\delta t)$ term in (26), which itself vanishes in the limit $\delta t \to 0$. While this example illustrates the origin in nonnormality of the destabilization of stochastic dynamical systems, stochastic destabilization in the continuous limit $\delta t \to 0$ requires, as we will show below, further assumptions about the noise process in the limit $\delta t \to 0$.

5. Generalized parametric instability of atmospheric flows

The tangent linear system that governs evolution of small initial perturbations to a solution trajectory of the generally nonlinear equations of motion of a dynamical system can be cast as a nonautonomous linear system of form (1). We can decompose this nonautonomous operator as $\mathbf{A}(t) = \bar{\mathbf{A}} + \mathbf{A}'(t)$, where $\bar{\mathbf{A}}$ is the time-mean operator which is assumed to exist, and $\mathbf{A}'(t)$ is the temporal variation of the operator arising from externally and internally produced deviations from the mean state. We will model the time-dependent operator using a noise process

$$\frac{d\mathbf{u}}{dt} = \bar{\mathbf{A}}\mathbf{u} + \epsilon \sum_i \xi_i(t)\mathbf{B}_i\mathbf{u}, \quad (27)$$

where ϵ is the rms magnitude of the fluctuations, $\mathbf{B}_i$ are time-independent noise matrices, and the noise pro-

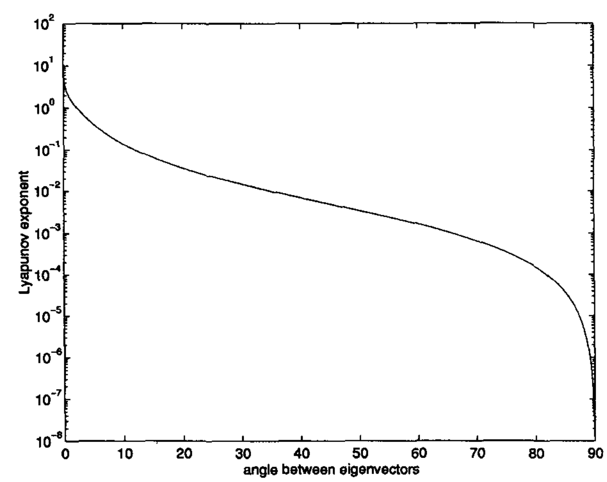

FIG. 7. Lyapunov exponent of the illustrative simple 2×2 dynamical system produced by random neutrally stable matrices with trace 0, as a function of the angle θ subtended by the eigenvectors. Although each individual matrix produces dynamics with 0 spectral abscissa, the nonautonomous dynamical system produced by sampling these neutrally stable matrices every unit interval of time has a positive Lyapunov exponent. It is seen in this figure that the Lyapunov exponent is an increasing function of θ, which is a measure of the nonnormality of the operator (it is only 0 when $\theta = 0$ and the system is normal).

cesses $\xi_i(t)$ are identically distributed and delta correlated with zero mean:

$$\langle \xi_i(t)\xi_j(s)\rangle = \delta_{ij}\delta(t-s), \qquad (28)$$

where $\langle \cdot \rangle$ denotes an ensemble average.

In order to address sensitivity to initial conditions, we must first decide on the definition of stability to be used (cf. Has'minskii 1980). Two candidate measures are sample asymptotic stability, which is equivalent in a stochastic system to the almost sure existence of a negative Lyapunov exponent λ [cf. (6)]; and mean square stability, which is equivalent to a negative Lyapunov exponent associated with the second moment, that is,

$$\mu_2 = \limsup_{t\to\infty} \frac{\ln\langle \|\mathbf{\Phi}(t)\mathbf{u}(0)\|^2\rangle}{t} < 0. \qquad (29)$$

In deterministic systems these two measures are equivalent but this equivalence is lost in purely stochastic systems for which it is only necessary that $\mu_2 \geqslant 2\lambda$ (Arnold 1984). It follows that in order to establish sample instability, it is necessary to demonstrate the existence of a positive Lyapunov exponent directly because the often more easily established instability of the second moment need not imply a positive Lyapunov exponent.

A physical dynamical system of the form (27) has necessarily a positive Lyapunov exponent for noise of sufficient magnitude ϵ, provided the dimensionality of the system is greater or equal to 2 and the noise matrices, $\mathbf{B}_i$, are neither skew-symmetric nor commute with the deterministic operator, $\bar{\mathbf{A}}$ (Has'minskii 1980; Colonius and Kliemann 1993; for Hamiltonian systems refer to Fustenberg 1963). This universal stochastic instability resulting from the nonnormality of the dynamical operator underlies the asymptotic increase in separation of initially adjacent trajectories in atmospheric flows. Moreover, it can be verified that increasing the degree of nonnormality of the deterministic background state operator $\bar{\mathbf{A}}$ reduces the noise amplitude, ϵ, required for asymptotically unstable perturbation growth.

Stochastic dynamical systems with multiplicative noise require careful physically informed analysis to properly account for the degree of correlation between the state of the system and the noise process at each time instant. White noise is a mathematical idealization that can be interpreted either as associated with a nonvanishing correlation between the state and the noise process or with these processes being independent. The former results in the physically appropriate continuous limit of a discrete process and is referred to as the Stratonovich interpretation, the latter is appropriate to discrete Markov processes and is referred to as the Ito interpretation (Arnold 1992).

Multiplicative noise in the physically appropriate continuous system implies correlation between the noise process and the state:

$$\langle \xi_i \mathbf{u}\rangle \neq \mathbf{0}. \qquad (30)$$

Physical linear stochastic systems like (27) can be transformed to equivalent Markov systems in which there is assumed to be no correlation between the white noise and the state vector by constructing the equivalent stochastic dynamical system with the augmented deterministic operator

$$\mathbf{A}_{\mathrm{I}} = \bar{\mathbf{A}} + \frac{\epsilon^2}{2}\sum_i \mathbf{B}_i^2, \qquad (31)$$

with $\langle \xi_i \mathbf{u}\rangle = \mathbf{0}$ (this is the Ito correction; Arnold 1992).

In a numerical solution of stochastic differential equations the correlation between the multiplicative noise process and the state vector at each instant is reflected in the discretization; in addition, the correlation of the noise processes themselves is also reflected in the discretization. Therefore, in order that the solution of the model system in the limit $\delta t \to 0$ approach, the physical solution care must be taken in the discretization process (Kloeden and Platen 1992). We have found that the propagator for physical linear stochastic systems (27) can be cast in the computationally advantageous form

$$\mathbf{\Phi}_{[t,t_0]} = \lim_{\delta t\to 0}\prod_\alpha \exp\left[\left(\bar{\mathbf{A}} + \epsilon\sum_i \frac{\xi_i(t^\alpha+\delta t)+\xi_i(t^\alpha)}{2(\delta t)^{1/2}}\mathbf{B}_i\right)\delta t\right], \qquad (32)$$

with t^α equally spaced on the interval between the initial and final time and ξ_i normally distributed as $N(0, 1)$. This formulation preserves the physically appropriate Stratonovich interpretation of (27) and the delta correlation of the noise process and is the generalization of the deterministic propagator (4).

The continuous limit is obtained as $\delta t \to 0$ because dividing the discrete noise variables $\xi(t^\alpha)$ by $(\delta t)^{1/2}$ preserves in the limit the delta function correlation of the noise processes [cf. (28)]. It is this scaling of the noise processes that enables traceless dynamical systems to exhibit stochastic parametric destabilization in the limit $\delta t \to 0$, while examples in the previous section resulted in vanishing Lyapunov exponents in this limit. Consider the example of the stochastically perturbed damped simple harmonic oscillator, which in the notation of (27) has mean operator

$$\bar{\mathbf{A}} = \begin{pmatrix} 0 & 1 \\ -1 & -2\gamma \end{pmatrix}, \qquad (33)$$

where γ is the coefficient of damping, and noise matrix

$$\mathbf{B} = \begin{pmatrix} 0 & 0 \\ -\epsilon & 0 \end{pmatrix}, \qquad (34)$$

where ϵ is the rms amplitude of the noise. The noise matrix satisfies $\mathbf{B}^2 = 0$ so that in this case the Ito correction (31) vanishes and there is no correlation between the noise and state vectors. We proceed as in (26) to obtain the small δt limit of the Lyapunov exponent, making use of the expression for the stochastic propagator as in (32):

$$\lambda = \frac{\left\langle \ln\left\|\left[\mathbf{I} + \left(\bar{\mathbf{A}} + \frac{\xi(t^\alpha + \delta t) + \xi(t^\alpha)}{2(\delta t)^{1/2}}\mathbf{B}\right)\delta t\right]\mathbf{x}\right\|\right\rangle}{\delta t}$$

$$= \left\langle \mathbf{x}^\dagger \frac{\bar{\mathbf{A}}^\dagger + \bar{\mathbf{A}}}{2}\mathbf{x}\right\rangle + \left\langle \mathbf{x}^\dagger \frac{\mathbf{B}^\dagger\mathbf{B}}{4}\mathbf{x}\right\rangle + O(\delta t). \quad (35)$$

In contrast to the limiting behavior of the Lyapunov exponent of the piecewise continuous random dynamical system shown in (26), the $1/\sqrt{\delta t}$ scaling of the noise process maintains the non-normality of the infinitesimal propagators in (35), preserving asymptotic instability in the limit $\delta t \to 0$, as shown in Fig. 8.

Stochastic parametric destabilization in an atmospheric context can be demonstrated using the three-layer model of the baroclinic midlatitude atmospheric jet. The three-layer model is chosen because the tangent linear system associated with the two-layer model results in a noise matrix $\mathbf{B}_i$, which commutes with the mean operator $\bar{\mathbf{A}}$. When $\bar{\mathbf{A}}$ and $\mathbf{B}_i$ commute, multiplicative noise cannot alter the stability properties of the mean operator because a change to normal coordinates simultaneous diagonalizes both $\bar{\mathbf{A}}$ and $\mathbf{B}_i$ and such a normal system cannot be destabilized by noise. This result explains the lack of noise induced instability in the canonical two-layer and Eady neutral mode models, the latter of which was studied by Hart (1971) (Ioannou 1996, unpublished manuscript).

In the three-layer model the geostrophic streamfunction is assumed to be of harmonic form in three vertical layers, $\psi_i(t)e^{ikx+ily}$, with x the zonal and y the meridional directions. The dynamical equation for the three components of the streamfunction is

$$\frac{d}{dt}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \mathbf{P}^{-1}\left[-ik\left(\mathbf{A} + \frac{\beta}{\alpha^2}\mathbf{I}\right) - r\mathbf{I}\right]\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}, \quad (36)$$

with

$$\mathbf{A} = \begin{pmatrix} -(U_1 + U_2\lambda^2/\alpha^2) & U_1\lambda^2/\alpha^2 & 0 \\ U_2\lambda^2/\alpha^2 & -(U_2 + (U_1 + U_3)\lambda^2/\alpha^2) & U_2\lambda^2/\alpha^2 \\ 0 & U_2\lambda^2/\alpha^2 & -(U_3 + U_2\lambda^2/\alpha^2) \end{pmatrix},$$

$$\mathbf{P} = \begin{pmatrix} -(1 + \lambda^2/\alpha^2) & \lambda^2/\alpha^2 & 0 \\ \lambda^2/\alpha^2 & -(1 + \lambda^2/\alpha^2) & \lambda^2/\alpha^2 \\ 0 & \lambda^2/\alpha^2 & -(1 + \lambda^2/\alpha^2) \end{pmatrix}, \quad (37)$$

and $\mathbf{I}$ the identity matrix (Staley 1986).

The top layer is indexed first and each layer velocity is considered to be of the form $U_i = \bar{U}_i + \epsilon\xi_i(t)$, with ξ_i being an independent white noise process as in (28). The mean velocities (m s^{-1}) have been expressed in terms of the mean shear over each layer, ΔU, as $\bar{U}_i = 10 + (3 - i)\Delta U$ $(i = 1, 2, 3)$. The total horizontal wavenumber is $\alpha = (k^2 + l^2)^{1/2}$; the meridional wavelength $l = \pi/Y_c$ corresponds to the gravest mode in a channel of meridional extent $Y_c = 4000$ km. The Rossby deformation wavenumber is $\lambda = f_0/(\sigma^{1/2}\delta p)$ with $f_0 = 10^{-4}$ s^{-1} and $\beta = 1.65 \times 10^{-11}$ m^{-1} s^{-1}, the midlatitude value of Coriolis parameter and its northward derivative, respectively, and $\sigma = 2 \times 10^{-6}$ Pa^{-2} s^{-2} m^{-2} the stratification parameter typical of the troposphere. Equally spaced pressure surfaces have been taken with $\delta p = 10^5/3$ Pa. The coefficient of potential vorticity damping is denoted by r. Equations (36) and (37) are presented in nondimensional form. Time has been nondimensionalized by $T_d = 1$ day; horizontal lengths by L_x, the length of the latitude circle at 45°; and velocities by L_x/T_d.

The first Lyapunov exponent as a function of shear ΔU and rms noise variance ϵ for global wavenumber 11 and nondimensional potential vorticity damping $r = 0.2$ corresponding to a damping time of 5 days is shown in Fig. 9. Note the destabilization of this system as noise increases. The threshold noise required to destabilize the system is seen to gradually decrease as the nonnormality of the mean operator, indicated by the shear, increases. A typical midlatitude jet shear corresponds in this three-layer model to $\Delta U = 10$ m s^{-1}. This simplified model of the midlatitude atmosphere suggests existence of a positive Lyapunov exponent for rms temporal fluctuations of the order of 10%, while for 30% fluctuations a Lyapunov exponent of the order of 1/5 day^{-1} is expected.

One implication of this result is that the atmosphere maintained at sufficiently low variability would have negative Lyapunov exponents, corresponding to a decrease rather than increase of asymptotic errors. Nev-

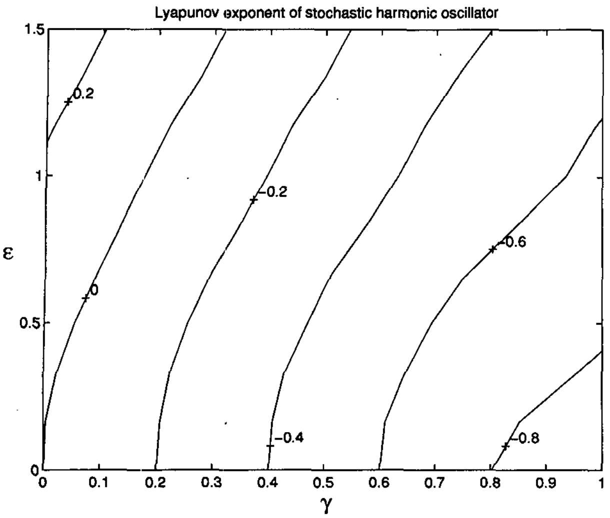

FIG. 8. Lyapunov exponent of the simple harmonic oscillator with randomly modulated perturbative restoring force as a function of the damping coefficient γ and the rms level of the random perturbation ϵ.

ertheless, the high degree of nonnormality of the mean flow would still result in rapid error growth on the forecast timescale; however, under conditions of low temporal variability these perturbations would after a period of transient growth eventually decay. Another facet of this result concerns the interplay among damping, nonnormality, and temporal variability in determining asymptotic error growth. Damping of sufficient magnitude is able to asymptotically stabilize the Lyapunov exponents, even with high temporal variability, and an increase in the nonnormality of the background flow, corresponding to an increase in baroclinic shear in this model, is found to greatly reduce the temporal variability required to destabilize asymptotic error growth.

6. Conclusions

The stability theory of autonomous operators plays a central role in fluid dynamics. Generalized to include transient growth processes, the stability theory of autonomous operators addresses diverse phenomena including formation of cyclones, the growth of errors on synoptic forecast timescales, and, with the inclusion of additive noise, the statistically steady state of fully turbulent shear flow. There is, however, a class of physical phenomena the stability analysis of which requires inclusion of time dependence. One such problem is that of the asymptotic divergence of initially adjacent trajectories in a turbulent dynamical system. In this work the generalized stability theory of autonomous operators has been extended to address the stability of time-dependent operators arising in this and similar problems.

We find that most dynamical systems are destabilized by temporal variation of their underlying dynamical operators and that this destabilization requires progressively less variation of the operator the greater the nonnormality of the underlying mean operator and the less the system is damped. While certain forms of time dependence can be contrived that do not produce destabilization and special spatial correlations of the noise process can be formed that defeat the destabilization, *it is quite generic to find that multiplicative white noise effectively destabilizes a dynamical system.*

The short and intermediate time perturbation growth rates that are of greatest physical significance in autonomous systems are also found to typically exceed asymptotic growth rates in nonautonomous systems, suggesting that the asymptotic exponential growth of the Lyapunov vector is determinative of trajectory divergence only at timescales long compared to that of feasible deterministic forecast. This timescale for emergence of the Lyapunov vector as the dominant error growth structure depends on the degree of nonnormality of the underlying operator and the magnitude of the variance of the operator relative to dissipation. An estimate of $1/5$ day^{-1} for this growth rate made above implies timescales of the order of one month for asymptotic emergence of the Lyapunov vector in atmospheric error dynamics.

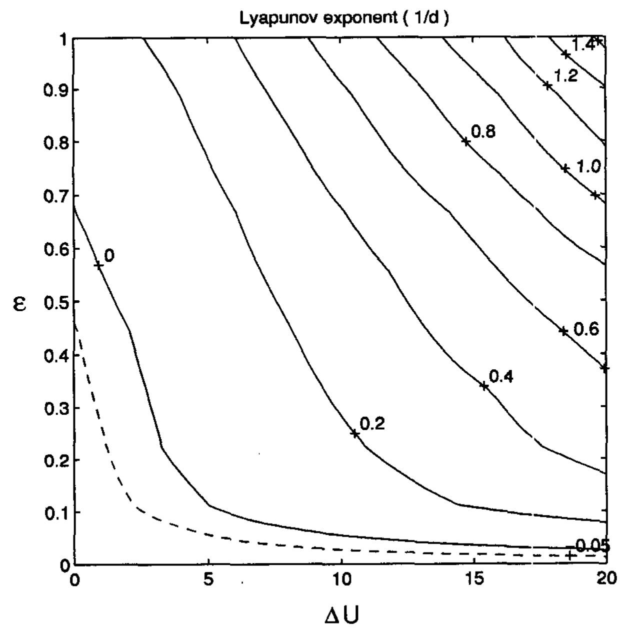

FIG. 9. The Lyapunov exponent for the baroclinic three-layer model as a function of shear Δu and strength of the multiplicative noise forcing ϵ. The global zonal wavenumber is 11, the meridional wavenumber is $l = \pi/4$, and dissipation corresponding to $r = 0.2$ has been included. A finite magnitude of both shear and parametric forcing is required to produce a positive Lyapunov exponent. The Lyapunov exponent increases with both the shear and the magnitude of multiplicative noise forcing.

The universal destabilization of dynamical systems by multiplicative noise analyzed in this work and demonstrated with the example of the three-layer model of the midlatitude jet has wide implications for the growth and maintenance of variance in randomly forced and turbulent systems. While coherent parametric modulation such as envisioned in the harmonic variation of the restoring force in the harmonic oscillator leading to instability in the Mathieu equation is specialized and of restricted application to physical problems, incoherent variation of parameters is characteristic of turbulent dynamics and the associated instabilities are correspondingly more likely to be of importance in physical problems.

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REFERENCES

Ahlquist, J. E., and J. K. Sivillo, 1995: Short range growth of linear perturbations. Preprints, *Tenth Conf. on Atmospheric and Oceanic Waves and Stability,* Big Sky, MT, Amer. Meteor. Soc., 241.

Arnold, L. G., 1984: Formula connecting sample and moment stability of linear stochastic systems. *SIAM J. Appl. Math.,* **44,** 793–802.

——, 1992: *Stochastic Differential Equations: Theory and Applications.* Krieger, 228 pp.

——, and W. Kliemann, 1983: Qualitative theory of stochastic systems. *Probabilistic Analysis and Related Topics.* Vol. III, A. T. Bharucha-Reid, Ed., Academic Press, 2–73.

——, G. Papanicolaou, and V. Wihstutz, 1986: Asymptotic analysis of the Lyapunov exponent and rotation of the random oscillator and applications. *SIAM J. Appl. Math.,* **46,** 427–450.

Borges, M. D., and D. L. Hartmann, 1992: Barotropic instability and optimal perturbations of observed nonzonal flows. *J. Atmos. Sci.,* **49,** 335–354.

Buizza, R., and T. N. Palmer, 1995: The singular vector structure of the atmospheric global circulation. *J. Atmos. Sci.,* **52,** 1434.

Carrier, G., 1970: Stochastically driven dynamical systems. *J. Fluid Mech.,* **44,** 249–264.

Coddington, E. A., and N. Levinson, 1955: *Theory of Ordinary Differential Equations.* McGraw Hill, 429 pp.

Colonius, F., and W. Kliemann, 1993: Minimal and maximal Lyapunov exponents of bilinear control systems. *J. Differential Equations,* **101,** 232–275.

DelSole, T. M., 1996: Can quasigeostrophic turbulence be modeled stochastically? *J. Atmos. Sci.,* **53,** 1617–1633.

——, and B. F. Farrell, 1995: A stochastically excited linear system as a model for quasigeostrophic turbulence: Analytic results for one- and two-layer fluids. *J. Atmos. Sci.,* **52,** 2531–2547.

Farrell, B. F., 1989: Optimal excitation of baroclinic waves. *J. Atmos. Sci.,* **46,** 1193–1206.

——, 1990: Small error dynamics and the predictability of atmospheric flows. *J. Atmos. Sci.,* **47,** 2409–2416.

——, and P. J. Ioannou, 1993a: Stochastic forcing of perturbation variance in unbounded shear and deformation flows. *J. Atmos. Sci.,* **50,** 200–211.

——, and ——, 1993b: Stochastic dynamics of baroclinic waves. *J. Atmos. Sci.,* **50,** 4044–4057.

——, and ——, 1993c: Stochastic forcing of the linearized Navier–Stokes equations. *Phys. Fluids A,* **5,** 2600–2609.

——, and ——, 1994a: Variance maintained by stochastic forcing of nonnormal dynamical systems associated with linearly stable shear flows. *Phys. Rev. Lett.,* **72,** 1188–1191.

——, and ——, 1994b: A theory for the statistical equilibrium energy spectrum and heat flux produced by transient baroclinic waves. *J. Atmos. Sci.,* **51,** 2685–2698.

——, and ——, 1995: Stochastic dynamics of the midlatitude atmospheric jet. *J. Atmos. Sci.,* **52,** 1642–1656.

Fustenberg, H., 1963: Noncommuting random products. *Trans. Amer. Math. Soc.,* **198,** 377–428.

Hart, J. E., 1971: A note on the baroclinic instability of general time-dependent basic fields of the Eady type. *J. Atmos. Sci.,* **28,** 808–809.

Hartmann, D. L., R. Buizza, and T. N. Palmer, 1995: Singular vectors: The effect of spatial scale on linear growth of disturbances. *J. Atmos. Sci.,* **52,** 3885–3894.

Has'minskii, R. Z., 1980: *Stochastic Stability of Differential Equations.* Sijthoff and Noordhoff, 341 pp.

Joseph, D. D., 1976: *Stability of Fluid Motion I.* Springer-Verlag, 282 pp.

Kloeden, P. E., and E. Platen, 1992: *Numerical Solution of Stochastic Differential Equations.* Springer-Verlag, 632 pp.

Lacarra, J.-F., and O. Talagrand, 1988: Short-range evolution of small perturbations in a barotropic model. *Tellus,* **40A,** 81–95.

Lorenz, E. N., 1965: A study of the predictability of a 28-variable atmospheric model. *Tellus,* **17,** 321–333.

Lyapunov, A. M., 1907: Probleme general de la stabilite du mouvement. *Ann. Fac. Sci. Univ. Touluse,* **9,** 203–475.

Moore, A. M., and B. F. Farrell, 1993: Rapid perturbation growth on spatially and temporally varying oceanic flows determined using an adjoint method: Application to the Gulf Stream. *J. Phys. Oceanogr.,* **23,** 1682–1702.

Orlanski, I., 1973: Trapeze instability as a source of internal gravity waves. Part I. *J. Atmos. Sci.,* **30,** 1007–1016.

Oseledec, V. I., 1968: The multiplicative ergodic theorem. The Lyapunov characteristic numbers of dynamical systems. *Trans. Moscow Math. Soc.,* **19,** 197–231.

Palmer, T. N., 1993: Extended-range atmospheric prediction and the Lorenz model. *Bull. Amer. Meteor. Soc.,* **74,** 49–65.

Penland, C., and P. Sardeshmukh, 1995: The optimal growth of sea surface anomalies. *J. Climate,* **8,** 1999–2024.

Reddy, S. C., and D. S. Henningson, 1993: Energy growth in viscous channel flows. *J. Fluid Mech.,* **252,** 209–238.

Staley, D. O., 1986: Baroclinic and barotropic instability spectra as functions of N in N-level models. *J. Atmos. Sci.,* **43,** 1817–1832.

Trefethen, L. N., A. E. Trefethen, S. C. Reddy, and T. A. Driscoll, 1993: Hydrodynamic stability without eigenvalues. *Science,* **261,** 578–584.

Yoden, S., and M. Nomura, 1993: Finite-time Lyapunov stability analysis and its application to atmospheric predictability. *J. Atmos. Sci.,* **50,** 1531–1543.

Zel'dovich, Y. B., A. A. Ruzmaikin, S. S. Molchanov, and D. D. Sokoloff, 1984: Kinematic dynamo problem in a linear velocity field. *J. Fluid Mech.,* **144,** 1–11.