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Applications of the Mathieu equation

Lawrence Ruby
Oregon Graduate Institute of Science & Technology, P.O. Box 91000, Portland, Oregon 97291-1000

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The properties of the Mathieu equation are reviewed in order to discuss some of the applications that have appeared in recent years. Those mentioned are: vibrations in an elliptic drum, the inverted pendulum, the radio frequency quadrupole, frequency modulation, stability of a floating body, alternating gradient focusing, the Paul trap for charged particles, and the mirror trap for neutral particles. © *1996 American Association of Physics Teachers.*

I. INTRODUCTION

With few exceptions, notably Morse and Feshbach,[1] and Mathews and Walker,[2] most authors of textbooks on mathematics for science and engineering choose to omit any discussion of the Mathieu equation. Until recently, this omission could be justified on the basis that the Mathieu equation had only limited scientific application. However, in recent years, many new applications have arisen. We will discuss the equation briefly, and then describe some of the scientific applications.

II. THE MATHIEU EQUATION

The Mathieu equation is a special case of a linear second-order homogeneous differential equation, such as occurs in many applications in physics and engineering. A variety of notations for the Mathieu equation exists in the literature, and we will adopt that of our primary reference, Abramowitz and Stegun,[3] which contains an extensive bibliography of early papers and books on the subject. The Mathieu equation is

$$\frac{d^2Y}{dX^2}+[a-2q\cos(2X)]Y=0, \qquad (1)$$

where "a" and "b" are real constants in physical applications. According to Floquet's theorem (proved in Refs. 1 and 2) a solution exists to Mathieu's equation of the form

$$Y_\gamma(X)=\exp\{i\gamma X\}P(X), \qquad (2)$$

where γ is a function of a and q, and $P(X)$ is a periodic function with the same period as the trigonometric function in Eq. (1), namely π. Unless γ is an integer, $Y_\gamma(-X)$ is independent of $Y_\gamma(X)$ and constitutes the second solution of the equation. If γ is an imaginary number, then the solution of Eq. (1) is unstable, i.e., infinite, at $X=\infty$ or at $X=-\infty$. If γ is a real number and is an integer, then the solution is periodic with period π or 2π; otherwise it is nonperiodic. However, if γ is a proper fraction m_1/m_2, then the solution is periodic of period at most $2\pi m_2$, but not π or 2π. In physical situations, X is often an angular variable, in which case the solution is required to repeat at intervals of $\pm 2\pi$, thereby requiring γ to be an integer. If γ is an integer, then $Y_\gamma(X)$ is proportional to $Y_\gamma(-X)$, and a second independent solution is required mathematically. As discussed in Refs. 1 and 3, the second solution in nonperiodic, and is therefore generally not useful in physical problems. The various types of solution correspond to different regions in (a,q) space. A plot of the latter is shown in Fig. 1, and other such plots are to be found in Refs. 1–3. By solving Mathieu's equation with $q=0$, is easy to see that the entire positive "a" axis must be within the stable region of solution. The curves which separate the stable and unstable regions correspond to the solutions for integer values of γ, i.e., the periodic solutions for "a" as a function of "q", are indicated in Fig. 1 as a_γ for the even solutions, or b_γ for the odd solutions. Reference 3 discusses the computation of γ, once a and q are known.

The general periodic solution to Eq. (1) is found by substituting the trial function

$$Y=\sum_0^\infty [A_k\cos(kX)+B_k\sin(kX)], \qquad (3)$$

with $B_0=0$. Then, after some manipulation, it will be evident that to satisfy the equation by making the coefficient of each cos(kx), as well as of each sin(kx), identically equal to zero, there must exist a three-term recursion relation for A_k,

$$(a-k^2)A_k-q(A_{k-2}+A_{k+2})+A_{k+2})=0, \qquad (4)$$

which holds for $k\geqslant 3$, together with $aA_0-qA_2=0$, $(a-1)A_1-q(A_1+A_3)=0$, and $(a-4)A_2 \quad -q(2A_0+A_4)=0$.

And similarly for B_k,

$$(a-k^2)B_k-q(B_{k-2}+B_{k+2})=0, \qquad (5)$$

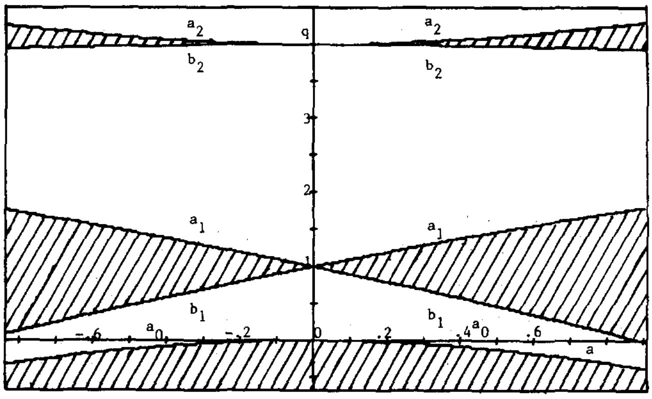

Fig. 1. Mathieu equation parameters "a" vs "q" with unstable areas shaded. Periodic solutions are curves labeled "a_n" and "b_n."

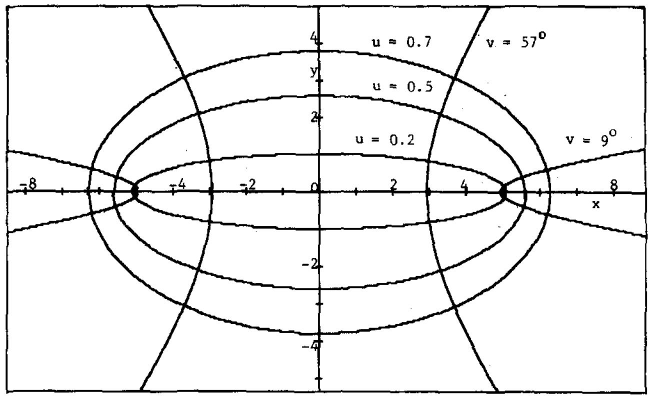

Fig. 2. Constant-coordinate curves for elliptic cylinder coordinates.

which holds for $k \geqslant 3$, together with $(a-4)B_2 - qB_4 = 0$, $(a-1)B_1 + q(B_1 - B_3) = 0$, and $(a-4)B_2 - qB_4 = 0$.

Starting from the known values of A_2/A_0, A_3/A_1, B_4/B_2, and B_3/B_1, Eqs. (4) and (5) may be solved as a continued fraction for "a" as a function of "q" by employing a method of successive approximations.[1,2] Although the method is tedious if applied by hand, the series data given in Ref. 3 can be obtained in this fashion. The required number of terms depends sensitively on the absolute value of q. Actually, both Eqs. (4) and (5) have multiple roots, depending on the value of k used to begin the continued fraction. For Eq. (4), the roots for a as a function of q are denoted as a_0, a_1, a_2, etc., and for Eq. (5) as b_1, b_2, b_3, etc. For example, to find a_0, the continued fraction to be solved is

$$\frac{a}{q} = \frac{2q}{(a-2^2) - \cfrac{q^2}{(a-4^2) - \cfrac{q^2}{(a-6^2) - \cdots}}}. \tag{6}$$

Correspondingly, the value of a_1 is found by solving

$$\frac{a-4}{q} - \frac{2q}{a} = \frac{q}{(a-4^2) - \cfrac{q^2}{(a-6^2) - \cfrac{q^2}{(a-8^2) - \cdots}}}. \tag{7}$$

By virtue of their symmetry properties, the various even and odd periodic solutions of Mathieu's equation are known as "Mathieu functions," and denoted as

$$\begin{aligned} ce_{2n}(X,q) &= \sum_0^\infty A_{2k} \cos(2kX), \\ ce_{2n+1}(X,q) &= \sum_0^\infty A_{2k+1} \cos([2k+1]X), \\ se_{2n}(X,q) &= \sum_0^\infty B_{2k} \sin(2kX), \\ se_{2n+1}(X,q) &= \sum_0^\infty B_{2k+1} \sin([2k+1]X). \end{aligned} \tag{8}$$

A. Vibrations in an elliptic drum

Historically, it was Mathieu's investigations of vibrations in an elliptic drum[3] which led to interest in what is now termed "Mathieu's equation." Elliptic cylinder coordinates[4] are defined by the transformations

$$x = \rho \cosh(u)\cos(v), \quad y = \rho \sinh(u)\sin(v), \quad z = z. \tag{9}$$

As shown in Fig. 2, the curves u=constant are confocal ellipses, and the curves v = constant are orthogonal hyperbolas. If the time dependence in the wave equation is eliminated through the substitution

$$\psi\{u,v,z,t\} = W\{u,v,z\}\exp(\pm ikt), \tag{10}$$

the result is a Helmholz equation of the form

$$\nabla^2 W + k^2 W = 0. \tag{11}$$

Substituting in Eq. (11) a solution of the form

$$W\{u,v,z\} = f\{u\}g\{v\}\phi\{z\} \tag{12}$$

gives for the equations in $f\{u\}$ and $g\{v\}$

$$a = (1/f)d^2f/du^2 + (k^2 - c)(\rho^2/2)\cosh(2u), \tag{13}$$

$$a = -(1/g)d^2g/dv^2 + (k^2 - c)(\rho^2/2)\cos(2v), \tag{14}$$

where a and c are separation constants. Substitution in Eq. (13) of $Y = f$, $X = u$, and $2q = (k^2 - c)\rho^2/2$, produces the modified Mathieu equation, i.e.,

$$\frac{d^2Y}{dX^2} - [a - 2q\cosh(2X)]Y = 0. \tag{15}$$

Equation (15) does not possess periodic solutions.

Substitution in Eq. (14) of $Y = g$, $X = v$, and $2q = (k^2 - c)\rho^2/2$ in Eq. (14) then produces the Mathieu equation, i.e., Eq. (1). Readers should be wary of errors in Eqs. (13) and (14) in original editions of Ref. 3 published by the U.S. Govt. Printing Office. The Mathieu and the modified Mathieu equations can be expected to appear in any problem involving a Helmholtz equation expressed in elliptic cylinder coordinates.

B. The inverted pendulum

According to Phelps and Hunter,[5] the history of this device appears to date from circa 1960. If the point of suspension of

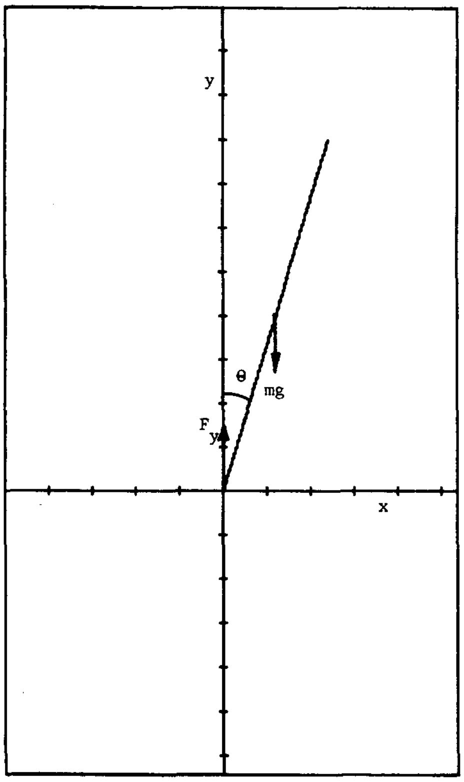

Fig. 3. Inverted pendulum showing point of suspension and center of mass.

a pendulum is vibrated vertically, then under certain conditions, the pendulum will execute stable oscillations about the vibrational axis, but with the center of mass always above the point of suspension. As shown in Fig. 3, the coordinates of the point of suspension are (x_0,y_0) at a given instant, and the pendulum, which is a uniform rod of mass m and length $2s$, points upward at an acute angle θ with the vertical. The corresponding coordinates of the pendulum center of mass are at (x,y). Following the derivation of Blitzer,[6] the equations of motion for the center of mass are

$$m\frac{d^2x}{dt^2}=F_x, \qquad m\frac{d^2y}{dt^2}=F_y-mg, \tag{16}$$

where F_x and F_y are forces acting at the point of suspension, and mg is the force of gravity acting at the center of mass. For rotation about the center of mass

$$I_c\frac{d^2\theta}{dt^2}=sF_y\sin(\theta)-sF_x\cos(\theta), \tag{17}$$

where I_c is the moment of inertia of the rod about its center of mass. Eliminating the forces between Eqs. (16) and (17) gives

$$I_c\frac{d^2\theta}{dt^2}=ms\left(\frac{d^2y}{dt^2}+g\right)\sin(\theta)-ms\frac{d^2x}{dt^2}\cos(\theta). \tag{18}$$

The center-of-mass coordinates in Eq. (18) can be eliminated through the relations

$$x=x_0+s\sin(\theta), \qquad y=y_0+s\cos(\theta), \tag{19}$$

which when differentiated and substituted in Eq. (18), gives

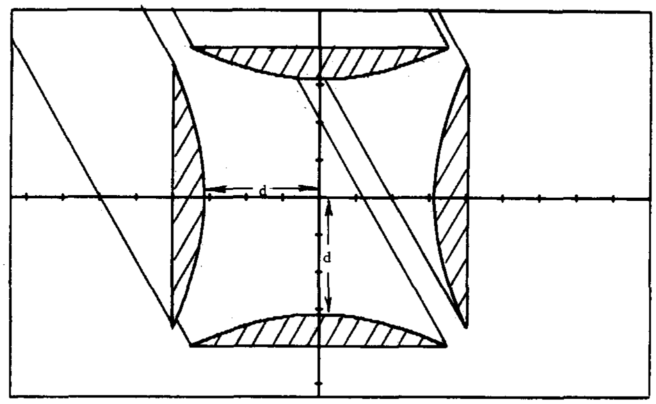

Fig. 4. Radiofrequency quadrupole. Opposite electrodes are connected electrically.

$$\frac{I}{ms}\frac{d^2\theta}{dt^2}+\frac{d^2x_0}{dt^2}\cos(\theta)-\left(\frac{d^2y_0}{dt^2}+g\right)\sin(\theta)=0, \tag{20}$$

where $I=I_c+ms^2$ is the moment of inertia about the point of suspension. Now, we will assume that the only motion of the point of suspension is vertical, which implies $d^2x_0/dt^2=0$. References 5 and 6 also discuss more general driving conditions. In our case, Eq. (20) becomes

$$\frac{I}{ms}\left(\frac{d^2\theta}{dt^2}\right)-\left(\frac{d^2y_0}{dt^2}+g\right)\sin(\theta)=0. \tag{21}$$

For a movement of the point of suspension, given by $y_0=A\cos(\omega t)$, Eq. (21) becomes

$$\frac{I}{ms}\left(\frac{d^2\theta}{dt^2}\right)+[\omega^2A\cos(\omega t)-g]\sin(\theta)=0. \tag{22}$$

In the limit of small oscillations, i.e., $\sin(\theta)\simeq\theta$, and with the substitutions $2X=\omega t$, $Y=-\theta$, $a=4msg/I\omega^2$, and $2q=4msA/I$, Eq. (22) becomes the Mathieu equation, i.e., Eq. (1). Phelps and Hunter[5] proceed to determine the range of the amplitude A for stable oscillations, and the coefficients in the power series solution for $\theta(t)$. A practical embodiment of the inverted pendulum is also described by Phelps and Hunter.[7]

C. Radio frequency quadrupole

The radio frequency (rf) quadrupole[8] is a device for both focusing and accelerating low-energy ion beams which are assumed to be traveling along the z axis. As shown in Fig. 4, the device appears as four hyperbolically shaped electrodes placed symmetrically about the ion beam, with the hyperbolas bisecting the x and y axes and extending to infinity along the lines $y=x$ and $y=-x$. Opposite electrodes are connected together electrically and between the pairs is impressed a rf potential varying as $\sin(\omega t)$. Let "d" represent the minimum electrode-to-axis distance. Then if $d\ll c/\omega$, the fields behave as in the electrostatic limit. Ideally, in the space between the electrodes, the electrostatic potential satisfies the equation

$$\phi=[E/(2d)](y^2-x^2), \tag{23}$$

where "d" is the minimum distance along an axis from the origin to an electrode, at which point the field is $\pm E$. In the

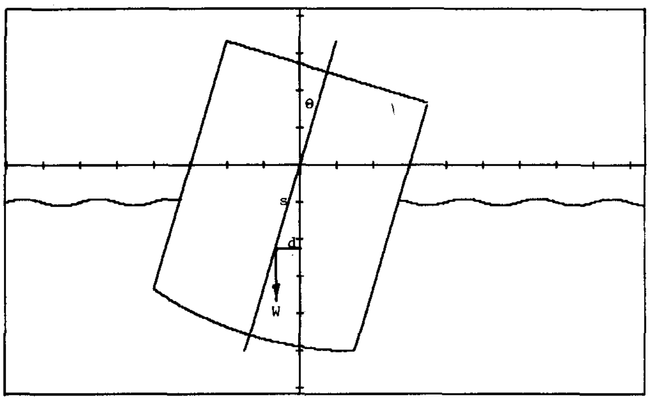

Fig. 5. Floating vessel. Line "d" connects vessel center of mass to center of buoyancy.

case of a rf electric field $E=E_0 \sin(\omega t)$, the force on a singly charged positive ion of mass m, in the x direction, is

$$m\frac{d^2x}{dt^2}=\left(\frac{eE_0x}{d}\right)\sin(\omega t) \quad (24)$$

and in the y direction is

$$m\frac{d^2y}{dt^2}=-\left(\frac{eE_0y}{d}\right)\sin(\omega t). \quad (25)$$

With the substitution in Eq. (24) of $Y=x$, $2X=\pi/2-\omega t$, and $2q=(4eE_0/m\omega^2d)$; and in Eq. (25) of $Y=y$, $2X=\pi/2-\omega t$, and $2q=-(4eE_0/m\omega^2d)$, both equations become the Mathieu equation, i.e., Eq. (1), with $a=0$.

The quadrupole can be modified to provide acceleration in the z direction, as well as focusing in the x and y directions. If the distance d in the horizontal plane is modulated sinusoidally with spatial period D in the z direction, and modulated similarly in the vertical plane, but 90° out of phase, then positive and negative traveling waves appear in the z direction. If the initial value of D is designed to match the incoming velocity of the particles, the latter are strongly coupled to the positive-traveling wave, and gain energy from it. The value of D is made to increase with length to keep the accelerating particle in phase, i.e., so that $v=D\omega/2\pi$.

D. Frequency modulation

McLachlan[3] notes that a method of producing a frequency-modulated carrier from an audio signal is to use a capacitor microphone as part of the tuned circuit for the carrier. In one type of microphone, when an audio signal of frequency 2ω impinges upon the diaphragm, the distance from the latter to the backplate varies as $d=d_0[1-\epsilon \cos(2\omega t)]$. Then, since capacitance varies inversely with d, the tuned circuit equation, with resistance neglected, becomes

$$\frac{d^2Q}{dt^2}+\left(\frac{Q}{(LC_0)}\right)[1-\epsilon \cos(2\omega t)]=0. \quad (26)$$

Letting $Y=Q$, $X=\omega t$, $a=1/(LC_0)$, and $2q=\epsilon/(\omega^2LC_0)$ then reproduces Mathieu's equation, i.e., Eq. (1).

Table I. Vector directions for a proton beam with B_z positive.

Position of ion	v_p	B_ϕ	$e(v_p\times B_\phi)$
Entering a hill with $z>0$	$+\rho$	$-\phi$	$-z$
Exiting a hill with $z>0$	$-\rho$	$+\phi$	$-z$
Entering a hill with $z<0$	$+\rho$	$+\rho$	$+z$
Exiting a hill with $z<0$	$-\rho$	$-\phi$	$+z$

E. Stability of a floating body

Allievi and Soudack[9] consider the effect of wave motion on a floating vessel which has symmetry with respect to a vertical bisecting plane. Figure 5 depicts a vessel with a list wherein the aforementioned plane is inclined to the vertical by an angle θ, the righting torque is, for small θ

$$M=-Wd=-Ws\sin(\theta)\simeq-Ws\theta, \quad (27)$$

where d is the horizontal distance from the vessel center of mass to the vertical through the center of buoyancy, i.e., the center of mass of the displaced water. The distance s is measured from the vessel center of mass, along the θ-inclined vessel symmetry axis, to the intersection of this axis with the vertical through the center of buoyancy. The point of intersection is termed the "transverse metacenter," which is independent of θ in the absence of waves. The gravitational force is assumed to act vertically at the vessel center of mass.

Allievi and Soudack next consider a two-dimensional wave which strikes the vessel and persists for a time T. They argue that such a wave will alter the distance "s" in a sinusoidal fashion so that the righting torque becomes

$$M(t)\simeq-W[s-(\Delta s/2)\sin(2\omega t)]\theta, \quad (28)$$

where $\omega=2\pi/T$. Under certain conditions, the wave can progressively augment the list angle θ, thereby leading to an instability. This can be seen by writing the equation of motion for the amount of list, i.e.,

$$I\frac{d^2\theta}{dt^2}+D\frac{d\theta}{dt}-M=0. \quad (29)$$

Combining Eqs. (28) and (29), and neglecting the damping, gives

$$\frac{d^2\theta}{dt^2}+\left(\frac{W}{I}\right)\left[s-\left(\frac{\Delta s}{2}\right)\sin(2\omega t)\right]\theta=0. \quad (30)$$

Then, substituting $Y=\theta$, $2X=\pi/2-2\omega t$, $a=Ws/(I\omega^2)$, and $2q=W\Delta s/(2I\omega^2)$ leads to the Mathieu equation, i.e., Eq. (1).

III. ALTERNATING GRADIENT FOCUSING

In a fixed-field alternating-gradient cyclotron[10] in which particles circulate in the x,y plane, the average magnetic field in the z direction, increases with radius in order to maintain constant frequency in the face of increasing mass. This results in axial, i.e., z direction, defocusing of the beam. Without compensation, the beam would soon expand and collide with the accelerating electrodes (the "dees"). Such a calamity can be prevented by an azimuthal variation of the magnetic field, as first pointed out by L. H. Thomas in 1938. The variations can be considered hill-valley combinations,

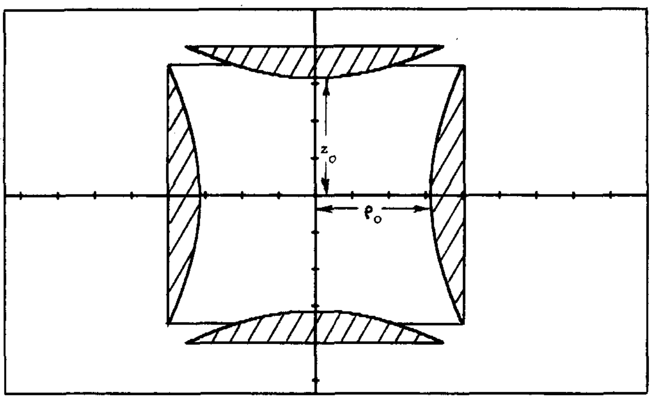

Fig. 6. Paul trap for charged particles. Electrodes comprise a cylindrical shell and two end caps connected electrically.

with at least three such combinations being required for stability. In one form of the design,[11] the field variation can be considered as harmonic, i.e.,

$$B_z\{\rho,\phi\}=B_0\{\rho\}[1+f\{\rho\}\sin(N\phi)], \tag{31}$$

where N is the number of hill-valley combinations and ρ is the radius. For a particular ρ, the larger B_z in the hill regions causes the orbits to resemble N-fold polygons with rounded corners. Focusing in the z direction is determined mainly by B_ϕ, which is given approximately by

$$B_\phi=-g\{\rho\}z\cos(N\phi). \tag{32}$$

As indicated in Table I, particles are focused in the z direction both in entering and in leaving a hill region, and sufficiently so to overcome the defocusing effect of B_0 which increases with ρ.

The equation of motion in the z direction is, for protons,

$$\frac{d^2z}{dt^2}-\left(\frac{ev_\rho}{m}\right)B_\phi=0. \tag{33}$$

Combining Eqs. (32) and (33) under the approximation of constant ρ, and substituting $\phi=\omega t$, $Y=z$, $2X=N\omega t$, and $-2q=4ev_\rho g/(mN^2\omega^2)$ produces the Mathieu equation, i.e., Eq. (1) with $a=0$. In modern cyclotrons, the azimuthal variation is achieved with spiral-shaped wedges, rather than by continuously tapered pole pieces producing a harmonic variation. The orbits can then be analyzed[10] by paraxial-ray theory, based on wedge-focusing parameters.

A. The Paul trap for charged particles

A very efficient trap for charged particles, devised by W. Paul, has been described by Winter and Ortjohann.[12] As shown in Fig. 6, the trap consists of a cylindrical electrode plus end-cap electrodes at both ends. All three electrodes are hyperbolically shaped. Between the end caps, which are connected together electrically, and the cylindrical electrode, is impressed a combination of a dc and an ac potential. For some measurements, a magnetic field along the cylinder axis is also used. Dehmelt[13] has termed the latter configuration "a Penning trap" although the original use of the latter name referred to a geometry in which the cylindrical electrode was merely a ring, and the end caps were flat. A "modified Paul trap" would be a preferable designation in Dehmelt's case.

Depending on the sign of the ac potential, the Paul-trap electric field is such as to produce stable oscillations in the z direction and unstable motion in the ρ direction, or vice versa. The combined effect of the ac is to produce confinement in both the z and ρ directions. Dehmelt[13] has made use of the Paul trap, modified to include a magnetic field, to confine electrons, positrons, and atomic ions, and thereby has measured their gyromagnetic ratios. The same positron was observed in a trap by Dehmelt and co-workers for a period of three months. For the positron, Dehmelt measured the gyromagnetic ratio to a precision 30 000 times more than had been previously observed. In 1989, Paul and Dehmelt were awarded the Nobel prize in Physics for their combined contributions.

Referring to our previous discussion of the rf quadrupole, we will assume that E_z and E_ρ are given by the relations:

$$E_z=-\frac{\partial V}{\partial z}=-\frac{[V_{\rm dc}-V_{\rm ac}\cos(\omega t)]z}{z_0^2}, \tag{34}$$

$$E_\rho=\frac{-\partial V}{\partial \rho}=\frac{+[V_{\rm dc}-V_{ac}\cos(\omega t)]\rho}{\rho_0^2}. \tag{35}$$

Since V must satisfy Laplace's equation, this implies that

$$\rho_0^2=2z_0^2. \tag{36}$$

Then, by assuming that the cylindrical electrode is grounded, and integrating dV from $[V_{\rm dc}-V_{\rm ac}\cos(\omega t)]/2$ to $[V_{\rm dc}-V_{\rm ac}\cos(\omega t)]$, the curves for the endcaps turn out to be

$$z^2=z_0^2-\rho^2/2 \tag{37}$$

and by integrating dV from $[V_{\rm dc}-V_{\rm ac}\cos(\omega t)]/2$ to 0, the curve for the cylindrical electrode comes out to be

$$z^2=(\rho^2-\rho_0^2)/2. \tag{38}$$

The general integral for dV gives

$$V\{z,\rho\}=[V_{\rm dc}-V_{ac}\cos(\omega t)](4z_0^2)^{-1}[2z^2+(\rho_0^2-\rho^2)]. \tag{39}$$

The equations of motion for a particle of charge e and mass m, in this field, become

$$\frac{d^2z}{dt^2}+\left(\frac{e}{m}\right)\frac{[V_{\rm dc}-V_{\rm ac}\cos(\omega t)]z}{z_0^2}=0. \tag{40}$$

$$\frac{d^2\rho}{dt^2}-\left(\frac{e}{m}\right)\frac{[V_{\rm dc}-V_{\rm ac}\cos(\omega t)]\rho}{2z_0^2}=0. \tag{41}$$

Then, substituting $Y=z$, $2X=\omega t$, $a=4eV_{\rm dc}/(mz_0^2\omega^2)$, and $2q=4eV_{\rm dc}/(mz_0^2\omega^2)$ into Eq. (40); and substituting $Y=\rho$, $2X=\omega t$, $a=-2eV_{\rm dc}/(mz_0^2\omega^2)$, and $2q=-2eV_{\rm dc}/(mz_0^2\omega^2)$ into Eq. (41), converts both equations into the Mathieu equation, i.e., Eq. (1).

B. The mirror trap for neutral particles

A trap for neutral particles, analogous to the modified Paul trap for charged particles, has been proposed by Lovelace *et al.*, and described by Sackett *et al.*.[14] Whereas the Paul trap couples to the particle's charge, the Lovelace design acts on the particle's magnetic dipole moment. The Lovelace design, modified by the addition of a steady axial magnetic field, has been used successfully to trap groups of cesium atoms, followed by laser cooling of the trapped particles.

The Lovelace design is simply a loop placed in the x,y plane, with its center at the origin. A combination of a steady plus a harmonically varying current I is made to flow in the loop. If R denotes the radius of the loop, and ρ denotes a radius in the x,y plane, close to the origin, and z denotes a point on the z axis, also close to origin, then

$$B_z\{\rho,0\}=[\mu_0 IR/(4\pi)]2\int_0^\pi d\phi[R-\rho\cos(\phi)]/[\rho^2+R^2-2\rho R\cos(\phi)]^{3/2}$$

and since $\rho/R\ll 1$

$$B_z\{\rho,0\}\simeq[\mu_0 I/(2R)][1+3\rho^2/(4R^2)+\cdots]. \quad (43)$$

Similarly

$$B_z\{0,z\}=[\mu_0 IR/(4\pi)]\int_0^{2\pi} d\phi[R/(R^2+z^2)^{3/2}] \quad (44)$$

and since $z/R\ll 1$

$$B_z\{0,z\}\simeq[\mu_0 I/(2R)][1-3z^2/(2R^2)+\cdots]. \quad (45)$$

In view of Eqs. (43) and (45), it is reasonable to assume that, close to the origin

$$B_z\{\rho,z\}\simeq[\mu_0 I/(2R)][1+3\rho^2/(4R^2)-3z^2/(2R^2)]. \quad (46)$$

With the addition of the large steady field B_{0z}, the magnetic field is predominantly $B_z\{\rho,z,t\}$, where

$$B_z\{\rho,z,t\}=B_{0z}+[\mu_0(I_{dc}+I_{ac}\cos(\Omega t))/(2R)]\times[1+3\rho^2/(4R^2)-3z^2/(2R^2)]. \quad (47)$$

At this point, we shall assume that the trapped dipole is aligned with its magnetic moment parallel to B_z, and that the gradient of B_z then serves to move the dipole with respect to the origin. The equations of motion are

$$\frac{d^2\rho}{dt^2}=\left(\frac{1}{m}\right)\left(\frac{-\partial(-\mu B_z)}{\partial\rho}\right), \quad (48)$$

$$\frac{d^2z}{dt^2}=\left(\frac{1}{m}\right)\left(\frac{-\partial(-\mu B_z)}{\partial z}\right), \quad (49)$$

where μ is the dipole moment. Letting $I=[I_{dc}+I_{ac}\cos(\Omega t)]$ gives

$$\frac{d^2\rho}{dt^2}=+\left(\frac{3\mu\mu_0}{(4mR^3)}\right)[I_{dc}+I_{ac}\cos(\Omega t)]\rho, \quad (50)$$

$$\frac{d^2z}{dt^2}=-\left(\frac{3\mu\mu_0}{(2mR^3)}\right)[I_{dc}+I_{ac}\cos(\Omega t)]z. \quad (51)$$

Thus, for $I_{ac}=0$, the motion is bounded in z and unbounded in ρ, and the effect of I_{ac} is to reverse this trend periodically so that the overall motion is bounded in both coordinates.

In Eq. (50), letting $2X=\Omega t$, $Y=\rho$, $a=-3\mu\mu_0 I_{dc}/(\Omega^2 mR^3)$, and $2q=3\mu\mu_0 I_{ac}/(\Omega^2 mR^3)$ then gives the Mathieu equation, i.e., Eq. (1). Similarly, in Eq. (51), letting $2X=\Omega t$, $Y=z$, $a=6\mu\mu_0 I_{dc}/(\Omega^2 mR^3)$, and $2q=-6\mu\mu_0 I_{ac}/(\Omega^2 mR^3)$ then also gives the Mathieu equation.

IV. CONCLUSION

The applications of the Mathieu equation discussed above probably only scratch the surface of what might be said in this regard. Phelps and Hunter[7] mention several others, i.e., pulsating flow superimposed on a steady flow, and elastic oscillations of a ferromagnetic substance. In view of the number of new applications described or mentioned above, it appears that the Mathieu equation deserves the attention of the authors of future textbooks on mathematics for scientist and engineers.

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MECHANICAL APTITUDE

I'd always been good in math, but when it came to mechanical things I was less than gifted. In high school I tried to rebuild the engine on my minibike after it froze when I let the oil get too low, but that project ended in my taking the minibike to the lawnmower shop. The model rocket I built in sixth grade went up 6 inches, veered to the right, and crashed.

Pepper White, *The Idea Factory—Learning to Think at MIT* (Penguin Books, New York, 1991), p. 79.