or, by collecting the leading terms,

$$\varepsilon^{2\alpha}\underbrace{\left(c_0^2U_{\xi\xi}-c_0^2U_{\xi\xi}\right)}_{=0}-\varepsilon^{4\alpha}\left(2c_0U_{\tau\xi}+\frac{c_0^2}{12}U_{\xi\xi\xi\xi}\right)-\varepsilon^{2\alpha+1}ac_0^2(U^2)_{\xi\xi} = \mathcal{O}(\varepsilon^{6\alpha},\varepsilon^{4\alpha+1}). \qquad (1.49)$$

One notices that the choice $\alpha = 1/2$ brings the leading nonlinear and dispersive terms to the same order ε^2 and that, at this order, we obtain the equation

$$-c_0\left(2U_\tau+\frac{c_0}{12}U_{\xi\xi\xi}+ac_0(U^2)_\xi\right)_\xi = 0. \qquad (1.50)$$

This appears to be an appropriate choice if we look for soliton-like solutions since we expect that the effect of dispersion and nonlinearity should cancel each other.

Equation (1.50) can be readily integrated once to give

$$\frac{\partial U}{\partial \tau}+ac_0\,U\frac{\partial U}{\partial \xi}+\frac{c_0}{24}\frac{\partial^3 U}{\partial \xi^3}=0. \qquad (1.51)$$

The integration constant has been chosen to be zero because we look for spatially localised solutions. This equation can be reduced to the KdV equation by introducing scaling factors for the time and amplitude, $\tau = c_0T/24$ and $\phi = 4aU$. Its solution

$$\phi = A\,\mathrm{sech}^2\left[\sqrt{\frac{A}{2}}(\xi-2A\tau)\right] \qquad (1.52)$$

confirms that, if the amplitude of the wave is of the order ε, its spatial evolution is of the order $\varepsilon^{1/2}$ while its time dependence is of the order $\varepsilon^{3/2}$ as expected. The result is therefore coherent with the spatial scale and timescale chosen in Equation (1.47).

As the path from Boussinesq to KdV may seem tortuous, it is important to check that the results which can be derived from the KdV approximation are correct. If we consider a soliton solution moving at speed c close to c_0 such that $c = c_0(1+\alpha')$ where $\alpha' \ll 1$, the comparison of the solutions of the two equations shows that they are identical to first order in α'. When we studied the Boussinesq equation, we saw that the speed $c = 1.05\,c_0$ was the upper limit of the validity of this equation, which imposes to restrict our attention to $\alpha' \leq 0.05$. In this range the two equations, Boussinesq and KdV, give results which differ at most by a few percent. The error which is introduced by the KdV approximation is comparable to the experimental errors in the measurements on the electrical chain. But the gain is very significant because all the mathematical tools that come with the KdV equation become available. It fully justifies this approximation.

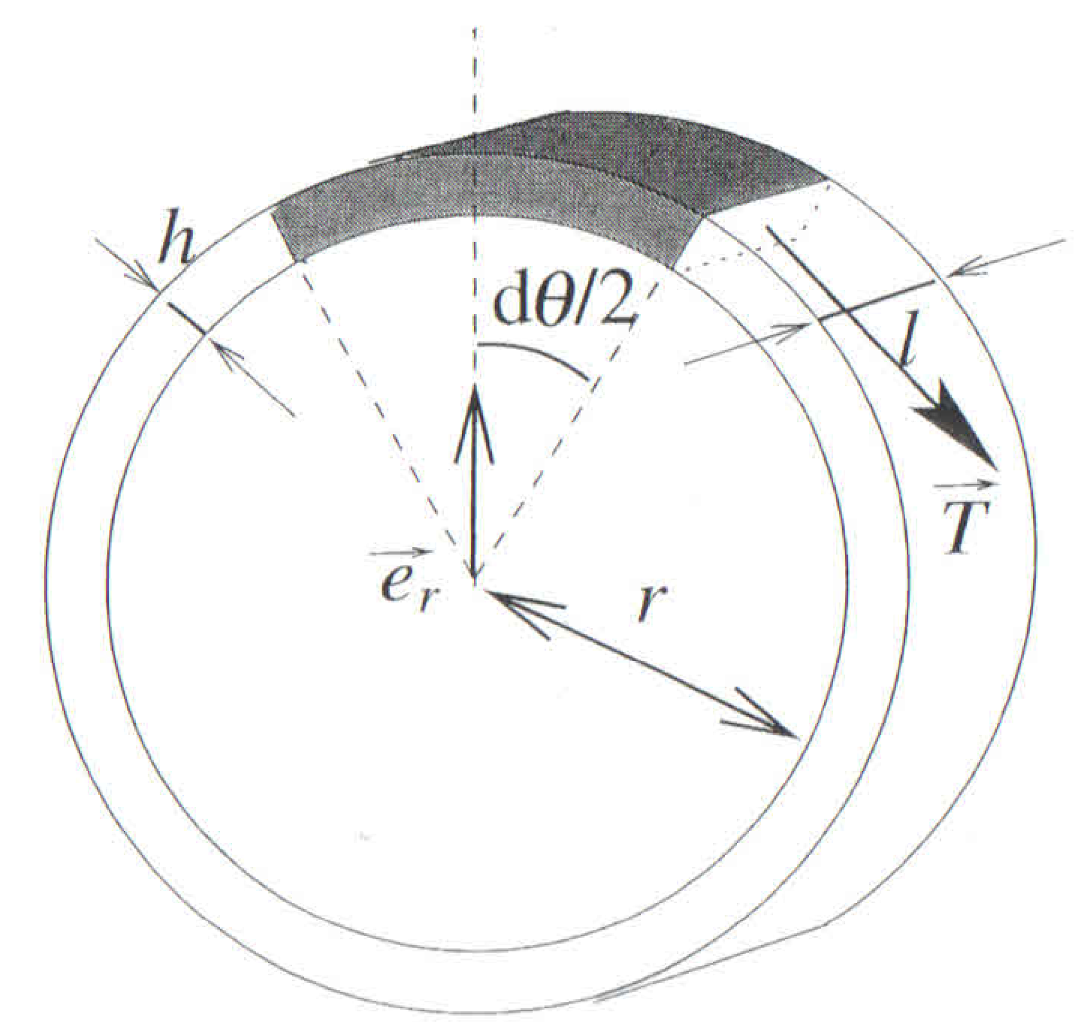

Figure 1.16. Schematic view of a small piece of artery of length ℓ along the axis of the artery, and of thickness h.

1.5 Blood pressure waves

Let us now consider a nonstandard case for which one can derive the KdV equation [143, 144]. Although it is certainly a rough approximation in this case, the KdV equation can answer a simple question: why is it possible to feel the blood pulse at the wrist or the ankle? The heart sends a pressure wave in the arteries. It moves along the arteries and induces a local expansion of the vessels. It is this deformation that we perceive when we feel the blood pulse. It is remarkable that it can propagate to the ends of the limbs without being noticeably dispersed. This can be explained by an equilibrium between the nonlinearity coming from the hydrodynamics of the blood flux and the dispersion associated with the elasticity of the arteries, as we shall see from a simple model.

A small piece of artery, of length ℓ, can be viewed as an elastic ring, having a radius r_0 at equilibrium and a thickness h (Figure 1.16). The time evolution of its radius r is determined on one hand by the pressure inside the artery and on the other hand by the elastic stress inside the material of the ring. Let us consider these two aspects successively.

The force $\mathrm{d}\vec{f_p}$ exerted by the blood pressure on a small part of the ring (shaded in Figure 1.16), defined by its angular aperture $\mathrm{d}\theta$, is directed along the radial vector $\vec{e}_r$ and is given by

$$\mathrm{d}\vec{f_p} = p\,r\mathrm{d}\theta\,\ell\,\vec{e}_r, \qquad (1.53)$$

where p is the blood pressure inside the ring, or, more precisely, the difference between the maximum pressure when the heart contracts (systolic blood pressure) and the pressure in the artery when the heart is at rest (diastolic blood pressure).

When the ring radius changes from its equilibrium value r_0 to another value r, the relative stretching of the part of the ring that we are considering is $(r\mathrm{d}\theta - r_0\mathrm{d}\theta)/r_0\mathrm{d}\theta$. The elastic stress inside the material of the ring gives rise to force $\overrightarrow{T}$ applied by the remainder of the ring on the two ends of the part under study. If we denote by E the Young modulus of the material of the ring, the tensile force T is given by

$$\frac{r\mathrm{d}\theta - r_0\mathrm{d}\theta}{r_0\mathrm{d}\theta} = \frac{1}{E}\frac{T}{\ell h} \tag{1.54}$$

since ℓh is the area of the section of the ring on which T is applied. The radial component of the tensile elastic forces on this part of the ring is therefore

$$\mathrm{d}\overrightarrow{f_T} = -2T\sin\frac{\mathrm{d}\theta}{2}\,\vec{e}_r \simeq -T\mathrm{d}\theta\,\vec{e}_r = -E\ell h\frac{r-r_0}{r_0}\mathrm{d}\theta\,\vec{e}_r. \tag{1.55}$$

Writing Newton's law for the small part of the ring, and projecting it along the radial vector $\vec{e}_r$, we get

$$\mathrm{d}m\,\frac{\partial^2 r}{\partial t^2} = \mathrm{d}f_p + \mathrm{d}f_T, \tag{1.56}$$

where the mass of the ring section is $\mathrm{d}m = \rho_0\ell h r_0\mathrm{d}\theta$ if we denote by ρ_0 the mass of a unit volume of artery tissue. After some simplifications we obtain

$$\rho_0 h r_0\frac{\partial^2 r}{\partial t^2} = p\,r - Eh\frac{r-r_0}{r_0}. \tag{1.57}$$

In order to connect the deformations of the artery to the variations of the flux of the blood that it carries, it is convenient to introduce the section of the tube $A = \pi r^2$. Its time evolution obeys

$$\frac{\partial^2 A}{\partial t^2} = 2\pi r\frac{\partial^2 r}{\partial t^2} + 2\pi\left(\frac{\partial r}{\partial t}\right)^2 \simeq 2\pi r\frac{\partial^2 r}{\partial t^2} \tag{1.58}$$

if we only keep the linear part of the expression, taking into account the small values of the radial velocities. On the other hand, as the variations of r are small, we have

$$A - \pi r_0^2 = \pi(r+r_0)(r-r_0) \simeq 2\pi r_0(r-r_0), \tag{1.59}$$

which suggests rewriting Equation (1.57) as

$$2\pi r_0\frac{\partial^2 r}{\partial t^2}\rho_0 h = 2\pi p\,r_0 + 2\pi\frac{r-r_0}{r_0}\left[p r_0 - hE\right]. \tag{1.60}$$

Dividing by $\rho_0 h$, we get

$$2\pi r_0\frac{\partial^2 r}{\partial t^2} = \frac{2\pi p\,r_0}{\rho_0 h} - E2\pi\frac{r-r_0}{\rho_0 r_0} \tag{1.61}$$

if we make the approximation $pr_0 \ll hE$ which is valid for an artery because typical values [143] of p are in the range 20–40 mm Hg, r_0 in the range 2 – 10 mm, $h \simeq 0.1\,r_0$ and E in the range 20 – 130 N cm^{-2}. We finally get

$$\frac{\partial^2 A}{\partial t^2} = \frac{2\pi p\,r_0}{\rho_0 h} - \frac{E}{\rho_0 r_0^2}\left(A - \pi r_0^2\right). \tag{1.62}$$

This equation coming from the elasticity of the artery must be completed by the hydrodynamic equations of the blood flow. Viscosity, which damps the initial pulse, leads to a perturbation of the equation. We shall neglect it here to simplify the calculations, and we shall discuss below how it is actually balanced in the blood flow.

Noticing that only the z component along the axis of the artery enters into play, the Euler equation reads

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}. \tag{1.63}$$

It must be completed by the equation describing the mass conservation across a small piece of artery

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho A(z,t)\,\mathrm{d}z\right] = \underbrace{\rho A(z,t)v(z,t)}_{\text{incoming mass}} - \underbrace{\rho A(z+\mathrm{d}z,t)v(z+\mathrm{d}z,t)}_{\text{outgoing mass}}. \tag{1.64}$$

With the standard hypothesis of an incompressible flow (ρ constant), we finally get

$$\rho\frac{\partial A}{\partial t} + \rho\frac{\partial(Av)}{\partial z} = 0. \tag{1.65}$$

The system of Equations (1.57), (1.62) and (1.65) determines how the blood pulse propagates, but it cannot be solved in its present form. In order to control the approximations, as usual it is convenient to change to a set of dimensionless variables by defining

$$\widetilde{A} = \frac{A}{\pi r_0^2}, \quad \widetilde{p} = \frac{2r_0}{Eh}p, \quad \widetilde{v} = \frac{v}{L\Omega}, \quad \xi = \frac{z}{L} \quad \text{and} \quad \tau = \Omega t, \tag{1.66}$$

where we introduce

$$\Omega = \sqrt{\frac{E}{\rho_0 r_0^2}} \quad \text{and} \quad L = \sqrt{\frac{\rho_0}{\rho}\frac{r_0 h}{2}}. \tag{1.67}$$

The system of Equations (1.57), (1.62) and (1.65) reduces to

$$\frac{\partial^2 \widetilde{A}}{\partial \tau^2} + (\widetilde{A} - 1) = \widetilde{p} \tag{1.68}$$

$$\frac{\partial \widetilde{v}}{\partial \tau} + \widetilde{v}\frac{\partial \widetilde{v}}{\partial \xi} = -\frac{\partial \widetilde{p}}{\partial \xi} \tag{1.69}$$

$$\frac{\partial \widetilde{A}}{\partial \tau} + \frac{\partial \widetilde{A}\widetilde{v}}{\partial \xi} = 0. \tag{1.70}$$

As it is obviously nonlinear, soliton solutions can be expected provided it is also dispersive. In order to check this point, let us linearise the equations around the equilibrium values

$$\widetilde{A}_{\mathrm{eq}} = 1, \quad \widetilde{p}_{\mathrm{eq}} = 0 \quad \text{and} \quad \widetilde{v}_{\mathrm{eq}} = 0. \tag{1.71}$$

The equations for the variations $\delta\widetilde{A}$, $\delta\widetilde{v}$ and $\delta\widetilde{p}$ around these equilibrium values are therefore

$$\frac{\partial^2 \delta\widetilde{A}}{\partial \tau^2} + \delta\widetilde{A} = \delta\widetilde{p} \tag{1.72}$$

$$\frac{\partial \delta\widetilde{v}}{\partial \tau} = -\frac{\partial(\delta\widetilde{p})}{\partial \xi} \tag{1.73}$$

$$\frac{\partial \delta\widetilde{A}}{\partial \tau} + \frac{\partial(\delta\widetilde{v})}{\partial \xi} = 0. \tag{1.74}$$

They have plane wave solutions

$$(\delta\widetilde{A}, \delta\widetilde{v}, \delta\widetilde{p}) = (A_0, v_0, p_0)\, \mathrm{e}^{\mathrm{i}(q\xi - \omega t)} \tag{1.75}$$

if the determinant

$$\begin{vmatrix} 1-\omega^2 & 0 & -1 \\ 0 & -\mathrm{i}\omega & \mathrm{i}q \\ -\mathrm{i}\omega & \mathrm{i}q & 0 \end{vmatrix} \tag{1.76}$$

vanishes. This gives the dispersion relation

$$\omega^2 = \frac{q^2}{1+q^2} \tag{1.77}$$

which yields the phase velocity $v_\varphi = \omega/q = 1/\sqrt{1+q^2}$, which indeed depends on the wavevector q. This confirms that the system is dispersive. Moreover we can notice that v_φ decreases as q increases, i.e. the dispersion relation is qualitatively similar to the dispersion relations of shallow water waves or signals in the electrical chain.

The search for soliton solutions can then be carried out with the same methods that lead from the Boussinesq to the KdV equation in the electrical chain. One first analyses the different orders of magnitude involved and then changes to a frame moving at the speed of long-wavelength linear waves, i.e. here $v_\varphi = 1$.

We still consider the case of a weak nonlinearity, i.e. we look for solutions having a magnitude of the order of ε, assumed to be small with respect to the equilibrium values. However it turns out that the expansion must be carried up to order 2 to give interesting results.

Therefore we look for solutions of the form

$$\widetilde{A} = 1 + \varepsilon A_1 + \varepsilon^2 A_2, \quad \widetilde{v} = \varepsilon v_1 + \varepsilon^2 v_2 \quad \text{and} \quad \widetilde{p} = \varepsilon p_1 + \varepsilon^2 p_2. \tag{1.78}$$

As with the study of the Boussinesq equation, changing to the moving frame amounts to looking for solutions which vary more slowly in time than in space. Thus we define the new variables

$$\chi = \varepsilon^{1/2}(\xi - \tau) \quad \text{and} \quad \eta = \varepsilon^{3/2}\tau, \tag{1.79}$$

which are then introduced into Equations (1.68), (1.69) and (1.70). The lowest order terms simply give $A_1 = v_1 = p_1$ if we look for spatially localised solutions. The next order terms lead to

$$\frac{\partial p_1}{\partial \eta} + \frac{3}{2}p_1\frac{\partial p_1}{\partial \chi} + \frac{1}{2}\frac{\partial^3 p_1}{\partial \chi^3} = 0 \tag{1.80}$$

which is indeed the KdV equation.

Thus it turns out that the blood pressure waves appear as KdV solitons, within the approximate description that we introduced. An equilibrium between dispersion and nonlinearity explains why the blood pulse stays localised after travelling a long distance along the arteries, even down to the ends of the limbs. Introducing an appropriate value for the maximal excess of pressure generated by the heart (such as $p_{\max} = 2500$ Pa) we find a propagation speed of about 5 m s^{-1} for the blood pulse, and a spatial extent of the soliton of about 1 cm.

Besides the curiosity of considering blood pressure waves as solitons, this modelling can give useful results because it links the parameters of the soliton to the properties of the artery, such as its Young modulus. For a normal blood flow the volume of blood in the initial impulse is fixed. It is proportional to the area of the soliton solution (Figure 1.17). However, as the width and the amplitude of the soliton strongly depend on the Young modulus, it is easy to check that, if the artery becomes more rigid (owing to the effects of cigarette smoking for instance!) so that its Young modulus increases, the width of the pulse decreases while its amplitude increases, and therefore the blood pressure measured by a physician

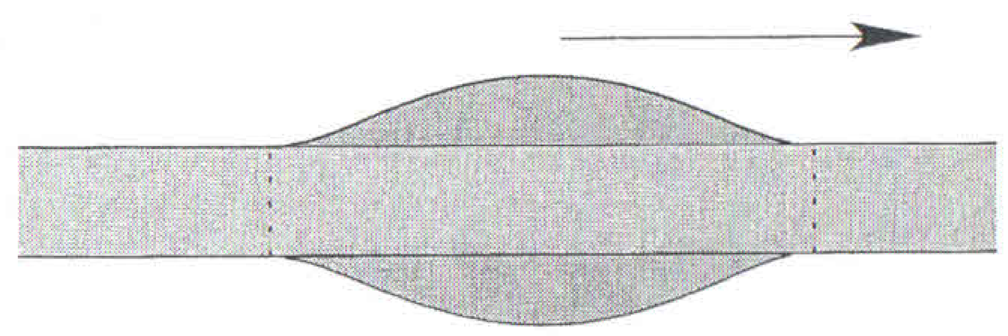

Figure 1.17. Schematic plot of the localised deformation of an artery associated with the propagation of the blood pulse. The volume of blood carried by the pulse (marked by the darker shaded area on the plot) is proportional to the area of the soliton solution.

increases. This is the effect that can lead to the breaking of small arteries in the brain.

Neglecting viscosity may seem a crude approximation, and indeed if one simply adds viscous effects to the above analysis, one finds that the blood pulse is quickly damped. But the reality of the blood flow is more subtle because there is a second effect that comes into play to balance the influence of viscosity, the conicity of the arteries. Their section gradually decreases as one moves away from the heart. A more realistic model, which includes the effect of conicity as well as viscosity shows that two perturbations of the KdV equation are generated which have opposite effects on the properties of the soliton. The balance of the two maintains a localised solution, as you can easily check by feeling your pulse.

1.6 Internal waves in oceanography

The Andaman sea, part of the Indian Ocean on the western coast of Thailand, is a place where one can observe impressive hydrodynamic solitons. The salt concentration is stratified along the vertical. Like surface waves on the sea which exist thanks to the difference between the density of water and air, *internal gravity waves* can exist within the ocean, due to the variation of the density of the water from the bottom to the surface (Figure 1.18). But, as the density difference between neighbouring layers is very small, the periods and amplitudes of these internal waves are much larger than those of surface waves. For instance amplitudes as large as 200 metres have been observed.

Sailors have known for centuries that this sea could show stripes where the surface was significantly above the average sea level. This phenomenon is for instance described in Maury's book [126] published in 1861.

> The ripplings are seen in calm weather approaching from a distance, and in the night their noise is heard a considerable time before they come near. They beat against the sides of a ship with great violence … and a small boat could not always resist in the turbulence of these remarkable ripplings.

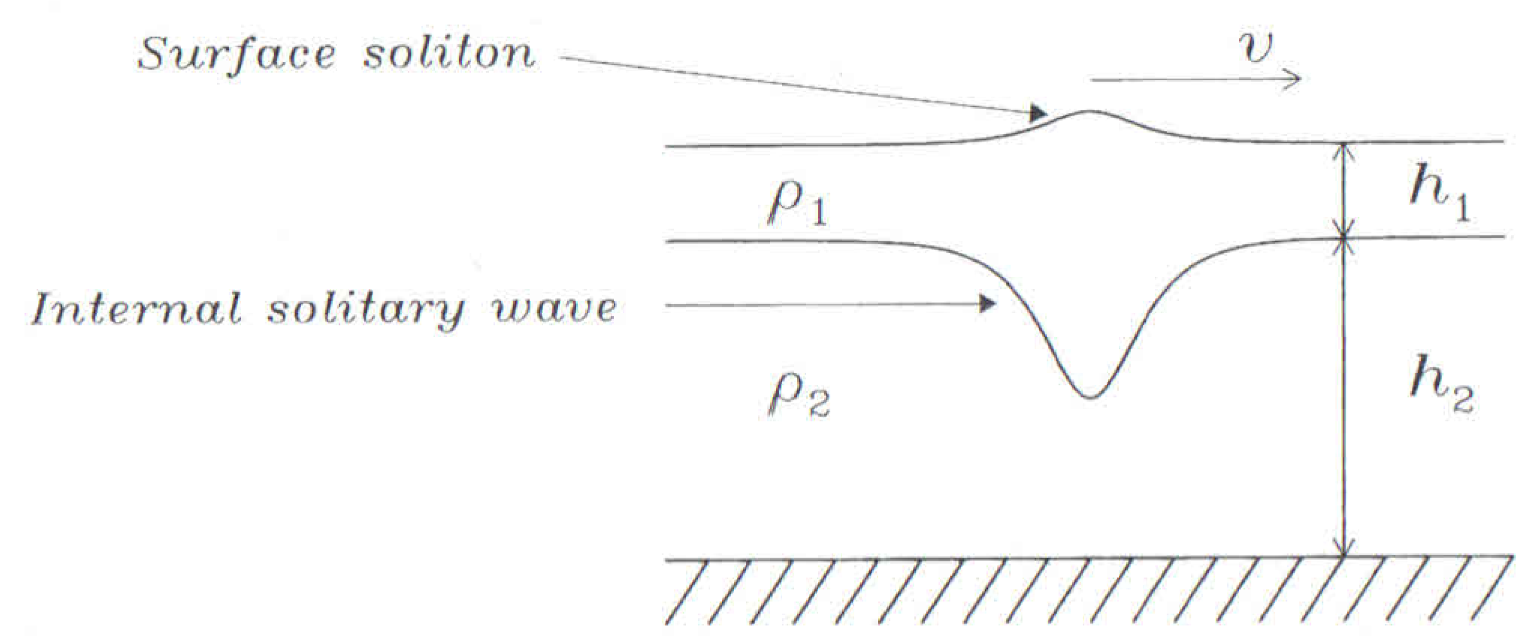

Figure 1.18. A solitary wave of amplitude η_0 in a two-layer fluid of densities $\rho_1 < \rho_2$ is a dip when the thickness of the layers are such that $h_1 < h_2$. It induces a small surface soliton which has an amplitude approximately equal to $(\rho_2 - \rho_1)\eta_0$.

As they exist in deep water far from the coast these waves cannot be attributed to standard tide phenomena. In 1965, a systematic series of measurements carried out by Perry and Schimke [148] showed that these stripes are the surface signature of some large internal waves. In 1980, Osborne and Burch analysed the data collected by the American company Exxon [141] on these waves which can damage off-shore oil-rigs. They concluded that they are indeed induced by internal solitary waves which are well described by the KdV equation. When the sea comprises layers which have different, but close, densities the internal waves also affect the surface as shown in Figure 1.18. These solitary waves are created by the tides near the coast, but then they can propagate over several hundreds of kilometres.

Measurements showed [141] that those solitons are not isolated. Groups of solitons are generated, the first solitons in a group coming with a time interval of about 40 minutes, which then decreases for the solitons which are in the tail of the group. The amplitude of the internal waves is about 60 m, and they induce surface waves which are about 2 m high. The width of one of these waves is of the order of 1 km so that they show up in a spectacular way on satellite images as shown in Figure 1.19. A measurement station installed in 1995 provides data on the creation and propagation of these remarkable water waves.

Similar internal waves exist in many oceans in the world, and for instance at the straits of Gibraltar, which connect the Mediterranean sea and the Atlantic Ocean. Such waves are presumed to be at the origin of the loss of the American submarine USS Thresher which disappeared in 1969 after a too sudden descent.

1.7 Generality of the KdV equation

There are numerous examples in physics which can be approximately described by the KdV equation. We shall meet later (see Chapter 8) the example of an atomic lattice which played a prominent role in the introduction of the soliton concept.