SECTION 5

A TRANSLATION OF HOPF'S ORIGINAL PAPER

BY L. N. HOWARD AND N. KOPELL

"Abzweigung einer periodischen Lösung von einer stationären Lösung eines Differentialsystems" Berichten der Mathematisch-Physischen Klasse der Sächsischen Akademie der Wissenschaften zu Leipzig. XCIV. Band Sitzung vom 19. Januar 1942.

Bifurcation of a Periodic Solution from a Stationary Solution of a System of Differential Equations

by

Eberhard Hopf

Dedicated to Paul Koebe on his 60th birthday

1. Introduction

Let

$$\dot{x}_i = F_i(x_1,\dots,x_n,\mu) \quad (i = 1,\dots,n)$$

or, in vector notation,

$$\dot{\underline{x}} = \underline{F}(\underline{x},\mu) \tag{1.1}$$

be a real system of differential equations with real parameter μ, where $\underline{F}$ is analytic in $\underline{x}$ and μ for $\underline{x}$ in a domain G and $|\mu| < c$. For $|\mu| < c$ let (1.1) possess an analytic family of stationary solutions $\underline{x} = \underline{\tilde{x}}(\mu)$ lying in G:

$$\underline{F}(\underline{\tilde{x}}(\mu),\mu) = 0.$$

As is well known, the characteristic exponents of the stationary solution are the eigenvalues of the eigenvalue problem

$$\lambda\underline{a} = \underline{L}_{\mu}\underline{a}$$

where $\underline{L}_{\mu}$ stands for the linear operator, depending only on μ, which arises after neglect of the nonlinear terms in the series expansion of F about $\underline{x} = \underline{\tilde{x}}$. The exponents are either real or pairwise complex conjugate and depend on μ.

Suppose one assumes simply that there is a stationary solution $\underline{x}_0$ in G for the special value $\mu = 0$ and that none of the characteristic exponents is 0; then, as is well known, it automatically follows that there is a unique stationary solution $\underline{\tilde{x}}(\mu)$ in a suitable neighborhood of $\underline{x} = \underline{x}_0$ for every sufficiently small $|\mu|$, and $\underline{\tilde{x}}(\mu)$ is analytic at $\mu = 0$.

On passing through $\mu = 0$ let us now assume that none of the characteristic exponents vanishes, but a conjugate pair crosses the imaginary axis. This situation commonly occurs in nonconservative mechanical systems, for example, in hydrodynamics. The following theorem asserts, that with this hypothesis, there is always a periodic solution of equation (1.1) in the neighborhood of the values $\underline{x} = \underline{x}_0$ and $\mu = 0$.

Theorem. *For* $\mu = 0$, *let exactly two characteristic exponents be pure imaginary. Their continuous extensions* $\alpha(\mu)$, $\bar{\alpha}(\mu)$ *shall satisfy the conditions*

$$\alpha(0) = -\bar{\alpha}(0) \neq 0, \quad \mathrm{Re}(\alpha'(0)) \neq 0. \tag{1.2}$$

Then, there exists a family of real periodic solutions $\underline{x} = \underline{x}(t,\varepsilon)$, $\mu = \mu(\varepsilon)$ *which has the properties* $\mu(0) = 0$ *and* $\underline{x}(t,0) = \underline{\tilde{x}}(0)$, *but* $\underline{x}(t,\varepsilon) \neq \underline{\tilde{x}}(\mu(\varepsilon))$, *for all sufficiently small* $\varepsilon \neq 0$. $\varepsilon(\mu)$ *and* $\underline{x}(t,\varepsilon)$ *are analytic at the point* $\varepsilon = 0$ *and correspondingly at each point* $(t,0)$. *The same holds for the period* $T(\varepsilon)$ *and*

$$T(0) = 2\pi/|\alpha(0)|.$$

For arbitrarily large L *there are two positive numbers* a *and* b *such that for* $|\mu| < b$, *there exist no periodic solutions besides the stationary solution and the solutions of the semi-family* $\varepsilon > 0$ *whose period is smaller than* L *and which lie entirely in* $|\underline{x}-\underline{\tilde{x}}(\mu)| < a$.*

For sufficiently small μ, the periodic solutions generally exist only for $\mu > 0$ or only for $\mu < 0$; it is also possible that they exist only for $\mu = 0$.

As is well known, the characteristic exponents of the periodic solution $\underline{x}(t,\varepsilon)$ are the eigenvalues of the eigenvalue problem

$$\dot{\underline{v}} + \lambda\underline{v} = \underline{L}_{t,\varepsilon}(\underline{v}) \tag{1.3}$$

where $\underline{v}(t)$ has the same period $T = T(\varepsilon)$ as the solution.

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*The other half-family must represent the same solution curves.

$\underline{L}$ is the linear operator obtained by linearizing around the periodic solution. It depends periodically on t with the period T and at $\varepsilon = 0$ is analytic in t and ε. The characteristic exponents are only determined $\mathrm{mod}(2\pi i/T)$ and depend continuously on ε. One of them, of course, is zero; for $\underline{F}$ does not depend explicitly on t, so

$$\lambda = 0, \quad \underline{v} = \dot{\underline{x}}(t,\varepsilon)$$

is a solution of the eigenvalue problem. For $\varepsilon \to 0$ the exponents, $\mathrm{mod}(2\pi i/T_0)$, go continuously into those of the stationary solution $\underline{x}(0)$ of (1.1) with $\mu = 0$. By assumption then exactly two exponents approach the imaginary axis. One of them is identically zero. The other $\beta = \beta(\varepsilon)$ must be real and analytic at $\varepsilon = 0$, $\beta(0) = 0$. It follows directly from the above theorem that the coefficients μ_1 and β_1 in the power series expansion

$$\mu = \mu_1\varepsilon + \mu_2\varepsilon^2 + \ldots$$

$$\beta = \beta_1\varepsilon + \beta_2\varepsilon^2 + \ldots$$

satisfy $\mu_1 = \beta_1 = 0$. In addition to that it will be shown below that the simple relationship

$$\beta_2 = -2\mu_2 \mathrm{Re}(\alpha'(0)) \tag{1.4}$$

holds; I have not run across it before.

In the general case $\mu_2 \neq 0$, this relationship gives information about the stability conditions. If, for example, for $\mu < 0$ all the characteristic exponents of the stationary solution $\underline{x} = \tilde{\underline{x}}(\mu)$ have a negative real part (stability, a small neighborhood of $\tilde{\underline{x}}$ collapses onto $\tilde{\underline{x}}$ as $t \to \infty$), then there are the following alternatives. Either the periodic solutions branch off after the destabilization of the stationary solution ($\mu > 0$); in this case all characteristic exponents of the periodic solution have negative real part (stability; a thin tube around the periodic solutions collapses onto these as $t \to \infty$). Alternatively, the family exists before, that is for $\mu < 0$; then the periodic solutions are unstable.*

Since in nature only stable solutions can be observed for a sufficiently long time of observation, the bifurcation of a periodic solution from a stationary solution is observable only through the latter becoming unstable. Such observations are well known in hydromechanics. For example, in the flow around a solid body; the motion is stationary if the velocity of the oncoming stream is low enough; yet if the latter is sufficiently large it can become periodic (periodic vortex shedding). Here we are talking about examples of nonconservative systems (viscosity of the fluid).+ In conservative systems, of course, the hypothesis (1.2) is never fulfilled; if λ is a characteristic exponent, $-\lambda$ always is as well.

In the literature, I have not come across the bifurcation problem considered on the basis of the hypothesis

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*In $n = 2$ dimensions, this is immediately clear.

+I do not know of a hydrodynamical example of the second case. One could conclude the existence of the unstable solutions if, with the most careful experimenting, (very slow variation of the parameters) one always observes a sudden breaking off of the stationary motion at exactly the same point.

(1.2). However, I scarcely think that there is anything essentially new in the above theorem. The methods have been developed by Poincaré perhaps 50 years ago,* and belong today to the classical conceptual structure of the theory of periodic solutions in the small. Since, however, the theorem is of interest in non-conservative mechanics it seems to me that a thorough presentation is not without value. In order to facilitate the extension to systems with infinitely many degrees of freedom, for example the fundamental equations of motion of a viscous fluid, I have given preference to the more general methods of linear algebra rather than special techniques (e.g. choice of a special coordinate system).

Of course, it can equally well happen that at $\mu = 0$ a real characteristic exponent $\alpha(\mu)$ of the stationary solution $\underline{\tilde{x}}(\mu)$ crosses the imaginary axis, i.e.,

$$\alpha(0) = 0, \quad \alpha'(0) \neq 0$$

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*Les méthodes nouvelles de la mécanique céleste. The above periodic solutions represent the simplest limiting case of Poincaré's periodic solutions of the second type ("genre"). Compare Vol. III, chapter 28, 30-31. Poincaré, having applications to celestial mechanics in mind, has only thoroughly investigated these solutions (with the help of integral invariants) in the case of canonical systems of differential equations, where the situation is more difficult than above. Poincaré uses the auxiliary parameter ε in Chap. 30 in the calculation of coefficients (the calculation in our §4 is essentially the same), but not in the proof of existence which thereby becomes simpler.

In a short note in Vol. I, p. 156, Painlevé is touched upon: Les petits mouvements périodiques des systèmes, Comptes Rendus Paris XXIV (1897), p. 1222. The general theorem stated there refers to the case $\mu = 0$ in our system (.), but it cannot be generally correct. For the validity of this statement $\underline{F}$ must satisfy special conditions.

while the others remain away from it. In this case it is not <u>periodic</u> but other stationary solutions which branch off.[*] We content ourselves with the statement of the theorems in this simpler case. There is an analytic family, $\underline{x} = \underline{x}^*(\varepsilon)$, $\mu = \mu^*(\varepsilon)$ of stationary solutions, different from $\tilde{x}$, with $\mu(0) = 0$, $x^*(0) = \tilde{x}(0)$. If $\mu_1 \neq 0$ (the general case) then the solutions exist for $\mu > 0$ <u>and</u> for $\mu < 0$. For the characteristic exponent $\beta(\varepsilon)$ which goes through zero, the analog of (1.4) holds:

$$\beta_1 = -\mu_1 \alpha'(0).$$

If $\underline{\tilde{x}}$ is stable for $\mu < 0$ and unstable for $\mu > 0$ then just the opposite holds for $\underline{x}^*$. (If one observes $\tilde{x}$ for $\mu < 0$, than one will observe x^* for $\mu > 0$.) In the exceptional case $\mu_1 = 0$, the situation is different. If $\mu_2 \neq 0$, then the new solutions exist only for $\mu > 0$ or only for $\mu < 0$. There are then two solutions for fixed μ, (one with ε positive, one with ε negative). Here we have

$$\beta_2 = -2\mu_2 \alpha'(0).$$

From this one can obtain statements about stability analogous to those above. In this case either both solutions x^* are stable or both are unstable.

2. <u>The Existence of the Periodic Solutions</u>.

Without restriction of generality one can assume that the stationary solution lies at the origin, i.e.,

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[*] An example from hydrodynamics is the fluid motion between two concentric cylinders (G. I. Taylor).

$$\underline{F}(0,\mu) = 0.$$

Let the development of $\underline{F}$ in powers of the x_i be

$$\underline{F}(\underline{x},\mu) = \underline{L}_\mu(\underline{x}) + \underline{Q}_\mu(\underline{x},\underline{x}) + \underline{K}_\mu(\underline{x},\underline{x},\underline{x}) + \ldots, \tag{2.1}$$

where the vector functions

$$\underline{L}_\mu(\underline{x}), \quad \underline{Q}_\mu(\underline{x},\underline{y}), \quad \underline{K}_\mu(\underline{x},\underline{y},\underline{z}), \ \ldots$$

are linear functions of each argument and also symmetric in these vectors.

The substitution

$$\underline{x} = \varepsilon\underline{y} \tag{2.2}$$

carries (1.1) into

$$\dot{\underline{y}} = \underline{L}_\mu(\underline{y}) + \varepsilon\underline{Q}_\mu(\underline{y},\underline{y}) + \varepsilon^2\underline{K}_\mu(\underline{y},\underline{y},\underline{y}) + \ldots \tag{2.3}$$

The right hand side is analytic in ε, μ, $\underline{y}$ at the point $\varepsilon = \mu = 0$, $\underline{y} = \underline{y}^0$ ($\underline{y}^0$ arbitrary). We consider the case $\varepsilon = 0$ in (2.3), that is, the homogeneous linear differential equation

$$\dot{\underline{z}} = \underline{L}_\mu(\underline{z}). \tag{2.4}$$

For the question of existence, this has the deciding significance.

The complex conjugate characteristic exponents $\alpha(\mu)$, $\overline{\alpha}(\mu)$, which were referred to in the hypothesis, are simple for all small $|\mu|$. In the associated solutions

$$e^{\alpha t}\underline{a}, \quad e^{\overline{\alpha} t}\overline{\underline{a}} \tag{2.5}$$

of (2.4), the complex vector $\underline{a}$ is consequently determined up to a complex scalar factor; $\overline{\underline{a}}$ is the conjugate vector.

Furthermore, there are no solutions of the form

$$e^{\alpha t}(t\underline{b} + \underline{c}), \quad \underline{b} \neq 0. \tag{2.6}$$

$\alpha(\mu)$ is analytic at $\mu = 0$. One can choose a fixed real vector $\underline{e} \neq 0$ so that for all small $|\mu|$, $\underline{a} \cdot \underline{e} \neq 0$ for $\underline{a} \neq 0$. $\underline{a} = \underline{a}(\mu)$ is then uniquely determined by the condition

$$\underline{a}(\mu)\cdot\underline{e} = \frac{1}{\alpha(\mu)-\overline{\alpha}(\mu)} \quad (\overline{\underline{e}} = \underline{e} \neq 0). \tag{2.7}$$

By hypothesis,

$$\overline{\alpha}(0) = -\alpha(0) \neq 0. \tag{2.8}$$

$\underline{a}(\mu)$ is analytic at $\mu = 0$.

The real solutions of (2.4), which are linear combinations of (2.5), have the form

$$\underline{z} = ce^{\alpha t}\underline{a} + \overline{c}e^{\overline{\alpha}t}\overline{\underline{a}} \tag{2.9}$$

with complex scalar c. They form a family depending on two real parameters; one of these parameters is a proportionality factor, while the other represents an additive constant in t (the solutions form only a one parameter family of curves). Because $\overline{\underline{e}} = \underline{e}$, we have

$$\left.\begin{aligned} \underline{z}\cdot\underline{e} &= c\ \underline{a}\cdot\underline{e} + \overline{c}\ \overline{\underline{a}\cdot\underline{e}} \\ \dot{\underline{z}}\cdot\underline{e} &= c\ \alpha\ \underline{a}\cdot\underline{e} + \overline{c}\ \overline{\alpha}\ \overline{\underline{a}\cdot\underline{e}} \end{aligned}\right\} \quad \text{at} \quad t = 0.$$

For $c = 1$, (2.9) is

$$\underline{z} = e^{\alpha t}\underline{a} + e^{\overline{\alpha}t}\overline{\underline{a}} = \underline{z}(t,\mu); \tag{2.10}$$

because of (2.7), this $\underline{z}$ satisfies the conditions:

$$t = 0: \quad \underline{z}\cdot\underline{e} = 0, \quad \frac{d}{dt}(\underline{z}\cdot\underline{e}) = 1. \tag{2.11}$$

This is the unique solution of the form (2.9) satisfying these conditions; for from

$$t = 0: \quad \underline{z}\cdot\underline{e} = \dot{\underline{z}}\cdot\underline{e} = 0$$

and from (2.9), (2.7) and (2.8) it follows that $c = 0$; thus $\underline{z} = 0$.

By hypothesis, for $\mu = 0$, α, $\overline{\alpha}$ are the only ones among the characteristic exponents which are pure imaginary. Hence, for $\mu = 0$, (2.9) gives all the real and periodic solutions of (2.4). Their period is

$$T_0 = \frac{2\pi}{|\alpha(0)|} \quad . \tag{2.12}$$

In particular, for $\mu = 0$, (2.10) is the only real and periodic solution with the properties (2.11).

For later use we also notice that, for $\mu = 0$, (2.4) can have no solutions of the form

$$t\,\underline{p}(t) + \underline{q}(t)$$

where $\underline{p}$ and $\underline{q}$ have a common period and $\underline{p}$ is not identically zero. Otherwise (2.4) would break up into the two equations

$$\dot{\underline{p}} = \underline{L}_0(\underline{p}), \quad \underline{p} + \dot{\underline{q}} = \underline{L}_0(\underline{q})$$

and $\underline{p}$ would be a nontrivial linear combination of the solutions (2.5). The Fourier expansion of $\underline{q}(t)$ would then lead to a solution of the form (2.6).

By differentiation of (2.4) with respect to μ at $\mu = 0$ one obtains the non-homogeneous differential equation

$$\dot{\underline{z}}' = \underline{L}_0(\underline{z}') + \underline{L}_0'(\underline{z}); \quad \underline{L}_0' = \frac{d}{d\mu}\underline{L}_\mu, \quad \mu = 0, \tag{2.13}$$

for the μ-derivative of (2.10):

$$\underline{z}' = t(\alpha' e^{\alpha t}\underline{a} + \overline{\alpha}' e^{\overline{\alpha} t}\overline{\underline{a}}) + (e^{\alpha t}\underline{a}' + e^{\overline{\alpha} t}\overline{\underline{a}}') \quad (\mu = 0).$$

The factor of t is a solution of (2.4). If one expresses it linearly in terms of the solution (2.10) and $\dot{\underline{z}}$, it follows from (2.8) that

$$\underline{z}' = t(\mathrm{Re}(\alpha')\underline{z} + \frac{\mathrm{Im}(\alpha')}{\alpha}\dot{\underline{z}}) + \underline{h}(t) \tag{2.14}$$

with

$$\underline{h}(t + T_0) = \underline{h}(t). \tag{2.15}$$

Now let

$$\underline{y} = \underline{y}(t,\mu,\varepsilon,\underline{y}^0)$$

be the solution of (2.3), which satisfies the initial condition $y = y^0$ for $t = 0$. According to well known theorems it depends analytically on all its arguments at each point $(t,0,0,\underline{y}^0)$. It is periodic with the period T if and only if the equation

$$\underline{y}(T,\mu,\varepsilon,\underline{y}^0) - \underline{y}^0 = 0 \tag{2.16}$$

is satisfied. If one denotes by $\underline{z}^0$ the fixed initial value of the fixed solution (2.10) of (2.4), $\mu = 0$, then (2.16) is satisfied by the values

$$T = T_0,\ \mu = \varepsilon = 0,\ \underline{y}^0 = \underline{z}^0. \tag{2.17}$$

The problem is: for given ε, solve equation (2.16) for T, μ and $\underline{y}^0$. These are n equations with $n + 2$ unknowns. In order to make the solution unique, we add the two equations

$$\underline{y}^0 \cdot \underline{e} = 0,\quad \dot{\underline{y}}^0 \cdot \underline{e} = 1 \tag{2.18}$$

where $\underline{e}$ is the real vector introduced above and where

$\dot{\underline{y}}^0 = \dot{\underline{y}}$ for $t = 0$. The introduction of these conditions implies no restriction on the totality of solutions in the small, as will be demonstrated in the next section. For the initial values $\mu = \varepsilon = 0$, $\underline{y}^0 = \underline{z}^0$, it follows from (2.11) that these equations are satisfied by the solution (2.10).

Now for all sufficiently small $|\varepsilon|$, (2.16) and (2.18) have exactly one solution

$$T = T(\varepsilon), \quad \mu = \mu(\varepsilon), \quad \underline{y}^0 = \underline{y}^0(\varepsilon) \tag{2.19}$$

in a suitable neighborhood of the system of values

$$T = T_0, \quad \mu = 0, \quad \underline{y}^0 = \underline{z}^0, \tag{2.20}$$

if the following is the case: the system of linear equations formed by taking the differential (at the place (2.17)) with respect to the variables T, μ, ε, $\underline{y}^0$ is uniquely solvable for given $d\varepsilon$. Equivalently, there are such functions (2.19) if these linear equations for $d\varepsilon = 0$ have only the zero solution $dT = d\mu = d\underline{y}^0 = 0$. This is the case, as will now be shown.

We have

$$\dot{\underline{y}} = \underline{L}_\mu(\underline{y}), \quad \underline{y} = \underline{y}(t,\mu,0,\underline{y}^0). \tag{2.21}$$

In particular

$$\underline{y}(t,\mu,0,\underline{z}^0) = \underline{z}(t,\mu) \tag{2.22}$$

is the solution to (2.10). The differential $d\underline{y}(t,\mu,0,\underline{y}^0)$ is the sum of the differentials with respect to the separate arguments when the others are all fixed. If we introduce for the differentials

$$dt, \quad d\mu, \quad d\underline{y}^0$$

as independent constants or vectors the notations:

$$\rho, \quad \sigma, \quad \underline{u}^0$$

then, the differential referred to becomes

$$\rho\dot{\underline{y}} + \sigma\underline{y}' + \underline{u}.$$

Here $\dot{\underline{y}}$ and $\underline{y}' = \partial\underline{y}/\partial\mu$ are taken at $T = T_0$, $\mu = 0$, $\underline{y}^0 = \underline{z}^0$ and $\underline{u}$ is the solution of

$$\dot{\underline{u}} = \underline{L}_0(\underline{u})$$

with the initial value $\underline{u}^0$ for $t = 0$. According to (2.22), $\dot{\underline{y}} = \dot{\underline{z}}(t,0)$. If one sets $\underline{y}' = \underline{v}$, then $\underline{v}(t)$ is the solution of

$$\dot{\underline{v}} = \underline{L}_0(\underline{v}) + \underline{L}_0'(\underline{z}), \quad \underline{v}(0) = 0. \tag{2.23}$$

The linear vector equation arising from (2.16) is then

$$\rho\dot{\underline{z}}(T_0) + \sigma\underline{v}(T_0) + \underline{u}(T_0) - \underline{u}(0) = 0, \tag{2.24}$$

where $\underline{z}(t)$ denotes the solution (2.10) of

$$\dot{\underline{z}} = \underline{L}_0(\underline{z}). \tag{2.25}$$

$\underline{u}(t)$ is any solution of this homogeneous linear differential equation with constant $\underline{L}_0$, and $\underline{v}(t)$ is the solution of (2.23). We show now that (2.24) is possible only if $\rho = \sigma = 0$ and $\underline{u}(t) = 0$.

Now for all t

$$\rho\dot{\underline{z}}(t) + \sigma[\underline{v}(t+T_0) - \underline{v}(t)] + \underline{u}(t+T_0) - \underline{u}(t) = 0. \tag{2.26}$$

This is true because $\underline{z}(t)$ has period T_0, so it follows from (2.23) that the square bracket is a solution of (2.25). $\dot{z}$ is also a solution of (2.25)+, so the whole left

+ In the original, this number is (2.23).

side of (2.26) is a solution of (2.25). By (2.24) and the fact that $\underline{v}(0) = 0$, the initial value of this solution is zero, and thus it is identically zero. Now from (2.13) and (2.23) it follows that

$$\underline{v}(t) = \underline{z}'(t) + \underline{g}(t), \quad \dot{\underline{g}} = \underline{L}_0(\underline{g}).$$

Thus, by (2.14) and (2.15), the square bracket in (2.26) has the value

$$T_0[\mathrm{Re}(\alpha')\underline{z}(t) + \frac{\mathrm{Im}(\alpha')}{\alpha}\dot{\underline{z}}(t)] + [\underline{g}(t+T_0) - \underline{g}(t)].$$

If one sets $\underline{u} + \sigma\underline{g} = \underline{w}$ and

$$\sigma T_0 \mathrm{Re}(\alpha')\underline{z}(t) + [\rho+\sigma T_0 \frac{\mathrm{Im}(\alpha')}{\alpha}]\dot{\underline{z}}(t) = \tilde{\underline{z}}(t), \tag{2.27}$$

it follows that

$$\tilde{\underline{z}}(t) + \underline{w}(t+T_0) - \underline{w}(t) = 0,$$

where $\underline{w}(t)$ is a solution and $\tilde{\underline{z}}(t)$ a periodic solution of (2.25). This means that

$$\underline{w}(t) = -\frac{t}{T_0}\tilde{\underline{z}}(t) + q(t)$$

with periodic q. However, as we stated before, such solutions cannot exist unless $\tilde{\underline{z}} = 0$.

Since $\underline{z}$, $\dot{\underline{z}}$ are linearly independent, it follows from (2.27) and from the hypothesis (1.2) that $\sigma = 0$ and $\rho = 0$. Thus, by (2.24), $\underline{u}(t)$ has period T_0.

Finally, since $d\underline{y}^0 = \underline{u}^0$, and $d\dot{\underline{y}}^0 = \dot{\underline{u}}$, at $t = 0$, it follows from the equations (2.18) that

$$\underline{u}\cdot\underline{e} = \dot{\underline{u}}\cdot\underline{e} = 0, \quad \text{at} \quad t = 0.$$

A periodic solution of $\dot{\underline{u}} = \underline{L}_0(\underline{u})$ with these properties must vanish, as we have stated above. With this the proof of the existence of a periodic family is concluded.†

The solutions (2.19) are analytic at $\varepsilon = 0$

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\begin{aligned}
T &= T_0(1 + \tau_1\varepsilon + \tau_2\varepsilon^2 + \dots), \\
\mu &= \mu_1\varepsilon + \mu_2\varepsilon^2 + \dots \quad .
\end{aligned}
\tag{2.28}
$$

The periodic solutions $\underline{y}(t,\varepsilon)$ of (2.3), and the periodic family of solutions

$$\underline{x}(t,\varepsilon) = \varepsilon\underline{y}(t,\varepsilon) \tag{2.29}$$

of (1.1)+, are analytic at every point $(t,0)$.

One obtains exactly the same periodic solutions if one begins with a multiple mT_0 of the period instead of T_0, that is if one operates in a neighborhood of the system of values

$$T = mT_0,\ \mu = 0,\ y^0 = z^0 \tag{2.30}$$

instead of (2.20). Nothing essential is altered in the proof.

3. Completion of the Proof of the Theorem.

For arbitrarily large $L > T_0$ there are two positive numbers a and b with the following property. Every periodic solution $\underline{x}(t) \neq 0$ of (1.1)+, whose period is smaller than L, which belongs to a μ with $|\mu| < b$ and which lies in $|\underline{x}| < a$, belongs to the family (2.29), (2.28), $\varepsilon > 0$ if a suitable choice is made for the origin of t.

If this were not the case, there would be a sequence

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† See editorial comments in §5A below.

+ In the original paper, this number is (1).

of periodic solutions $\underline{x}_k(t) \neq 0$[++] having bounded periods $T_k < L$, and of corresponding μ-values, with

$$\kappa_k = \operatorname*{Max}_t |\underline{x}_k(t)| \to 0, \quad \mu_k \to 0 \tag{3.1}$$

and such that no pair $\underline{x}_k(t)$, μ_k belongs to the above family. We let

$$\underline{y}_k(t) = \frac{1}{\kappa_k}\, \underline{x}_k(t).$$

$\underline{y}_k$ is a solution of (2.3), with κ_k instead of ε, and $\underline{y}_k$ satisfies

$$\operatorname*{Max}_t |\underline{y}_k(t)| = 1.$$

One considers first a subsequence for which the initial values converge, $\underline{y}_k^{\,0} \to \underline{z}^0$. Then, uniformly for $|t| < L$, we have $\underline{y}_k(t) \to \underline{z}(t)$, where $\dot{\underline{z}} = \underline{L}_0(z)$ and $\underline{z}(0) = \underline{z}^0$. Since the maximum of $|\underline{z}| = 1$, $\underline{z}$ is not identically zero. $\underline{z}$ is of the form (2.9)[+], $c \neq 0$, and it has the fundamental period T_0. If one shifts the origin of t in $\underline{z}(t)$ to the place where $\underline{z} \cdot \underline{e} = 0$, one finds that $\dot{\underline{z}} \cdot \underline{e} \neq 0$ there. This quantity can be taken to be positive; otherwise, since

$$\underline{z}(t + \tfrac{1}{2} T_0) = -\underline{z}(t),$$

one could achieve this by shifting from $t = 0$ by $\frac{1}{2} T_0$. Consequently,

$$\underline{z}^0 \cdot \underline{e} = 0, \quad \dot{\underline{z}}^0 \cdot \underline{e} > 0.$$

From this it follows that in the neighborhood of $\underline{z}^0$, and for

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[++] In the original paper, the sequences x_k, T_k, etc. are not indexed.

[+] In the original, this number is (2.8).

small κ and $|\mu|$, all solutions of the differential equation (2.3) (κ instead of ε) cut the hyperplane $\underline{y} \cdot \underline{e} = 0$ once. In this intersection let $t = 0$. Then, for the sequence $\underline{y}_k(t)$, κ_k, μ_k under consideration, with this choice of origin, we always have $\underline{y}_k{}^0 \to \underline{z}^0$. Also

$$\dot{\underline{y}}_k^0 \cdot \underline{e} = 0, \ \rho_k = \dot{\underline{y}}_k^0 \cdot \underline{e} \to \dot{\underline{z}}^0 \cdot \underline{e} = \rho > 0$$

and $\kappa_k \to 0$, $\mu_k \to 0$. If one now sets

$$\tilde{\underline{y}}_k(t) = \frac{1}{\rho_k} \underline{y}_k(t) = \frac{1}{\rho_k \kappa_k} \underline{x}_k(t), \quad \rho_k \kappa_k = \varepsilon_k$$

then $\tilde{\underline{y}}_k$ is a solution of (2.3)+, for the parameter values $\varepsilon_k > 0$ and μ_k. For it, we have

$$\tilde{\underline{y}}_k \cdot \underline{e} = 0, \quad \dot{\tilde{\underline{y}}}_k \cdot \underline{e} = 1, \quad \text{at} \quad t = 0. \tag{2.18}$$

The periods in the sequence of solutions must converge to a multiple of T_0, mT_0. Furthermore $\varepsilon_k \to 0$. However, this implies that from some point on in the sequence one enters the neighborhood mentioned above of (2.20) or (2.30) in which, for all sufficiently small ε, there is only one solution of the system of equations under consideration. The solutions of our sequence must then belong to the above family and in fact with $\varepsilon > 0$, which conflicts with the assumption. Thus the assertion is proved.††

From the fact we have just proved it now follows that if $\mu(\varepsilon) \not\equiv 0$, then the first coefficient which is different from 0 in $\mu = \mu_1 \varepsilon + \mu_2 \varepsilon^2 + \ldots$ is of even order; the same

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+ In the original, this number is (3).

†† See editorial comments in §5A below.

holds for the expansion $T = T_0 \ (1 + \tau_1\varepsilon + \tau_2\varepsilon^2 + \ldots)$. For the solutions of the family corresponding to $\varepsilon < 0$, and the associated μ and T-values, must already be present among those for $\varepsilon > 0$.++ In particular we have

$$\mu_1 = \tau_1 = 0. \tag{3.2}$$

The periodic solutions exist, for sufficiently small $|\mu|$ and $|\underline{x}|$, only for $\mu > 0$, or only for $\mu < 0$, or only for $\mu = 0$.

4. Determination of the Coefficients.

We shall need the following result, which gives a criterion for the solvability of the inhomogeneous system of differential equations

$$\dot{\underline{w}} = \underline{L}(\underline{w}) + \underline{q}, \quad (\underline{L} = \underline{L}_0) \tag{4.1}$$

where $\underline{q}(t)$ has a period T_0. Let

$$\dot{\underline{z}}^* = -\underline{L}^*(\underline{z}^*) \tag{4.2}$$

be the differential equation which is adjoint to the homogeneous one; L^* is the adjoint operator to L (transposed matrix), defined by

$$\underline{L}(\underline{u}) \cdot \underline{v} = \underline{u} \cdot \underline{L}^*(\underline{v}).^+$$

Then (4.1) has a periodic solution $\underline{w}$ with the period T_0, if and only if

++ And indeed with a shift of the t-origin of approximately $T_0/2$.

+In the following, the inner product of two complex vectors a, b is defined by $\sum \bar{a}_i b_i$.

$$\int_0^{T_0} \underline{q}\cdot\underline{z}^* dt = 0 \tag{4.3}$$

for all solutions of (4.2) which have the period T_0.

This result follows from the known criterion for the solvability of an ordinary system of linear equations. The necessity follows directly from (4.1) and (4.2). That the condition is sufficient can be shown in the following way: The adjoint equation has the same characteristic exponents and therefore it also has two solutions of the form

$$e^{\alpha t}a^*, \quad e^{-\alpha t}\overline{a}^*, \quad \alpha = \alpha(0) = -\overline{\alpha}(0), \tag{4.4}$$

from which all periodic solutions can be formed by linear combinations. Furthermore, the development of $\underline{q}(t)$ in Fourier series shows that it suffices to consider the case

$$\underline{q} = e^{-\alpha t}\underline{b}$$

and the analogous case with α instead of $-\alpha$. In (4.1) let us insert

$$\underline{w} = e^{-\alpha t}\underline{c}.$$

(4.1) then becomes

$$(\alpha I + \underline{L})\underline{c} = \underline{b}.$$

(4.4) and (4.2) imply

$$(\alpha I + \underline{L})^*\underline{a}^* = 0$$

while (4.3) says $\underline{b}\cdot\underline{a}^* = 0$. From this everything follows with the help of the theorem referred to.

Secondly, we shall need the following fact. For any solution $\underline{z} \neq 0$ of $\dot{\underline{z}} = \underline{L}(\underline{z})$ having period T_0, there is always a solution $\underline{z}^*$ of the adjoint equation, with the same period, such that

$$\int_0^{T_0} \underline{z}\cdot\underline{z}^* dt \neq 0.^*$$

Otherwise, the equation $\dot{\underline{w}} = \underline{L}(\underline{w}) + \underline{z}$ would have a solution $\underline{w}$, and $\underline{w} - t\underline{z}^+$ would be a solution of the homogeneous differential equation, which contradicts the simplicity of the characteristic exponent α.

Let $\underline{z}_1^*$ and $\underline{z}_2^*$ be two linearly independent solutions of (4.2) with the period T_0. Let

$$[\underline{q}]_i = \int_0^{T_0} \underline{q}\cdot\underline{z}_i^* \, dt \qquad (i = 1,2).$$

Then the criterion for solvability of (4.1) under the given conditions is

$$[\underline{q}]_1 = [\underline{q}]_2 = 0. \tag{4.5}$$

We also note that z_1^*, z_2^* can be chosen in such a way that

$$[\underline{z}]_1 = [\dot{\underline{z}}]_2 = 1, \quad [\underline{z}]_2 = [\dot{\underline{z}}]_1 = 0 \tag{4.6}$$

where $\underline{z}$ is the solution (2.10) of (2.4) with $\mu = 0$ (biorthogonalization).

The problem of the determination of the coefficients for the power series representation of the periodic family can now be solved in a general way. If one defines the new independent variable s by

$$t = s(1 + \tau_2\varepsilon^2 + \tau_3\varepsilon^3 + \dots) \tag{4.7}$$

then according to (2.28) the period in the family of solutions

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* Also, the integrand is always constant.

+ In the original, the statement reads "w+tz", which is incorrect.

$\underline{y} = \underline{y}(s,\varepsilon)$ is constantly equal to T_0. $\underline{y}$, as a function of s (or t) and ε, is analytic at every point $(s,0)$. One has

$$\underline{y} = \underline{y}_0(s) + \varepsilon\underline{y}_1(s) + \varepsilon^2\underline{y}_2(s) + \dots, \tag{4.8}$$

where all the $\underline{y}_i$ have the period T_0. The derivative with respect to s will again be denoted by a dot. We write for simplicity

$$\underline{L}_0 = \underline{L},\ \underline{L}_0' = \underline{L}',\ \underline{Q}_0 = \underline{Q},\ \underline{K}_0 = \underline{K},\ \dots$$

Then, using (3.2), and inserting (4.7) and (4.8) in (2.3), one obtains the recursive equations

$$\dot{\underline{y}}_0 = \underline{L}(\underline{y}_0) \qquad (\underline{y}_0 = \underline{z}) \tag{4.9}$$

$$\dot{\underline{y}}_1 = \underline{L}(\underline{y}_1) + \underline{Q}(\underline{y}_0,\underline{y}_0) \tag{4.10}$$

$$\begin{aligned}-\tau_2\dot{\underline{y}}_0 + \dot{\underline{y}}_2 = \underline{L}(\underline{y}_2) + \mu_2\underline{L}'(\underline{y}_0) &+ 2\underline{Q}(\underline{y}_0,\underline{y}_1)\\ &+ \underline{K}(\underline{y}_0,\underline{y}_0,\underline{y}_0)\end{aligned} \tag{4.11}$$

$$\dots\dots\dots\dots\dots$$

from which the $\underline{y}_i$, μ_i, τ_i are to be determined. In addition to these, we have the conditions following from (2.18)

$$\underline{y}_k \cdot \underline{e} = \dot{\underline{y}}_k \cdot \underline{e} = 0, \quad \text{at} \quad s = 0 \tag{4.12}$$

for $k = 1,2,\dots$. In the equations, we again write t instead of s. By (4.10) and (4.12), y_1 is uniquely determined as a periodic function with period T_0. From (4.11) $\underline{L}'$ must first be eliminated with the help of (2.13). Since the parenthesis in the first summand of (2.14) is a solution of $\dot{\underline{z}} = \underline{L}(\underline{z})$, (2.13) can be written

$$\operatorname{Re}(\alpha')\underline{z} + \frac{\operatorname{Im}(\alpha')}{\alpha}\dot{\underline{z}} + \dot{\underline{h}} = \underline{L}(\underline{h}) + \underline{L}'(\underline{z}) \tag{4.13}$$

Let

$$\underline{y}_2 - \mu_2\underline{h} = \underline{v}, \tag{4.14}$$

which, according to (2.15), has the period T_0. Since $\underline{z} = \underline{y}_0$, it follows that

$$\begin{aligned} -\mu_2\operatorname{Re}(\alpha')\underline{y}_0 - (\tau_2 + \mu_2\frac{\operatorname{Im}(\alpha')}{\alpha})\dot{\underline{y}}_0 + \dot{\underline{v}} \\ = \underline{L}(\underline{v}) + 2\underline{Q}(\underline{y}_0,\underline{y}_1) + \underline{K}(\underline{y}_0,\underline{y}_0,\underline{y}_0). \end{aligned} \tag{4.15}$$

Thus, by (4.6)

$$\begin{aligned} \mu_2\operatorname{Re}(\alpha') &= -[2\underline{Q}(\underline{y}_0,\underline{y}_1) + \underline{K}(\underline{y}_0,\underline{y}_0,\underline{y}_0)]_1, \\ \tau_2 + \mu_2\frac{\operatorname{Im}(\alpha')}{\alpha} &= -[2Q(\underline{y}_0,\underline{y}_1) + K(\underline{y}_0,\underline{y}_0,\underline{y}_0)]_2. \end{aligned} \tag{4.16}$$

By hypothesis (1.2), μ_2 and τ_2 are determined from this. One then solves (4.15) for $\underline{v}$ and obtains $\underline{y}_2$ from (4.14) and (4.12), $k = 2$, in a unique way.

In an analogous fashion all the higher coefficients are obtained from the subsequent recursion formulas. In general $\mu_2 \neq 0$. If μ_2 is positive then the periodic solutions exist only for $\mu > 0$; the corresponding statement holds for $\mu_2 < 0$.†††

5. The Characteristic Exponents of the Periodic Solution.

In the following we shall sometimes make use of determinants; this, however, can be avoided. In the linearization about the periodic solutions of (2.3),

$$\underline{u} = \underline{L}_{t,\varepsilon}(\underline{u}) \tag{5.1}$$

†††See editorial comments in §5A below.

we have, by (2.3),

$$\underline{L}_{t,\varepsilon}(\underline{u}) = \underline{L}_{\mu}(\underline{u}) + 2\varepsilon\underline{Q}_{\mu}(\underline{y},\underline{u}) + 3\varepsilon^2\underline{K}_{\mu}(\underline{y},\underline{y},\underline{u})+\ldots \qquad (5.2)$$

A fundamental system $\underline{u}_i(t,\varepsilon)$ formed with fixed initial conditions depends analytically on (t,ε). The coefficients in $\underline{u}_i(T,\varepsilon) = \sum a_{i\nu}(\varepsilon)\underline{u}_{\nu}(0)$ are analytic at $\varepsilon = 0$. The determinantal equation

$$||a_{ik}(\varepsilon) - \zeta\delta_{ik}|| = 0, \quad \zeta = e^{\lambda T(\varepsilon)}, \qquad (5.3)$$

determines the characteristic exponents λ_k and the solutions $\underline{v}$, of (1.3), where

$$\underline{u} = e^{\lambda t}\underline{v} \quad .$$

Since (5.1) is solved by $\underline{u} = \dot{\underline{y}}$, $\zeta = 1$ is a root of (5.3). The exponent β, which was spoken of in the introduction, corresponds to a simple root of the equation obtained by dividing out $\zeta - 1$. $\beta(\varepsilon)$ is thus real and analytic at $\varepsilon = 0$, $\beta = \beta_2\varepsilon^2 + \ldots$ (β_1 is also equal to zero for the same reasons as μ_1 and τ_1). Now if β is not $\equiv 0$, then there is some minor of order $n - 1$ in the determinant (5.3) (with the corresponding ζ) which is not 0. From this it follows that (1.3), $\lambda = \beta$, has a solution $\underline{v} \not\equiv 0$ which is analytic at $\varepsilon = 0$. Even if $\beta \equiv 0$, there is a minor of order $n - 2$ which is not zero. As we know, in this case, there is a solution of (5.1), analytic at $\varepsilon = 0$, of the form $\underline{u} = t\underline{v} + \underline{w}$ with periodic $\underline{v}$, $\underline{w}$, where either $\underline{v} \not\equiv 0$, or $\underline{v} = 0$ and $\underline{w}$ is linearly independent of the solution $\underline{u} = \dot{\underline{y}}$.* That $t\underline{v} + \underline{w}$ is a solution implies that

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* Cf. e.g. F. R. Moulton, Periodic Orbits, Washington, 1920, p. 26.

$$\dot{\underline{v}} = \underline{L}_{t,\varepsilon}(\underline{v}), \quad \underline{v} + \dot{\underline{w}} = \underline{L}_{t,\varepsilon}(\underline{w}). \tag{5.4}$$

After these preliminary observations we shall calculate β_2. We assume here that $\mu_2 \neq 0$. $\beta \equiv 0$ is then impossible as will subsequently be proved. If we use (4.7) to introduce s as a new t into (1.3) we get

$$(1 - \tau_2\varepsilon^2 + \ldots)\dot{\underline{v}} + \beta\underline{v} = \underline{L}_{t,\varepsilon}(\underline{v}).$$

Also, we have (with the new t)

$$\underline{v} = \underline{v}_0(t) + \varepsilon\underline{v}_1(t) + \varepsilon^2\underline{v}_2(t) + \ldots$$

where all the $\underline{v}_i$ have the same period T_0. If we introduce the power series for μ, β, $\underline{v}$, $\underline{y}$, it follows (dropping the subscript zero on the operators as before) that

$$\dot{\underline{v}}_0 = \underline{L}(\underline{v}_0), \tag{5.5}$$

$$\dot{\underline{v}}_1 = \underline{L}(\underline{v}_1) + 2\underline{Q}(\underline{y}_0,\underline{v}_0), \tag{5.6}$$

$$\begin{aligned}\beta_2\underline{v}_0 - \tau_2\dot{\underline{v}}_0 + \dot{\underline{v}}_2 &= \underline{L}(\underline{v}_2) + \mu_2\underline{L}'(\underline{v}_0) \\ &\quad + 2\underline{Q}(\underline{y}_1,\underline{v}_0) + 2\underline{Q}(\underline{y}_0,\underline{v}_1) + 3\underline{K}(\underline{y}_0,\underline{y}_0,\underline{v}_0).\end{aligned} \tag{5.7}$$

These equations have the trivial solution

$$\beta_i = 0, \quad v_i = \dot{y}_i \quad (i = 0,1,\ldots). \tag{5.8}$$

Thus, one has

$$\ddot{\underline{y}}_1 = \underline{L}(\dot{\underline{y}}_1) + 2\underline{Q}(\underline{y}_0,\dot{\underline{y}}_0). \tag{5.9}$$

$$\begin{aligned}-\tau_2\ddot{y}_0 + \ddot{\underline{y}}_2 &= \underline{L}(\dot{\underline{y}}_2) + \underline{\mu}_2 L'(\dot{\underline{y}}_0) \\ &\quad + 2\underline{Q}(\underline{y}_1,\dot{\underline{y}}_0) + 2\underline{Q}(\underline{y}_0,\dot{\underline{y}}_1) + 3\underline{K}(\underline{y}_0,\underline{y}_0,\dot{\underline{y}}_0).\end{aligned} \tag{5.10}$$

Since we may assume that $\underline{v}_0 \neq 0$

$$\underline{v}_0 = \rho\underline{y}_0 + \sigma\dot{\underline{y}}_0,^* \tag{5.11}$$

where at least one of the coefficients is not 0. If we set

$$\underline{v}_1 - 2\rho\underline{y}_1 - \sigma\underline{y}_1 = \underline{w}, \tag{5.12}$$

it follows from (4.10), (5.6) and (5.9) that $\dot{\underline{w}} = \underline{L}(\underline{w})$, thus

$$\underline{w} = \rho'\underline{y}_0 + \sigma'\dot{\underline{y}}_0. \tag{5.13}$$

If one forms the combination

$$(5.7) - \rho(4.11) - \sigma(5.10),$$

in which $\underline{L}'$ cancels out, and sets

$$\underline{v}_2 - \rho\underline{y}_2 - \sigma\dot{\underline{y}}_2 = \underline{u},$$

then, using (5.11) and (5.12), one obtains:

$$\beta_2\underline{v}_0 + \dot{\underline{u}} = \underline{L}(\underline{u}) + 2\rho(2\underline{Q}(\underline{y}_0,\underline{y}_1) + \underline{K}(\underline{y}_0,\underline{y}_0,\underline{y}_0))+\underline{R} \tag{5.14}$$

with

$$\underline{R} = 2\underline{Q}(\underline{y}_0,\underline{w}).$$

If we now apply the bracket criterion of the previous section to (4.10) and (5.9), it follows from (5.13) that

$$[\underline{R}]_1 = [\underline{R}]_2 = 0.$$

If we apply it to (5.14), in which $\underline{u}$ has the period T_0, it follows from (4.6) (with $\underline{z} = \underline{y}_0$) that

$$\rho\beta_2 = 2\rho[2\underline{Q}(\underline{y}_0,\underline{y}_1) + \underline{K}(\underline{y}_0,\underline{y}_0,\underline{y}_0)]_1.$$

* The ρ and σ in (5.11) are unrelated to the symbols ρ and σ as used in Section 2.

Hence, by (4.16),

$$\rho\beta_2 = -2\rho\mu_2 \underline{R}(\alpha').$$

Likewise, it follows that

$$\sigma\beta_2 = -2\rho(\tau_2 + \mu_2 \frac{\mathrm{Im}(\alpha')}{\alpha}).$$

From this, either β_2 is given by (1.4) (and then β_2 is not zero since $\mu_2 \neq 0$) or else $\beta_2 = 0$. In either case $\rho:\sigma$ is completely determined (in the second case $\rho = 0$).

To check that the first case really occurs we must undertake a somewhat longer consideration. One may think of the process as schematized in the following manner. The equation for β and $\underline{v}$ (namely the equation which follows equation (5.4)) should be divided by the factor in parenthesis. It is then once again of the form

$$\dot{\underline{v}} + \beta\underline{v} = \underline{L}_{t,\varepsilon}(\underline{v})$$

with

$$\underline{L}_{t,\varepsilon} = \underline{L}_0 + \varepsilon\underline{L}_1 + \varepsilon^2\underline{L}_2 + \ldots,$$

where $\underline{L}_0$ is a constant operator, while L_i, $i > 0$, depend on t with the period T_0. The coefficients of $1, \varepsilon$ are not altered by the division. Introduction of the power series leads to

$$\begin{aligned} \dot{\underline{v}}_0 &= \underline{L}_0(\underline{v}_0), \\ \dot{\underline{v}}_1 &= \underline{L}_0(\underline{v}_1) + \underline{L}_1(\underline{v}_0), \end{aligned} \quad * \tag{5.15}$$

$$\beta_2\underline{v}_0 + \dot{\underline{v}}_2 = \underline{L}_0(\underline{v}_2) + \underline{L}_1(\underline{v}_1) + \underline{L}_2(\underline{v}_0),$$

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*One does not really have to assume $\beta_1 = 0$. From the bracket criterion this is a consequence of (5.17).

$$\beta_3\underline{v}_0 + \beta_2\underline{v}_1 + \dot{\underline{v}}_3 = \underline{L}_0(\underline{v}_3) + \underline{L}_1(\underline{v}_2) + \underline{L}_2(\underline{v}_1) + \underline{L}_3(\underline{v}_0)$$

and so forth. The situation is the following. For $\varepsilon = 0$ there are two solutions $\underline{z}$, $\dot{\underline{z}}$ with period T_0. Furthermore

$$\underline{v}_0 = \rho\underline{z} + \sigma\dot{\underline{z}} \tag{5.16}$$

and

$$[\underline{L}_1(\underline{z})] = [\underline{L}_1(\dot{\underline{z}})] = 0 \tag{5.17}$$

for both bracket subscripts. It follows that

$$\underline{v}_1 = \rho\underline{g} + \sigma h + \rho'\underline{z} + \sigma'\underline{z} \tag{5.18}$$

with fixed periodic $\underline{g}$ and $\underline{h}$. For the third equation of (5.15), the bracket criterion gives

$$\begin{aligned} \beta_2\rho &= A_1\rho + B_1\sigma \\ \beta_2\sigma &= A_2\rho + B_2\sigma \end{aligned} \tag{5.19}$$

with

$$\begin{aligned} A_i &= [\underline{L}_1(\underline{g}) + \underline{L}_2(\underline{z})]_i, \\ B_i &= [\underline{L}_1(\underline{h}) + \underline{L}_2(\dot{\underline{z}})]_i, \end{aligned} \tag{5.20}$$

while (5.17) implies that ρ', σ' drop out. The situation now is that the equations (5.19) with the unknowns β_2, ρ, σ have two distinct real solutions β_2.* To them belong two linearly independent pairs (ρ,σ). Each of the two solution systems leads now to a unique determination of the β_i and $\underline{v}_i$ through the recursion formulas, if one suitably normalizes

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*In the general case, that is if the special condition (5.17) is not fulfilled, the splitting into two cases occurs already at the second equation (5.15). The solution of the problem in this case is found in F. R. Houlton, Periodic Orbits. Compare Chapter 1, particularly pages 34 and 40.

$\underline{v}$. To this end choose a constant vector $\underline{a} \neq 0$ in such a way that $\underline{v}_0 \cdot \underline{a} = 1$ $(t = 0)$ for both pairs (ρ,σ) in (5.16). One concludes then that

$$\underline{v} \cdot \underline{a} = 1, \quad t = 0,$$

that is, $\underline{v}_i \cdot \underline{a} = 0$ at $(t = 0)$ for $i > 0$. Let

$$\underline{z} \cdot \underline{a} = C, \quad \dot{\underline{z}} \cdot \underline{a} = D \quad (t = 0).$$

Then, for either of the two values of β_2, the system of equations

$$\begin{aligned} (A_1 - \beta_2)\rho + B_1\sigma &= 0 \\ A_2\rho + (B_2 - \beta_2)\sigma &= 0 \\ C\rho + D\sigma &= 1 \end{aligned} \tag{5.21}$$

uniquely determines the unknowns ρ and σ. Up to now β_2, ρ, σ, $\underline{v}_0$ are determined. Using the definition of $\underline{g}$, $\underline{h}$ and (5.18), one obtains from the third equation of (5.15)

$$\underline{v}_2 = \rho'\underline{g} + \sigma'\underline{h} + \rho''\underline{z} + \sigma''\dot{\underline{z}} + \ldots, \tag{5.22}$$

where the terms omitted are already known. From the fourth equation of (5.15) one obtains the equations

$$\begin{aligned} \rho\beta_3 - (A_1 - \beta_2)\rho' - B_1\sigma' &= \ldots, \\ \sigma\beta_3 - A_2\rho' - (B_2 - \beta_2)\sigma' &= \ldots \end{aligned}$$

by using (5.18), (5.20), (5.22) and the bracket criterion. Since $\underline{v}_1 \cdot a = 0$ $(t = 0)$, we add to these equations the equation

$$C\rho' + D\sigma' = \ldots$$

Through the three equations, the three quantities β_2, ρ', σ' are now uniquely determined. With the help of (5.21), the determinant is found to be

$$A_1 + B_2 - 2\beta_2.$$

It is not equal to zero, since by hypothesis, (5.19) has two distinct solutions β_2. From this β_3, ρ', σ' and $\underline{v}_1$ are determined.

It is now easy to see that at the next step β_4, ρ'', σ'' are determined by equations with exactly the same left hand side, and that by the further analogous steps everything is determined.

We return now to the special problem which interests us, and assume that by suitable normalization two different formal power series pairs $(\beta,\underline{v})$ exist which solve the equation

$$(1 - \tau_2\varepsilon^2 + \ldots)\dot{\underline{v}} + \beta\underline{v} = \underline{L}_{t,\varepsilon}(\underline{v}).$$

On the other hand it was previously demonstrated that under the assumption $\beta \not\equiv 0$, two actual solutions exist, of which one is known, namely (5.8). Under this assumption the second (normalized) solution can thus be represented by the power series and the formula (1.4) for β_2 does in fact hold. To dispose of this completely we must still show that $\beta \equiv 0$ cannot occur if $\mu_2 \neq 0$. We show this also in terms of the schematic problem. Since (5.19) has the solution $\beta_2 = \rho = 0$ and the second $\beta_2 \neq 0$,

$$B_1 = B_2 = 0, \quad A_1 \neq 0. \tag{5.23}$$

If β were $\equiv 0$, then (5.4) would have a solution with the properties given there.

Setting in the power series for $\underline{v}$, $\underline{w}$ gives

$$\begin{aligned} \underline{v}_0 + \dot{\underline{w}}_0 &= \underline{L}_0(\underline{w}_0) \\ \underline{v}_1 + \dot{\underline{w}}_1 &= \underline{L}_0(\underline{w}_1) + \underline{L}_1(\underline{w}_0) \\ \underline{v}_2 + \dot{\underline{w}}_2 &= \underline{L}_0(\underline{w}_2) + \underline{L}_1(\underline{w}_1) + \underline{L}_2(\underline{w}_0). \end{aligned} \tag{5.24}$$

We have

$$\underline{w}_0 = \rho\underline{z} + \sigma\dot{\underline{z}}. \tag{5.25}$$

Since v_0 is also of this form, according to the bracket criterion $\underline{v}_0$ must be equal to zero. By (5.17), it follows analogously that $\underline{v}_1 = 0$. Similarly, as in (5.18), we find

$$\underline{w}_1 = \rho\underline{g} + \sigma\underline{h} + \rho'\underline{z} + \sigma'\dot{\underline{z}}.$$

It has been demonstrated above that $\dot{\underline{v}} = \underline{L}_{t,\varepsilon}(\underline{v})$ has a solution $(\underline{y})$ of period T_0, unique up to a factor. Thus we certainly have

$$\underline{v}_2 = \lambda\dot{\underline{z}}.$$

As above, using (5.20), application of the bracket rule to (5.24) gives the equations

$$0 = A_1\rho + B_1\sigma,$$

$$\lambda = A_2\rho + B_2\sigma.$$

(in which ρ', σ' once again fall out). According to (5.23) it thus follows that $\rho = \lambda = 0$, and from this $\underline{v}_2 = 0$. According to (5.25) $\underline{w}_0 = \sigma\dot{z}$. If one subtracts from the second equation of (5.4) the solution $\sigma\dot{\underline{y}}$ of $\dot{\underline{w}} = \underline{L}(\underline{w})$ and divides by ε, then the whole process can be repeated, and we find successively that the $\underline{v}_i = 0$, and thus $\underline{v} = 0$. With this it is demonstrated that β cannot be equal to zero.

The verification of the formula (1.4) is thus complete under the assumption $\mu_2 \neq 0$. This assumption could be replaced by $\mu \not\equiv 0$. The considerations would be changed only in that in the calculation of the coefficients the case of splitting will occur later.

The difficulties of these considerations could be avoided in the following manner. One first calculates purely formally as above the coefficient of the power series for β and $\underline{v}$ and then shows the convergence directly by a suitable application of the method of majorants. This would correspond to our intention of facilitating the application to partial differential systems. But one can also carry out the discussion of the case of splitting and the proof of (1.4) exclusively with determinants.††††

†††† See editorial comments in Section 5A.

SECTION 5A

EDITORIAL COMMENTS

BY L. N. HOWARD AND N. KOPELL

(†) 1. Hopf's argument can be considerably simplified. After "blowing up" the equation (1.1) to (2.3), one wishes to show that for each sufficiently small ε there is a $\mu(\varepsilon)$, a period $T(\varepsilon)$ and initial conditions $\underline{y}^0(\varepsilon)$ (suitably normalized), so that (2.16) holds; the family of solutions to (1.1) asserted in the theorem is then $\underline{x}(t,\varepsilon) = \varepsilon\underline{y}(t,\mu(\varepsilon),\varepsilon,\underline{y}^0)$. Now (2.16) is satisfied if $\mu = \varepsilon = 0$, $\underline{y}^0 = \underline{z}^0$. Hence, the existence of the functions $\mu(\varepsilon)$, $T(\varepsilon)$, $\underline{y}^0(\varepsilon)$ follows from an implicit function theorem argument, provided that the $n \times n$ matrix

$$\left(\frac{\partial \underline{y}}{\partial t}, \frac{\partial \underline{y}}{\partial \mu}, \frac{\partial \underline{y}}{\partial \underline{y}^0}\right)\Bigg|_{t = T_0,\ \mu = 0,\ \varepsilon = 0,\ \underline{y}^0 = \underline{z}^0}$$

has maximal rank. (Here $\frac{\partial \underline{y}}{\partial \underline{y}^0}$ is an $n \times (n-2)$ matrix representing the derivative of $\underline{y}$ with respect to $(n-2)$ initial directions; there are two restrictions on the initial conditions from the normalization.) We show below how the rank of this matrix may be computed more easily.

Let $\underline{r}$ and $\underline{\ell}$ be the right and left eigenvectors corresponding to a pure imaginary eigenvalue of L_0; by rescaling time, we may assume that this eigenvalue is i. ($\overline{\underline{r}}$ and $\overline{\underline{\ell}}$ are eigenvectors for $-i$.) We may also assume that $\underline{\ell} \cdot \underline{r} = 1$. Let $L' = \frac{d}{d\mu} L_\mu \big|_{\mu = 0}$.

We note that hypothesis (1.2) may be rephrased: $\mathrm{Re}(\underline{\ell} \cdot L'\underline{r}) \neq 0$. (To see this, let $\underline{e}(\mu)$ be the eigenvector of L_μ which corresponds to the eigenvalue $\alpha(\mu)$ near a pure imaginary eigenvalue, normalized by $\underline{\ell} \cdot \underline{e} = 1$, so $\underline{e}(0) = \underline{r}$. Differentiating $L_\mu \underline{e} = \alpha(\mu)\underline{e}$ with respect to μ at $\mu = 0$, we get

$$L_0 \frac{d\underline{e}}{d\mu} + L'\underline{r} = \alpha'(0)\underline{r} + \alpha(0)\frac{d\underline{e}}{d\mu} \tag{5A.1}$$

Now $\underline{\ell} \cdot \frac{d\underline{e}}{d\mu} = 0$ and $\underline{\ell} L_0 = \alpha(0)\underline{\ell}$. Hence, if (5A.1) is multiplied by $\underline{\ell}$ on the left, we get $\underline{\ell} \cdot L'\underline{r} = \alpha'(0)$.)

Let $\underline{y}$ be defined by $\underline{y} = \underline{x}/\varepsilon$. t is replaced by $s = t/(1+\tau)$, where τ is to be adjusted (for each ε) so that the period in s is 2π. Then (1.1) becomes

$$\frac{d\underline{y}}{ds} = (1+\tau)[L_\mu \underline{y} + \varepsilon S(\underline{y},\varepsilon,\mu)].$$

For each ε, τ and μ we construct the solution with initial condition $\underline{y}(0)$, normalized by requiring $\underline{y}(0) = \frac{1}{2}(\underline{r}+\overline{\underline{r}}) + \underline{z}$, where $\underline{\ell} \cdot \underline{z} = \overline{\underline{\ell}} \cdot \underline{z} = 0$. (Hence, the initial conditions are parameterized by points in the $n-2$ dimensional space $W = (\underline{\ell} \oplus \overline{\underline{\ell}})^{\perp}$. Note that by the simplicity of the imaginary eigenvalues, W is transverse to $\underline{r} \oplus \overline{\underline{r}}$.) This solution we denote by $\underline{y}(s,\tau,\mu,\underline{z},\varepsilon)$.

Let $\underline{V}(\tau,\mu,\underline{z},\varepsilon) = \underline{y}(2\pi,\tau,\mu,\underline{z},\varepsilon) - \underline{y}(0,\tau,\mu,\underline{z},\varepsilon)$. At $\mu = \tau = \varepsilon = 0$, $\underline{z} = \underline{0}$, we have $\underline{y}(s) \equiv \underline{y}_0(s) = \mathrm{Re}(\underline{r}e^{is})$ and $v = \underline{0}$. To show that there is a family of 2π-periodic functions with $\tau = \tau(\varepsilon)$, $\mu = \mu(\varepsilon)$, $\underline{z} = \underline{z}(\varepsilon)$, it suffices to show that $\partial\underline{V}/\partial(\tau,\mu,\underline{z})$ has rank n at $\mu = \tau = \varepsilon = 0$, $\underline{z} = \underline{0}$.

Let $\underline{y}_\tau(s) = \dfrac{\partial\underline{y}}{\partial\tau}(s,0,0,\underline{0},0)$. Then $\underline{y}_\tau$ satisfies the variational equation

$$\frac{d}{ds}\,\underline{y}_\tau = L_0\underline{y}_\tau + L_0\underline{y}_0$$

with initial condition $\underline{y}_\tau(0) = \underline{0}$. The solution to this equation is $\underline{y}_\tau = s\,\dfrac{d\underline{y}_0}{ds}$ which implies that $\dfrac{\partial\underline{V}}{\partial\tau} = 2\pi\dfrac{i}{2}(\underline{r}-\underline{\bar{r}}) = -2\pi\mathrm{Im}\;\underline{r}$.

We next calculate $\underline{y}_\mu(s) = \dfrac{\partial\underline{y}}{\partial\mu}(s,0,0,\underline{0},0)$, which satisfies the variational equation

$$\frac{d}{ds}\,\underline{y}_\mu = L_0\underline{y}_\mu + L'\underline{y}_0$$

with initial condition $\underline{y}_\mu(0) = \underline{0}$. Since $L'\underline{y}_0 = \mathrm{Re}\;L'\underline{r}e^{is}$, $\underline{y}_\mu$ is the real part of $\underline{\eta}$, where $\underline{\eta}$ satisfies

$$\frac{d}{ds}\,\underline{\eta} - L_0\underline{\eta} = L'\underline{r}e^{is} \tag{5A.2}$$

with initial condition $\underline{\eta}(0) = \underline{0}$.

A particular solution to (5A.2) is $\underline{y} = s(\underline{\ell}\cdot L'\underline{r})\underline{r}e^{is} + \underline{b}e^{is}$, where $\underline{b}$ is any complex vector which satisfies:

$$(i - L_0)\underline{b} + (\underline{\ell}\cdot L'\underline{r})\underline{r} = L'\underline{r}. \tag{5A.3}$$

Now $(i-L_0)$ is singular, but (5A.3) may be solved for $\underline{b}$ since $\underline{\ell}\cdot(L'\underline{r} - (\underline{\ell}\cdot L'\underline{r})\underline{r}) = 0$. The solutions $\underline{b}$ all have the form $\underline{b}_0 + k\underline{r}$ for any k. There is a unique such value of $\underline{b}$ for which $\underline{\ell}\cdot\mathrm{Re}\,\underline{b} = \overline{\underline{\ell}}\cdot\mathrm{Re}\,\underline{b} = 0$. (Take $k = -\underline{\ell}\cdot(\underline{b}_0 + \overline{\underline{b}}_0)$.) We use this value of $\underline{b}$. The solution to (5A.2) satisfying $\underline{\eta}(0) = 0$ then has real part

$$\underline{\eta}_\mu = \mathrm{Re}\{(s(\underline{\ell}\cdot L'\underline{r})\underline{r} + \underline{b})e^{is}\} + \underline{\gamma}$$

where $\frac{d\gamma}{ds} = L_0\underline{\gamma}$ and $\underline{\gamma}(0) = -\mathrm{Re}\,\underline{b}$. Hence $\frac{\partial \underline{V}}{\partial \mu} = 2\pi\,\mathrm{Re}(\underline{\ell}\cdot L'\underline{r})\underline{r} + \underline{\gamma}(2\pi) - \underline{\gamma}(0)$. We note that $\underline{\gamma}(2\pi) - \underline{\gamma}(0) = \underline{\gamma}_1$ satisfies $\underline{\ell}\cdot\gamma_1 = \overline{\underline{\ell}}\cdot\gamma_1 = 0$. (This follows since $\frac{d}{ds}(\underline{\ell}\cdot\underline{\gamma}) = \underline{\ell}\cdot L_0\underline{\gamma} = i\underline{\ell}\cdot\underline{\gamma}$. Since $\underline{\ell}\cdot\underline{\gamma}(0) = -\underline{\ell}\cdot\mathrm{Re}\,\underline{b} = 0$, $\underline{\ell}\cdot\underline{\gamma}(s) \equiv 0$. Similarly, $\overline{\underline{\ell}}\cdot\underline{\gamma}(s) \equiv 0$.)

Finally, we compute $\frac{\partial \underline{V}}{\partial \underline{z}}$. Let $\underline{\delta y}$ be the variation in $\underline{y}$ due to the variation $\underline{\delta z}$ in initial conditions. Then $\underline{\delta y}(s)$ satisfies $\frac{d}{ds}(\underline{\delta y}) = L_0(\underline{\delta y})$, $\underline{\delta y}(0) = \underline{\delta z}$, and $\underline{\ell}\cdot\underline{\delta z} = \overline{\underline{\ell}}\cdot\underline{\delta z} = 0$. This implies that $\frac{\partial \underline{V}}{\partial \underline{z}}(\underline{\delta z}) = (e^{2\pi L_0} - I)\underline{\delta z}$. Now $\underline{\delta z}$ is in the subspace W orthogonal to $\underline{\ell}$ and $\overline{\underline{\ell}}$. Since there are no other pure imaginary eigenvalues for L_0 (in particular, no integer multiples of $\pm i$), the matrix $(e^{2\pi L_0} - I)$ is invertible on W. Hence $\frac{\partial \underline{V}}{\partial \underline{z}}$ has rank $n-2$.

Now R^n is the direct sum of W and the span of $\mathrm{Re}\,\underline{r}$ and $\mathrm{Im}\,\underline{r}$. (This follows from the simplicity of the pure imaginary eigenvalues.) The range of $(e^{2\pi L_0} - I)$ is W, so $\partial\underline{V}/\partial(\tau,\mu,\underline{z})$ has rank n if and only if $\mathrm{Im}\,\underline{r}$ and $\mathrm{Re}(\underline{\ell}\cdot L'\underline{r})\underline{r}$ are independent. This is true if $\mathrm{Re}(\underline{\ell}\cdot L'\underline{r}) \neq 0$.

2. The argument in this section does not require analyticity; it merely sets up the hypotheses of an implicit function theorem. Hence this argument provides a proof for

a C^r version of this theorem. More specifically, suppose that $\underline{F}(\underline{x},\mu)$ is r times differentiable in $\underline{x}$ and μ. Then the right hand side of (2.3) is r times differentiable in $\underline{x}$ and μ, but only C^{r-1} in ε. The function $\underline{V}(\tau,\mu,\underline{z},\varepsilon)$ defined above is C^{r-1} in ε and at least C^r in the other variables. Hence, the implicit function theorem says that the functions $\tau(\varepsilon)$, $\mu(\varepsilon)$, $\underline{z}(\varepsilon)$ and $\underline{y}(s,\varepsilon) \equiv \underline{y}(s,\mu(\varepsilon),\varepsilon,\underline{z}(\varepsilon))$ are all C^{r-1}. The periodic solutions to (1.1), namely $x(t,\varepsilon) = y(\frac{t}{1+\tau(\varepsilon)},\varepsilon)$, are C^r.

(††) The uniqueness proved in this argument is weaker than that of Theorem 3.15 of these notes. That is, it is not proved in Hopf's paper that the periodic solutions which are found are the only ones in some neighborhood of the critical point. For example, Hopf's argument does not rule out a sequence of periodic functions $x_k(t)$ such that $\max|x_k(t)| \to 0$, the associated $\mu_k \to 0$, and the periods $T_k \to \infty$. Such behavior _is_ ruled out by the center manifold theorem, which says that any point not on the center manifold must eventually leave a sufficiently small neighborhood (at least for a while) or tend to the center manifold as $t \to \infty$. Thus the center manifold contains all sufficiently small closed orbits; since the center manifold is three dimensional (including the parameter dimension), the uniqueness of the periodic solutions is a consequence of the uniqueness for the two-dimensional theorem.

(†††) Formulas equivalent to Hopf's but somewhat easier to apply can be obtained in a simpler manner. The main point is to use the "e^{is}" form of the solutions more explicitly and thereby avoid introducing the bracket criterion.

We again assume that time has been scaled so that the pure imaginary eigenvalues of L_0 are $\pm i$, and we use the notation introduced in (†). Following Hopf we further rescale time by $t = (1+\tau(\varepsilon))s$, $\tau(0) = 0$, and let $\underline{y} = \varepsilon\underline{x}$. Then (1.1) becomes

$$\dot{\underline{y}} = [1+\tau(\varepsilon)][L_0\underline{y} + \mu L'\underline{y} + \varepsilon Q(\underline{y},\underline{y}) + \varepsilon^2 K(\underline{y},\underline{y},\underline{y}) + \ldots] \quad (5A.4)$$

where Q and K are respectively the quadratic and cubic terms when $\mu = 0$.

Let the 2π-periodic solution of (5A.4) be $\underline{y}(s,\varepsilon) = \underline{y}_0 + \varepsilon\underline{y}_1 + \varepsilon^2\underline{y}_2 + \ldots$, where, as before, $\underline{y}_0 = \mathrm{Re}(e^{is}\underline{r})$, and the $\underline{y}_i$ are 2π-periodic with $\underline{\ell}\cdot\underline{y}_i(0) = \underline{\bar{\ell}}\cdot\underline{y}_i(0) = 0$ for $i \geq 1$. (Since the $\underline{y}_i$ are real, we may simply require $\underline{\ell}\cdot\underline{y}_i(0) = 0$.)

To get recursive equations for the $\underline{y}_i$, the series for $\underline{y}(s,\varepsilon)$ is inserted in (5A.4) and like powers of ε are collected. We use the fact that $\tau_1 = \mu_1 = 0$, so $\tau = \varepsilon^2\tau_2 + \ldots$ and $\mu = \varepsilon^2\mu_2 + \ldots$. We find that $\underline{y}_1$ should satisfy $\dot{\underline{y}}_1 = A\underline{y}_1 + \underline{Q}(\underline{y}_0,\underline{y}_0)$, and $\underline{Q}(\underline{y}_0,\underline{y}_0) = \frac{1}{2}\underline{Q}(\underline{r},\underline{\bar{r}}) + \frac{1}{2}\mathrm{Re}[e^{2is}\underline{Q}(\underline{r},\underline{r})]$. A periodic solution to this equation is $\underline{a} + \mathrm{Re}(\underline{c}e^{2is})$ where $\underline{a}$ and $\underline{c}$ are constant vectors satisfying

$$\begin{aligned} -L_0\underline{a} &= \frac{1}{2}\underline{Q}(\underline{r},\underline{\bar{r}}) \\ (2i - L_0)\underline{c} &= \frac{1}{2}\underline{Q}(\underline{r},\underline{r}). \end{aligned} \quad (5A.5)$$

(Since $-L_0$ and $2i - L_0$ are non-singular, these formulas do determine $\underline{a}$ and $\underline{c}$.) Thus $\underline{y}_1 = \underline{a} + \mathrm{Re}(\underline{c}e^{2is}) + \mathrm{Re}(C_1\underline{r}e^{is})$, where the complex number C_1 is to be chosen so that $\underline{\ell}\cdot\underline{y}_1(0) = 0$; using equations (5A.5), one readily finds that

$$C_1 = \frac{1}{2}\,\underline{\ell}\cdot[\underline{Q}(\underline{r},\overline{\underline{r}}) - \frac{1}{2}\,\underline{Q}(\underline{r},\underline{r}) + \frac{1}{6}\,\underline{Q}(\overline{\underline{r}},\overline{\underline{r}})].$$

Now $\underline{y}_2$ is a periodic solution of

$$\dot{\underline{y}}_2 = L_0\underline{y}_2 + \mu_2 L'\underline{y}_0 + 2\underline{Q}(\underline{y}_0,\underline{y}_1) + \underline{K}(\underline{y}_0,\underline{y}_0,\underline{y}_0) + \tau_2 L_0\underline{y}_0$$

Hence

$$\dot{\underline{y}}_2 - L_0\underline{y}_2 = \mathrm{Re}\,\mu_2 L'\underline{r}e^{is} + \mathrm{Re}\,\underline{Q}(\underline{r}e^{is}+\overline{\underline{r}}e^{-is},\ \underline{a}+\underline{c}e^{2is}+C_1\underline{r}e^{is})$$

$$+ \frac{1}{4}\,\mathrm{Re}[K(\underline{r},\underline{r},\underline{r})e^{3is} + 3\underline{K}(\underline{r},\underline{r},\overline{\underline{r}})e^{is}] + \tau_2\mathrm{Re}[i\underline{r}e^{is}]$$

$$= \mathrm{Re}\{C_1\underline{Q}(\overline{\underline{r}},\underline{r}) + e^{is}[(\mu_2 L'+i\tau_2)\underline{r} + 2\underline{Q}(\underline{r},\underline{a}) + \underline{Q}(\underline{r},\underline{c})$$

$$+ \frac{3}{4}\,\underline{K}(\underline{r},\underline{r},\overline{\underline{r}})] + e^{2is}C_1\underline{Q}(\underline{r},\underline{r}) + e^{3is}[\underline{Q}(\underline{r},\underline{c})+\frac{1}{4}\underline{K}(\underline{r},\underline{r},\underline{r})]\}.$$

These equations have a periodic solution if and only if there is no resonance, which requires that the coefficient of e^{is} in this last formula should be orthogonal to $\underline{\ell}$ (the bracket criteria in disguise). Thus

$$\mu_2(\underline{\ell}\cdot L'\underline{r}) + i\tau_2 = -2\underline{\ell}\cdot\underline{Q}(\underline{r},\underline{a}) - \underline{\ell}\cdot\underline{Q}(\overline{\underline{r}},\underline{c}) - \frac{3}{4}\,\underline{\ell}\cdot\underline{K}(\underline{r},\underline{r},\overline{\underline{r}}) \equiv B.$$

Hence we get the formulas for μ_2 and τ_2:

$$\begin{aligned} \mu_2\,\mathrm{Re}(\underline{\ell}\cdot L'\underline{r}) &= -\mathrm{Re}\,B \\ \tau_2 &= -\mu_2\mathrm{Im}(\underline{\ell}\cdot L'\underline{r}) - \mathrm{Im}\,B \end{aligned} \qquad (5A.6)$$

where $\underline{a}$ and $\underline{c}$ are the solutions of (5A.5). [These formulas are unchanged if the eigenvalue is $i\omega$ instead of i, except that, instead of the second equation (5A.5), $\underline{c}$ is the solution of $(2i\omega - L_0)\underline{c} = \frac{1}{2}\,\underline{Q}(\underline{r},\underline{r})$. Also the value of C_1 given above should be divided by ω.]

The formulas (5A.6) are equivalent to Hopf's (4.16). The determination of the left eigenvector $\underline{\ell}$ and the solution of the linear equations (5A.5) for $\underline{a}$ and $\underline{c}$ takes the place of finding the adjoint eigenfunctions and evaluating the integrals implied by the bracket symbols.

(††††) 1. The translators must admit that they have found this section somewhat less transparent than the rest of the paper. In their article, Joseph and Sattinger [1] point out an apparent circularity in a part of Hopf's argument; they also show there that it can be rectified rather easily.

2. The relationship of β, the Floquet exponent near zero (of the periodic solution), to the coefficient μ_2 can be found with relatively little calculation, as follows. The argumented system

$$\begin{aligned} \dot{\underline{x}} &= F_\mu(\underline{x}) \\ \dot{\mu} &= 0 \end{aligned} \tag{5A.7}$$

has the origin as a critical point. There are three eigenvalues of this critical point with zero real part; a zero eigenvalue with the μ-axis as eigenvector, and the conjugate pair of imaginary eigenvalues $\pm i$ (after suitably rescaling the time variable) with eigenvectors $\underline{r}$ and $\overline{\underline{r}}$. All other eigenvalues are off the imaginary axis, so this critical point has a 3 dimensional center manifold. This center manifold must contain the μ-axis, the periodic solutions given by Hopf's Theorem, and any trajectories of (5A.7) which for all time remain close to the origin; it is tangent to the linear space generated by the μ-axis and the real and imaginary parts of $\underline{r}$. Let us set $\underline{x} = \varepsilon(\zeta\underline{r}+\overline{\zeta\underline{r}}) + \underline{x}_2$

where $\underline{\ell}\cdot x_2 = \underline{\overline{\ell}}\cdot x_2 = 0$. $\varepsilon \geq 0$ is regarded as a replacement for μ, given by the function $\mu(\varepsilon)$ of Hopf's Theorem: $\mu = \mu(\varepsilon) = \mu_2\varepsilon^2 + \ldots$, where we are now assuming that $\mu_2 \neq 0$. Thus we may think of the real and imaginary parts of ζ, and ε, as parameters on the center manifold. For any $(\underline{x},\mu)$ on this manifold $\underline{x}_2 = 0(\varepsilon^2)$ since it is at least quadratic in $\underline{\ell}\cdot\underline{x} = \varepsilon\zeta$ and $\underline{\overline{\ell}}\cdot x = \varepsilon\overline{\zeta}$. Thus we may write the equations of the center manifold as $\mu = \mu(\varepsilon)$, $\underline{x}_2 = \varepsilon^2\underline{g}(\zeta,\overline{\zeta},\varepsilon)$, where $\underline{\ell}\cdot\underline{g} = \underline{\overline{\ell}}\cdot\underline{g} = 0$ and $\underline{g}$ is at least quadratic in ζ and $\overline{\zeta}$. For any trajectory on the center manifold we then have, with the notations of (†) and (†††),

$$\begin{aligned}\underline{r}\dot{\zeta} + \underline{\overline{r}}\,\dot{\overline{\zeta}} + \varepsilon(\underline{g}_{\zeta}\dot{\zeta} + \underline{g}_{\overline{\zeta}}\dot{\overline{\zeta}}) &= i\zeta\underline{r} - i\overline{\zeta}\,\underline{\overline{r}}\\ &+ \varepsilon L_0\underline{g} + \mu_2\varepsilon^2 L'(\zeta\underline{r} + \overline{\zeta}\,\underline{\overline{r}}) + \varepsilon Q(\zeta\underline{r}+\overline{\zeta}\,\underline{\overline{r}}) \qquad (5A.8)\\ &+ 2\varepsilon^2 Q(\zeta\underline{r} + \overline{\zeta}\,\underline{\overline{r}},\underline{g}) + \varepsilon^2 C(\zeta\underline{r} + \overline{\zeta}\,\underline{\overline{r}}) + 0(\varepsilon^3).\end{aligned}$$

By multiplying on the left with $\underline{\ell}$, we obtain

$$\begin{aligned}\dot{\zeta} &= i\zeta + \mu_2\varepsilon^2\underline{\ell}\cdot L'(\zeta\underline{r} + \overline{\zeta}\,\overline{r})\\ &+ \varepsilon\underline{\ell}\cdot[\zeta^2 Q(\underline{r},\underline{r}) + 2\zeta\overline{\zeta}Q(\underline{r},\underline{\overline{r}}) + \overline{\zeta}^2 Q(\underline{\overline{r}},\underline{\overline{r}})] \qquad (5A.9)\\ &+ \varepsilon^2\gamma(\zeta,\overline{\zeta}) + 0(\varepsilon^3)\end{aligned}$$

where γ is cubic in ζ and $\overline{\zeta}$.

We now introduce the function

$$I(\zeta,\overline{\zeta}) = \frac{1}{2}\zeta\overline{\zeta} + \varepsilon\mathrm{Re}[\frac{i\zeta^3}{3}\underline{\overline{\ell}}\cdot Q(\underline{r},\underline{r}) + i\zeta^2\overline{\zeta}(2\underline{\overline{\ell}}\cdot Q(\underline{r},\underline{\overline{r}})+\underline{\ell}\cdot Q(\underline{r},\underline{r})].$$

As we will see below, $I(\zeta,\overline{\zeta})$ is approximately invariant along trajectories lying on the center manifold. For any trajectory on the center manifold we have, using (5A.9) and its complex conjugate,

$$\frac{dI}{dt} = \mathrm{Re}\{\overline{\zeta}\dot{\zeta} + \varepsilon[i\zeta^2\dot{\zeta}\underline{\overline{\ell}}\cdot Q(\underline{r},\underline{r}) + i(\zeta^2\dot{\overline{\zeta}} + 2\zeta\overline{\zeta}\,\dot{\zeta})\cdot$$

$$(2\underline{\overline{\ell}}\cdot Q(\underline{r},\underline{\overline{r}}) + \underline{\ell}\cdot Q(\underline{r},\underline{r}))]\}$$

$$= \mathrm{Re}\{i\zeta\overline{\zeta} + \varepsilon\underline{\ell}\cdot[\overline{\zeta}\zeta^2 Q(\underline{r},\underline{r}) + 2\zeta\overline{\zeta}^2 Q(\underline{r},\underline{\overline{r}}) + \overline{\zeta}^3 Q(\underline{\overline{r}},\underline{\overline{r}})]$$

$$+ \varepsilon[-\zeta^3\underline{\overline{\ell}}\cdot Q(\underline{r},\underline{r}) + (\zeta^2\overline{\zeta} - 2\zeta^2\overline{\zeta})(2\underline{\overline{\ell}}\cdot Q(\underline{r},\underline{\overline{r}}) + \underline{\ell}\cdot Q(\underline{r},\underline{r}))]$$

$$+ \mu_2\varepsilon^2\overline{\zeta}\underline{\ell}\cdot L'(\zeta\underline{r} + \overline{\zeta}\,\underline{\overline{r}}) + \varepsilon^2\overline{\zeta}\gamma(\zeta,\overline{\zeta}) + \varepsilon^2\delta(\zeta,\overline{\zeta})\} + 0(\varepsilon^3)$$

where δ is quartic in ζ and $\overline{\zeta}$. The terms of order ε in the above are $\varepsilon\mathrm{Re}\{\overline{\zeta}^3\underline{\ell}\cdot Q(\underline{\overline{r}},\underline{\overline{r}}) - \zeta^3\underline{\overline{\ell}}\cdot Q(\underline{r},\underline{r}) + \overline{\zeta}\zeta^2\underline{\ell}\cdot Q(\underline{r},\underline{r}) - \overline{\zeta}\zeta^2\underline{\ell}\cdot Q(\underline{r},\underline{r}) + 2\zeta\overline{\zeta}^2\underline{\ell}\cdot Q(\underline{r},\underline{\overline{r}}) - 2\overline{\zeta}\zeta^2\underline{\overline{\ell}}\cdot Q(\underline{r},\underline{\overline{r}})\} \equiv 0$. Thus

$$\frac{dI}{dt} = \mu_2\varepsilon^2\mathrm{Re}[\overline{\zeta}\underline{\ell}\cdot L'(\zeta r + \overline{\zeta}\,\underline{\overline{r}})] + \varepsilon^2\delta_1(\zeta,\overline{\zeta}) + 0(\varepsilon^3) \quad (5A.10)$$

where δ_1 is also quartic in ζ and $\overline{\zeta}$. Thus $\frac{dI}{dt}$ is of order ε^2.

($I_0 = \frac{1}{2}|\zeta|^2$ is also an approximate invariant, but $\frac{dI_0}{dt}$ is $0(\varepsilon)$ whereas $\frac{dI}{dt}$ is only $0(\varepsilon^2)$. If we consider a trajectory on the center manifold starting at $t = 0$ with $\zeta = \overline{\zeta} = c$, we see from (5A.9) that it is given to $0(\varepsilon)$ by $\zeta = ce^{it}$; thus, after a time of approximately 2π, it must once again return to $\mathrm{Im}\,\zeta = 0$. This arc is a circle to $0(\varepsilon)$, but is more accurately described (to $0(\varepsilon^2)$) as a curve of constant I.)

We see from (5A.10) that the change in I in going once around this way is given, to $0(\varepsilon^2)$, by

$$\Delta I = 2\pi[\mu_2\varepsilon^2 c^2\mathrm{Re}(\underline{\ell}\cdot L'\underline{r}) + \varepsilon^2 c^4\delta_2] \qquad (5A.11)$$

where $\delta_2 = \frac{1}{2\pi}\int_0^{2\pi}\delta_1(e^{it},e^{-it})dt$. However, we know that if $c = 1$ we get the periodic solution, for which $\Delta I = 0$; consequently $\delta_2 = -\mu_2 \mathrm{Re}(\underline{\ell}\cdot L'\underline{r}) = -\mu_2 \mathrm{Re}(\alpha'(0))$ as noted in (†). Thus, in general,

$$\Delta I = 2\pi\varepsilon^2\mu_2 \mathrm{Re}(\alpha'(0)(c^2-c^4) + 0(\varepsilon^3). \tag{5A.12}$$

Since $c = 1$ gives the periodic solution, the $0(\varepsilon^3)$ part is also divisible by $(c-1)$. Thus, for c near 1, (5A.12) may be written

$$\Delta I = 2\pi\varepsilon^2(c-1)[-2\mu_2 \mathrm{Re}\ \alpha'(0) + 0(c-1)]. \tag{5A.13}$$

For small ε, any trajectory on the center manifold with $\zeta = 0(1)$ must keep going around an approximate circle. However, it cannot be periodic unless it passes through $\zeta = 1$. Hence, it is apparent from (5A.12) that, when $\mu_2 \mathrm{Re}\ \alpha'(0) > 0$, all trajectories on the center manifold (at a given ε, i.e., μ) which are inside the periodic solution must spiral out towards it as $t \to +\infty$ (or as $t \to -\infty$ if $\mu_2 \mathrm{Re}\ \alpha'(0) < 0$). Since I is approximately $\frac{1}{2}|\zeta|^2$, $\Delta I \simeq (\zeta(2\pi)-c)c$. Thus (5A.13) implies that these trajectories asymptotically approach the periodic solution with exponential rate $\beta = -2\varepsilon^2\mu_2 \mathrm{Re}\ \alpha'(0) + 0(\varepsilon^3)$, and this must thus be the numerically smallest non-zero Floquet exponent.

3. Equation (5A.12) actually tells us more; it implies that we may approximately describe the trajectories on the center manifold as slowly expanding (or contracting if $\mu_2 \mathrm{Re}\ \alpha'(0) < 0$) circles whose radius c varies according to the formula

$$c^2 = \frac{1}{2}(1 + \tanh(\varepsilon^2\mu_2 \mathrm{Re}\ \alpha'(0)(t-t_1)))$$

where t_1 is the time at which $c^2 = 1/2$.

4. The function I is also of some use in relating the above to the "vague attractor" hypothesis. If we set $\mu = 0$, the n-dimensional system $\dot{\underline{x}} = F_0(x)$ has a two-dimensional center manifold for its critical point at 0, tangent to the linear space spanned by the real and imaginary parts of $\underline{r}$. As in a previous paragraph, we set $\underline{x} = \varepsilon(\zeta\underline{r} + \overline{\zeta}\,\overline{\underline{r}}) + \underline{x}_2$, where $\underline{\ell}\cdot\underline{x}_2 = \overline{\underline{\ell}}\cdot\underline{x}_2 = 0$. On this center manifold $\underline{x}_2 = 0(\varepsilon^2)$ and is at least quadratic in ζ and $\overline{\zeta}$; ε is now an arbitrary scaling parameter. For any trajectory on this center manifold one obtains the same formula (5A.9) except for the omission of the $\mu_2 L'$ term - the other terms written down all come from the μ-dependent pairs of F_μ. If we then consider the function I for a trajectory on this center manifold, we obtain (5A.10) again, with the μ_2 term omitted, but the same δ_1; integrating this around, we get (5A.11) without the μ_2 term but the same δ_2. Since $\delta_2 = -\mu_2 \mathrm{Re}\ \alpha'(0)$ we have for trajectories in the center manifold at $\mu = 0$, to order ε^2,

$$\Delta(\tfrac{1}{2}c^2) = -2\pi\ \varepsilon^2\mu_2 \mathrm{Re}\ \alpha'(0)c^4,$$

or

$$\Delta(\varepsilon c) = -2\pi\mu_2 \mathrm{Re}\ \alpha'(0)(\varepsilon c)^3.$$

Since $\Delta(\varepsilon c)$ is $V(x_1)$ (the Poincaré map minus identity), where $\varepsilon c = x_1$ is the coordinate $\mathrm{Re}(\underline{\ell}\cdot\underline{x})$, we see that $V'''(0) = -2\pi\mu_2 \mathrm{Re}(\alpha'(0))\cdot 6 = -2\pi \mathrm{Re}(\alpha'(0))\cdot 3\mu''(0)$. This relates the calculations here to the stability calculations done in §4.