FLUID MECHANICS

Second Edition

by

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Institute of Physical Problems, U.S.S.R. Academy of Sciences

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Translated from the Russian by

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CONTENTS

PREFACE TO THE SECOND ENGLISH EDITION

The content and treatment in this edition remain in accordance with what was said in the preface to the first edition (see below). My chief care in revising and augmenting has been to comply with this principle.

Despite the lapse of thirty years, the previous edition has, with very slight exceptions, not gone out of date. Its material has been only fairly slightly supplemented and modified. About ten new sections have been added.

In recent decades, fluid mechanics has undergone extremely rapid development, and there has accordingly been a great increase in the literature of the subject. The development has been mainly in applications, however, and in an increasing complexity of the problems accessible to theoretical calculation (with or without computers). These include, in particular, various problems of instability and its development, including non-linear regimes. All such topics are beyond the scope of our book; in particular, stability problems are discussed, as previously, mainly in terms of results.

There is also no treatment of non-linear waves in dispersive media, which is by now a significant branch of mathematical physics. The purely hydrodynamic subject of this theory consists in waves with large amplitude on the surface of a liquid. Its principal physical applications are in plasma physics, non-linear optics, various problems of electrodynamics, and so on, and in that respect they belong in other volumes of the *Course*.

There have been important changes in our understanding of the mechanism whereby turbulence occurs. Although a consistent theory of turbulence is still a thing of the future, there is reason to suppose that the right path has finally been found. The basic ideas now available and the results obtained are discussed in three sections (§§30–32) written jointly with M. I. Rabinovich, to whom I am deeply grateful for this valuable assistance. A new area in continuum mechanics over the last few decades is that of liquid crystals. This combines features of the mechanics of liquid and elastic media. Its principles are discussed in the new edition of *Theory of Elasticity*.

This book has a special place among those I had occasion to write jointly with L. D. Landau. He gave it a part of his soul. That branch of theoretical physics, new to him at the time, caught his fancy, and in a very typical way he set about thinking through it *ab initio* and deriving its basic results. This led to a number of original papers which appeared in various journals, but several of his conclusions or ideas were not published elsewhere than in the book, and in some instances even his priority was not established till later. In the new edition, I have added an appropriate reference to his authorship in all such cases that are known to me.

In the revision of this book, as in other volumes of the *Course*, I have had the help and advice of many friends and colleagues. I should like to mention in particular numerous discussions with G. I. Barenblatt, L. P. Pitaevskiĭ, Ya. G. Sinaĭ, and Ya. B. Zel'dovich. Several useful comments came from A. A. Andronov, S. I. Anisimov, V. A. Belokon', A. L. Fabrikant, V. P. Kraĭnov, A. G. Kulikovskiĭ, M. A. Liberman, R. V. Polovin, and A. V. Timofeev. To all of them I express my sincere gratitude.

Institute of Physical Problems E. M. Lifshitz
August 1984

PM-A*

PREFACE TO THE FIRST ENGLISH EDITION

The present book deals with fluid mechanics, i.e. the theory of the motion of liquids and gases.

The nature of the book is largely determined by the fact that it describes fluid mechanics as a branch of theoretical physics, and it is therefore markedly different from other textbooks on the same subject. We have tried to develop as fully as possible all matters of physical interest, and to do so in such a way as to give the clearest possible picture of the phenomena and their interrelation. Accordingly, we discuss neither approximate methods of calculation in fluid mechanics, nor empirical theories devoid of physical significance. On the other hand, accounts are given of some topics not usually found in textbooks on the subject: the theory of heat transfer and diffusion in fluids; acoustics; the theory of combustion; the dynamics of superfluids; and relativistic fluid dynamics.

In a field which has been so extensively studied as fluid mechanics it was inevitable that important new results should have appeared during the several years since the last Russian edition was published. Unfortunately, our preoccupation with other matters has prevented us from including these results in the English edition. We have merely added one further chapter, on the general theory of fluctuations in fluid dynamics.

We should like to express our sincere thanks to Dr Sykes and Dr Reid for their excellent translation of the book, and to Pergamon Press for their ready agreement to our wishes in various matters relating to its publication.

Moscow 1958

L. D. Landau
E. M. Lifshitz

EVGENIĬ MIKHAĬLOVICH LIFSHITZ (1915–1985)†

Soviet physics suffered a heavy loss on 29 October 1985 with the death of the outstanding theoretical physicist Academician Evgeniĭ Mikhaĭlovich Lifshitz.

Lifshitz was born on 21 February 1915 in Khar'kov. In 1933 he graduated from the Khar'kov Polytechnic Institute. He worked at the Khar'kov Physicotechnical Institute from 1933 to 1938 and at the Institute of Physical Problems of the USSR Academy of Sciences in Moscow from 1939 until his death. He was elected an associate member of the USSR Academy of Sciences in 1966 and a full member in 1979.

Lifshitz's scientific activity began very early. He was among L. D. Landau's first students and at 19 he co-authored with him a paper on the theory of pair production in collisions. This paper, which has not lost its significance to this day, outlined many methodological features of modern relativistically invariant techniques of quantum field theory. It includes, in particular, a consistent allowance for retardation.

Modern ferromagnetism theory is based on the "Landau-Lifshitz" equation, which describes the dynamics of the magnetic moment in a ferromagnet. A 1935 article on this subject is one of the best known papers on the physics of magnetic phenomena. The derivation of the equation is accompanied by development of a theory of ferromagnetic resonance and of the domain structure of ferromagnets.

In a 1937 paper on the Boltzmann kinetic equation for electrons in a magnetic field, E. M. Lifshitz developed a drift approximation extensively used much later, in the 50s, in plasma theory.

A paper published in 1939 on deuteron dissociation in collisions remains a brilliant example of the use of quasi-classical methods in quantum mechanics.

A most important step towards the development of a theory of second-order phase transitions, following the work by L. D. Landau, was a paper by Lifshitz dealing with the change of the symmetry of a crystal, of its space group, in transitions of this type (1941). Many years later the results of this paper came into extensive use, and the terms "Lifshitz criterion" and "Lifshitz point," coined on its basis have become indispensable components of modern statistical physics.

A decisive role in the detection of an important physical phenomenon, second sound in superfluid helium, was played by a 1944 paper by E. M. Lifshitz. It is shown in it that second sound is effectively excited by a heater having an alternating temperature. This was precisely the method used to observe second sound in experiment two years later.

A new approach to the theory of molecular-interaction forces between condensed bodies was developed by Lifshitz in 1954–1959. It is based on the profound physical idea that these forces are manifestations of stresses due to quantum and thermal fluctuations of an electromagnetic field in a medium. This idea was pursued to develop a very elegant and general theory in which the interaction forces are expressed in terms of electrodynamic material properties such as the complex dielectric permittivity. This theory of E. M.

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† By A. F. Andreev, A. S. Borovik-Romanov, V. L. Ginzburg, L. P. Gor'kov, I. E. Dzyaloshinskiĭ, Ya. B. Zel'dovich, M. I. Kaganov, L. P. Pitaevskiĭ, E. L. Feĭnberg, and I. M. Khalatnikov; published in Russian in *Uspekhi fizicheskikh nauk* **148**, 549–550, 1986. This translation is by J. G. Adashko (first published in *Soviet Physics Uspekhi* **29**, 294–295, 1986), and is reprinted by kind permission of the American Institute of Physics.

Lifshitz stimulated many studies and was confirmed by experiment. It gained him the M. V. Lomonosov Prize in 1958.

E. M. Lifshitz made a fundamental contribution in one of the most important branches of modern physics, the theory of gravitation. His research into this field started with a classical 1946 paper on the stability of cosmological solutions of Einstein's theory of gravitation. The perturbations were divided into distinctive classes—scalar, with variation of density, vector, describing vortical motion, and finally tensor, describing gravitational waves. This classification is still of decisive significance in the analysis of the origin of the universe. From there, E. M. Lifshitz tackled the exceedingly difficult question of the general character of the singularities of this theory. Many years of labor led in 1972 to a complete solution of this problem in papers written jointly with V. A. Belinskiĭ and I. M. Khalatnikov, which earned their authors the 1974 L. D. Landau Prize. The singularity was found to have a complicated oscillatory character and could be illustratively represented as contraction of space in two directions with simultaneous expansion in the third. The contraction and expansion alternate in time according to a definite law. These results elicited a tremendous response from specialists, altered radically our ideas concerning relativistic collapse, and raised a host of physical and mathematical problems that still await solution.

His life-long occupation was the famous Landau and Lifshitz *Course of Theoretical Physics*, to which he devoted about 50 years. (The first edition of *Statistical Physics* was written in 1937. A new edition of *Theory of Elasticity* went to press shortly before his last illness.) The greater part of the *Course* was written by Lifshitz together with his teacher and friend L. D. Landau. After the automobile accident that made Landau unable to work, Lifshitz completed the edition jointly with Landau's students. He later continued to revise the previously written volumes in the light of the latest advances in science. Even in the hospital, he discussed with visiting friends the topics that should be subsequently included in the *Course.*

The *Course of Theoretical Physics* became world famous. It was translated in its entirety into six languages. Individual volumes were published in 10 more languages. In 1972 L. D. Landau and E. M. Lifshitz were awarded the Lenin Prize for the volumes published by then.

The *Course of Theoretical Physics* remains a monument to E. M. Lifshitz as a scientist and a pedagogue. It has educated many generations of physicists, is being studied, and will continue to teach students in future generations.

A versatile physicist, E. M. Lifshitz dealt also with applications. He was awarded the USSR State Prize in 1954.

A tremendous amount of E.M. Lifshitz's labor and energy was devoted to Soviet scientific periodicals. From 1946 to 1949 and from 1955 to his death he was deputy editor-in-chief of the *Journal of Experimental and Theoretical Physics.* His extreme devotion to science, adherence to principles, and meticulousness greatly helped to make this journal one of the best scientific periodicals in the world.

E. M. Lifshitz accomplished much in his life. He will remain in our memory as a remarkable physicist and human being. His name will live forever in the history of Soviet physics.

NOTATION

ρ	density
p	pressure
T	temperature
s	entropy per unit mass
ε	internal energy per unit mass
$w =$	$\varepsilon + p/\rho$ heat function (enthalpy)
$\gamma =$	c_p/c_v ratio of specific heats at constant pressure and constant volume
η	dynamic viscosity
$\nu =$	η/ρ kinematic viscosity
κ	thermal conductivity
$\chi =$	$\kappa/\rho c_p$ thermometric conductivity
R	Reynolds number
c	velocity of sound
M	ratio of fluid velocity to velocity of sound (Mach number)

Vector and tensor (three-dimensional) suffixes are denoted by Latin letters $i, k, l, \ldots$. Summation over repeated ("dummy") suffixes is everywhere implied. The unit tensor is δ_{ik}:

References to other volumes in the *Course of Theoretical Physics:*

Fields = Vol. 2 (*The Classical Theory of Fields*, fourth English edition, 1975).
QM = Vol. 3 (*Quantum Mechanics*, third English edition, 1977).
SP 1 = Vol. 5 (*Statistical Physics*, Part 1, third English edition, 1980).
ECM = Vol. 8 (*Electrodynamics of Continuous Media*, second English edition, 1984).
SP 2 = Vol. 9 (*Statistical Physics*, Part 2, English edition, 1980).
PK = Vol. 10 (*Physical Kinetics*, English edition, 1981).

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CHAPTER I

IDEAL FLUIDS

§1. The equation of continuity

Fluid dynamics concerns itself with the study of the motion of fluids (liquids and gases). Since the phenomena considered in fluid dynamics are macroscopic, a fluid is regarded as a continuous medium. This means that any small volume element in the fluid is always supposed so large that it still contains a very great number of molecules. Accordingly, when we speak of infinitely small elements of volume, we shall always mean those which are "physically" infinitely small, i.e. very small compared with the volume of the body under consideration, but large compared with the distances between the molecules. The expressions *fluid particle* and *point in a fluid* are to be understood in a similar sense. If, for example, we speak of the displacement of some fluid particle, we mean not the displacement of an individual molecule, but that of a volume element containing many molecules, though still regarded as a point.

The mathematical description of the state of a moving fluid is effected by means of functions which give the distribution of the fluid velocity $\mathbf{v} = \mathbf{v}(x, y, z, t)$ and of any two thermodynamic quantities pertaining to the fluid, for instance the pressure $p(x, y, z, t)$ and the density $\rho(x, y, z, t)$. All the thermodynamic quantities are determined by the values of any two of them, together with the equation of state; hence, if we are given five quantities, namely the three components of the velocity $\mathbf{v}$, the pressure p and the density ρ, the state of the moving fluid is completely determined.

All these quantities are, in general, functions of the coordinates x, y, z and of the time t. We emphasize that $\mathbf{v}(x, y, z, t)$ is the velocity of the fluid at a given point (x, y, z) in space and at a given time t, i.e. it refers to fixed points in space and not to specific particles of the fluid; in the course of time, the latter move about in space. The same remarks apply to ρ and p.

We shall now derive the fundamental equations of fluid dynamics. Let us begin with the equation which expresses the conservation of matter. We consider some volume V_0 of space. The mass of fluid in this volume is $\int \rho \, dV$, where ρ is the fluid density, and the integration is taken over the volume V_0. The mass of fluid flowing in unit time through an element $d\mathbf{f}$ of the surface bounding this volume is $\rho\mathbf{v} \cdot d\mathbf{f}$; the magnitude of the vector $d\mathbf{f}$ is equal to the area of the surface element, and its direction is along the normal. By convention, we take $d\mathbf{f}$ along the outward normal. Then $\rho\mathbf{v} \cdot d\mathbf{f}$ is positive if the fluid is flowing out of the volume, and negative if the flow is into the volume. The total mass of fluid flowing out of the volume V_0 in unit time is therefore

$$\oint \rho \mathbf{v} \cdot d\mathbf{f},$$

where the integration is taken over the whole of the closed surface surrounding the volume in question.

Next, the decrease per unit time in the mass of fluid in the volume V_0 can be written

$$-\frac{\partial}{\partial t}\int \rho \, \mathrm{d}V.$$

Equating the two expressions, we have

$$\frac{\partial}{\partial t}\int \rho \, \mathrm{d}V = -\oint \rho \, \mathbf{v}\cdot \mathbf{df}. \tag{1.1}$$

The surface integral can be transformed by Green's formula to a volume integral:

$$\oint \rho \, \mathbf{v}\cdot\mathbf{df} = \int \operatorname{div}(\rho\mathbf{v})\, \mathrm{d}V.$$

Thus

$$\int \left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v})\right] \mathrm{d}V = 0.$$

Since this equation must hold for any volume, the integrand must vanish, i.e.

$$\partial\rho/\partial t + \operatorname{div}(\rho\mathbf{v}) = 0. \tag{1.2}$$

This is the *equation of continuity*. Expanding the expression div $(\rho\mathbf{v})$, we can also write (1.2) as

$$\partial\rho/\partial t + \rho \operatorname{div} \mathbf{v} + \mathbf{v}\cdot \mathbf{grad}\, \rho = 0. \tag{1.3}$$

The vector

$$\mathbf{j} = \rho \mathbf{v} \tag{1.4}$$

is called the *mass flux density*. Its direction is that of the motion of the fluid, while its magnitude equals the mass of fluid flowing in unit time through unit area perpendicular to the velocity.

§2. Euler's equation

Let us consider some volume in the fluid. The total force acting on this volume is equal to the integral

$$-\oint p \, \mathbf{df}$$

of the pressure, taken over the surface bounding the volume. Transforming it to a volume integral, we have

$$-\oint p \, \mathbf{df} = -\int \mathbf{grad}\, p \, \mathrm{d}V.$$

Hence we see that the fluid surrounding any volume element $\mathrm{d}V$ exerts on that element a force $-\mathrm{d}V\, \mathbf{grad}\, p$. In other words, we can say that a force $-\mathbf{grad}\, p$ acts on unit volume of the fluid.

We can now write down the equation of motion of a volume element in the fluid by equating the force $-\mathbf{grad}\, p$ to the product of the mass per unit volume (ρ) and the acceleration $\mathrm{d}\mathbf{v}/\mathrm{d}t$:

$$\rho \, \mathrm{d}\mathbf{v}/\mathrm{d}t = -\mathbf{grad}\, p. \tag{2.1}$$

The derivative $d\mathbf{v}/dt$ which appears here denotes not the rate of change of the fluid velocity at a fixed point in space, but the rate of change of the velocity of a given fluid particle as it moves about in space. This derivative has to be expressed in terms of quantities referring to points fixed in space. To do so, we notice that the change $d\mathbf{v}$ in the velocity of the given fluid particle during the time dt is composed of two parts, namely the change during dt in the velocity at a point fixed in space, and the difference between the velocities (at the same instant) at two points $d\mathbf{r}$ apart, where $d\mathbf{r}$ is the distance moved by the given fluid particle during the time dt. The first part is $(\partial\mathbf{v}/\partial t)dt$, where the derivative $\partial\mathbf{v}/\partial t$ is taken for constant x, y, z, i.e. at the given point in space. The second part is

$$dx\frac{\partial\mathbf{v}}{\partial x}+dy\frac{\partial\mathbf{v}}{\partial y}+dz\frac{\partial\mathbf{v}}{\partial z} = (d\mathbf{r}\cdot\mathbf{grad})\mathbf{v}.$$

Thus

$$d\mathbf{v} = (\partial\mathbf{v}/\partial t)dt + (d\mathbf{r}\cdot\mathbf{grad})\mathbf{v},$$

or, dividing both sides by dt,†

$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v}. \tag{2.2}$$

Substituting this in (2.1), we find

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = -\frac{1}{\rho}\mathbf{grad}\,p. \tag{2.3}$$

This is the required equation of motion of the fluid; it was first obtained by L. Euler in 1755. It is called *Euler's equation* and is one of the fundamental equations of fluid dynamics.

If the fluid is in a gravitational field, an additional force $\rho\mathbf{g}$, where $\mathbf{g}$ is the acceleration due to gravity, acts on any unit volume. This force must be added to the right-hand side of equation (2.1), so that equation (2.3) takes the form

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = -\frac{\mathbf{grad}\,p}{\rho} + \mathbf{g}. \tag{2.4}$$

In deriving the equations of motion we have taken no account of processes of energy dissipation, which may occur in a moving fluid in consequence of internal friction (viscosity) in the fluid and heat exchange between different parts of it. The whole of the discussion in this and subsequent sections of this chapter therefore holds good only for motions of fluids in which thermal conductivity and viscosity are unimportant; such fluids are said to be *ideal*.

The absence of heat exchange between different parts of the fluid (and also, of course, between the fluid and bodies adjoining it) means that the motion is adiabatic throughout the fluid. Thus the motion of an ideal fluid must necessarily be supposed adiabatic.

In adiabatic motion the entropy of any particle of fluid remains constant as that particle moves about in space. Denoting by s the entropy per unit mass, we can express the condition for adiabatic motion as

$$ds/dt = 0, \tag{2.5}$$

† The derivative d/dt thus defined is called the *substantial* time derivative, to emphasize its connection with the moving substance.

where the total derivative with respect to time denotes, as in (2.1), the rate of change of entropy for a given fluid particle as it moves about. This condition can also be written

$$\partial s/\partial t + \mathbf{v}\cdot\mathbf{grad}\, s = 0. \tag{2.6}$$

This is the general equation describing adiabatic motion of an ideal fluid. Using (1.2), we can write it as an "equation of continuity" for entropy:

$$\partial(\rho s)/\partial t + \operatorname{div}(\rho s\mathbf{v}) = 0. \tag{2.7}$$

The product $\rho s\mathbf{v}$ is the *entropy flux density.*

The adiabatic equation usually takes a much simpler form. If, as usually happens, the entropy is constant throughout the volume of the fluid at some initial instant, it retains everywhere the same constant value at all times and for any subsequent motion of the fluid. In this case we can write the adiabatic equation simply as

$$s = \text{constant}, \tag{2.8}$$

and we shall usually do so in what follows. Such a motion is said to be *isentropic.*

We may use the fact that the motion is isentropic to put the equation of motion (2.3) in a somewhat different form. To do so, we employ the familiar thermodynamic relation

$$\mathrm{d}w = T\,\mathrm{d}s + V\,\mathrm{d}p,$$

where w is the heat function per unit mass of fluid (enthalpy), $V = 1/\rho$ is the specific volume, and T is the temperature. Since s = constant, we have simply

$$\mathrm{d}w = V\,\mathrm{d}p = \mathrm{d}p/\rho,$$

and so $(\mathbf{grad}\, p)/\rho = \mathbf{grad}\, w$. Equation (2.3) can therefore be written in the form

$$\partial\mathbf{v}/\partial t + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = -\mathbf{grad}\, w. \tag{2.9}$$

It is useful to notice one further form of Euler's equation, in which it involves only the velocity. Using a formula well known in vector analysis,

$$\tfrac{1}{2}\mathbf{grad}\, v^2 = \mathbf{v}\times\mathbf{curl}\,\mathbf{v} + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v},$$

we can write (2.9) in the form

$$\partial\mathbf{v}/\partial t - \mathbf{v}\times\mathbf{curl}\,\mathbf{v} = -\mathbf{grad}\,(w + \tfrac{1}{2}v^2). \tag{2.10}$$

If we take the curl of both sides of this equation, we obtain

$$\frac{\partial}{\partial t}(\mathbf{curl}\,\mathbf{v}) = \mathbf{curl}\,(\mathbf{v}\times\mathbf{curl}\,\mathbf{v}), \tag{2.11}$$

which involves only the velocity.

The equations of motion have to be supplemented by the boundary conditions that must be satisfied at the surfaces bounding the fluid. For an ideal fluid, the boundary condition is simply that the fluid cannot penetrate a solid surface. This means that the component of the fluid velocity normal to the bounding surface must vanish if that surface is at rest:

$$v_n = 0. \tag{2.12}$$

In the general case of a moving surface, v_n must be equal to the corresponding component of the velocity of the surface.

At a boundary between two immiscible fluids, the condition is that the pressure and the velocity component normal to the surface of separation must be the same for the two fluids, and each of these velocity components must be equal to the corresponding component of the velocity of the surface.

As has been said at the beginning of §1, the state of a moving fluid is determined by five quantities: the three components of the velocity **v** and, for example, the pressure p and the density ρ. Accordingly, a complete system of equations of fluid dynamics should be five in number. For an ideal fluid these are Euler's equations, the equation of continuity, and the adiabatic equation.

PROBLEM

Write down the equations for one-dimensional motion of an ideal fluid in terms of the variables a, t, where a (called a *Lagrangian variable*†) is the x coordinate of a fluid particle at some instant $t = t_0$.

SOLUTION. In these variables the coordinate x of any fluid particle at any instant is regarded as a function of t and its coordinate a at the initial instant: $x = x(a, t)$. The condition of conservation of mass during the motion of a fluid element (the equation of continuity) is accordingly written $\rho\, dx = \rho_0\, da$, or

$$\rho\left(\frac{\partial x}{\partial a}\right)_t = \rho_0,$$

where $\rho_0(a)$ is a given initial density distribution. The velocity of a fluid particle is, by definition, $v = (\partial x/\partial t)_a$, and the derivative $(\partial v/\partial t)_a$ gives the rate of change of the velocity of the particle during its motion. Euler's equation becomes

$$\left(\frac{\partial v}{\partial t}\right)_a = -\frac{1}{\rho_0}\left(\frac{\partial p}{\partial a}\right)_t,$$

and the adiabatic equation is

$$(\partial s/\partial t)_a = 0.$$

§3. Hydrostatics

For a fluid at rest in a uniform gravitational field, Euler's equation (2.4) takes the form

$$\mathbf{grad}\, p = \rho \mathbf{g}. \tag{3.1}$$

This equation describes the mechanical equilibrium of the fluid. (If there is no external force, the equation of equilibrium is simply $\mathbf{grad}\, p = 0$, i.e. $p =$ constant; the pressure is the same at every point in the fluid.)

Equation (3.1) can be integrated immediately if the density of the fluid may be supposed constant throughout its volume, i.e. if there is no significant compression of the fluid under the action of the external force. Taking the z-axis vertically upward, we have

$$\partial p/\partial x = \partial p/\partial y = 0, \qquad \partial p/\partial z = -\rho g.$$

Hence

$$p = -\rho g z + \text{constant}.$$

If the fluid at rest has a free surface at height h, to which an external pressure p_0, the same at every point, is applied, this surface must be the horizontal plane $z = h$. From the condition $p = p_0$ for $z = h$, we find that the constant is $p_0 + \rho g h$, so that

$$p = p_0 + \rho g(h - z). \tag{3.2}$$

† Although such variables are usually called Lagrangian, the equations of motion in these coordinates were first obtained by Euler, at the same time as equations (2.3).

For large masses of liquid, and for a gas, the density ρ cannot in general be supposed constant; this applies especially to gases (for example, the atmosphere). Let us suppose that the fluid is not only in mechanical equilibrium but also in thermal equilibrium. Then the temperature is the same at every point, and equation (3.1) may be integrated as follows. We use the familiar thermodynamic relation

$$\mathrm{d}\Phi = -s\mathrm{d}T + V\mathrm{d}p,$$

where Φ is the thermodynamic potential (Gibbs free energy) per unit mass. For constant temperature

$$\mathrm{d}\Phi = V\mathrm{d}p = \mathrm{d}p/\rho.$$

Hence we see that the expression $(\mathbf{grad}\, p)/\rho$ can be written in this case as $\mathbf{grad}\, \Phi$, so that the equation of equilibrium (3.1) takes the form

$$\mathbf{grad}\, \Phi = \mathbf{g}.$$

For a constant vector $\mathbf{g}$ directed along the negative z-axis we have

$$\mathbf{g} \equiv -\mathbf{grad}\,(gz).$$

Thus

$$\mathbf{grad}\,(\Phi + gz) = 0,$$

whence we find that throughout the fluid

$$\Phi + gz = \text{constant}; \tag{3.3}$$

gz is the potential energy of unit mass of fluid in the gravitational field. The condition (3.3) is known from statistical physics to be the condition for thermodynamic equilibrium of a system in an external field.

We may mention here another simple consequence of equation (3.1). If a fluid (such as the atmosphere) is in mechanical equilibrium in a gravitational field, the pressure in it can be a function only of the altitude z (since, if the pressure were different at different points with the same altitude, motion would result). It then follows from (3.1) that the density

$$\rho = -\frac{1}{g}\frac{\mathrm{d}p}{\mathrm{d}z} \tag{3.4}$$

is also a function of z only. The pressure and density together determine the temperature, which is therefore again a function of z only. Thus, in mechanical equilibrium in a gravitational field, the pressure, density and temperature distributions depend only on the altitude. If, for example, the temperature is different at different points with the same altitude, then mechanical equilibrium is impossible.

Finally, let us derive the equation of equilibrium for a very large mass of fluid, whose separate parts are held together by gravitational attraction—a star. Let ϕ be the Newtonian gravitational potential of the field due to the fluid. It satisfies the differential equation

$$\triangle\phi = 4\pi G\rho, \tag{3.5}$$

where G is the Newtonian constant of gravitation. The gravitational acceleration is $-\mathbf{grad}\,\phi$, and the force on a mass ρ is $-\rho\,\mathbf{grad}\,\phi$. The condition of equilibrium is therefore

$$\mathbf{grad}\, p = -\rho\,\mathbf{grad}\,\phi.$$

Dividing both sides by ρ, taking the divergence of both sides, and using equation (3.5), we obtain

$$\operatorname{div}\left(\frac{1}{\rho}\mathbf{grad}\,p\right) = -4\pi G\rho. \tag{3.6}$$

It must be emphasized that the present discussion concerns only mechanical equilibrium; equation (3.6) does not presuppose the existence of complete thermal equilibrium.

If the body is not rotating, it will be spherical when in equilibrium, and the density and pressure distributions will be spherically symmetrical. Equation (3.6) in spherical polar coordinates then takes the form

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r}\right) = -4\pi G\rho. \tag{3.7}$$

§4. The condition that convection be absent

A fluid can be in mechanical equilibrium (i.e. exhibit no macroscopic motion) without being in thermal equilibrium. Equation (3.1), the condition for mechanical equilibrium, can be satisfied even if the temperature is not constant throughout the fluid. However, the question then arises of the stability of such an equilibrium. It is found that the equilibrium is stable only when a certain condition is fulfilled. Otherwise, the equilibrium is unstable, and this leads to the appearance in the fluid of currents which tend to mix the fluid in such a way as to equalize the temperature. This motion is called *convection*. Thus the condition for a mechanical equilibrium to be stable is the condition that convection be absent. It can be derived as follows.

Let us consider a fluid element at height z, having a specific volume $V(p, s)$, where p and s are the equilibrium pressure and entropy at height z. Suppose that this fluid element undergoes an adiabatic upward displacement through a small interval ξ; its specific volume then becomes $V(p', s)$, where p' is the pressure at height $z+\xi$. For the equilibrium to be stable, it is necessary (though not in general sufficient) that the resulting force on the element should tend to return it to its original position. This means that the element must be heavier than the fluid which it "displaces" in its new position. The specific volume of the latter is $V(p', s')$, where s' is the equilibrium entropy at height $z+\xi$. Thus we have the stability condition

$$V(p', s') - V(p', s) > 0.$$

Expanding this difference in powers of $s'-s = \xi\,\mathrm{d}s/\mathrm{d}z$, we obtain

$$\left(\frac{\partial V}{\partial s}\right)_p \frac{\mathrm{d}s}{\mathrm{d}z} > 0. \tag{4.1}$$

The formulae of thermodynamics give

$$\left(\frac{\partial V}{\partial s}\right)_p = \frac{T}{c_p}\left(\frac{\partial V}{\partial T}\right)_p,$$

where c_p is the specific heat at constant pressure. Both c_p and T are positive, so that we can write (4.1) as

$$\left(\frac{\partial V}{\partial T}\right)_p \frac{\mathrm{d}s}{\mathrm{d}z} > 0. \tag{4.2}$$

The majority of substances expand on heating, i.e. $(\partial V/\partial T)_p > 0$. The condition that convection be absent then becomes

$$ds/dz > 0, \tag{4.3}$$

i.e. the entropy must increase with height.

From this we easily find the condition that must be satisfied by the temperature gradient dT/dz. Expanding the derivative ds/dz, we have

$$\frac{ds}{dz} = \left(\frac{\partial s}{\partial T}\right)_p \frac{dT}{dz} + \left(\frac{\partial s}{\partial p}\right)_T \frac{dp}{dz} = \frac{c_p}{T}\frac{dT}{dz} - \left(\frac{\partial V}{\partial T}\right)_p \frac{dp}{dz} > 0.$$

Finally, substituting from (3.4) $dp/dz = -g/V$, we obtain

$$-dT/dz < g\beta T/c_p, \tag{4.4}$$

where $\beta = (1/V)(\partial V/\partial T)_p$ is the thermal expansion coefficient. For a column of gas in equilibrium which can be taken as a thermodynamically perfect gas, $\beta T = 1$ and (4.4) becomes

$$-dT/dz < g/c_p. \tag{4.5}$$

Convection occurs if these conditions are not satisfied, i.e. if the temperature decreases upwards with a gradient whose magnitude exceeds the value given by (4.4) and (4.5).†

§5. Bernoulli's equation

The equations of fluid dynamics are much simplified in the case of steady flow. By *steady flow* we mean one in which the velocity is constant in time at any point occupied by fluid. In other words, **v** is a function of the coordinates only, so that $\partial \mathbf{v}/\partial t = 0$. Equation (2.10) then reduces to

$$\tfrac{1}{2}\,\mathbf{grad}\,v^2 - \mathbf{v}\times\mathbf{curl}\,\mathbf{v} = -\mathbf{grad}\,w. \tag{5.1}$$

We now introduce the concept of *streamlines*. These are lines such that the tangent to a streamline at any point gives the direction of the velocity at that point; they are determined by the following system of differential equations:

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z}. \tag{5.2}$$

In steady flow the streamlines do not vary with time, and coincide with the paths of the fluid particles. In non-steady flow this coincidence no longer occurs: the tangents to the streamlines give the directions of the velocities of fluid particles at various points in space at a given instant, whereas the tangents to the paths give the directions of the velocities of given fluid particles at various times.

We form the scalar product of equation (5.1) with the unit vector tangent to the streamline at each point; this unit vector is denoted by **l**. The projection of the gradient on any direction is, as we know, the derivative in that direction. Hence the projection of **grad** w is $\partial w/\partial l$. The vector $\mathbf{v}\times\mathbf{curl}\,\mathbf{v}$ is perpendicular to **v**, and its projection on the direction of **l** is therefore zero.

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† For water at 20°C, the right-hand side of (4.4) is about one degree per 6.7 km; for air, the right-hand side of (4.5) is about one degree per 100 m.

Thus we obtain from equation (5.1)

$$\frac{\partial}{\partial l}(\tfrac{1}{2}v^2 + w) = 0.$$

It follows from this that $\frac{1}{2}v^2 + w$ is constant along a streamline:

$$\tfrac{1}{2}v^2 + w = \text{constant}. \tag{5.3}$$

In general the constant takes different values for different streamlines. Equation (5.3) is called *Bernoulli's equation.*†

If the flow takes place in a gravitational field, the acceleration **g** due to gravity must be added to the right-hand side of equation (5.1). Let us take the direction of gravity as the z-axis, with z increasing upwards. Then the cosine of the angle between the directions of **g** and **l** is equal to the derivative $-\mathrm{d}z/\mathrm{d}l$, so that the projection of **g** on **l** is

$$-g\,\mathrm{d}z/\mathrm{d}l.$$

Accordingly, we now have

$$\frac{\partial}{\partial l}(\tfrac{1}{2}v^2 + w + gz) = 0.$$

Thus Bernoulli's equation states that along a streamline

$$\tfrac{1}{2}v^2 + w + gz = \text{constant}. \tag{5.4}$$

§6. The energy flux

Let us choose some volume element fixed in space, and find how the energy of the fluid contained in this volume element varies with time. The energy of unit volume of fluid is

$$\tfrac{1}{2}\rho v^2 + \rho\varepsilon,$$

where the first term is the kinetic energy and the second the internal energy, ε being the internal energy per unit mass. The change in this energy is given by the partial derivative

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2 + \rho\varepsilon).$$

To calculate this quantity, we write

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2) = \tfrac{1}{2}v^2\frac{\partial\rho}{\partial t} + \rho\mathbf{v}\cdot\frac{\partial\mathbf{v}}{\partial t},$$

or, using the equation of continuity (1.2) and the equation of motion (2.3),

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2) = -\tfrac{1}{2}v^2\,\mathrm{div}(\rho\mathbf{v}) - \mathbf{v}\cdot\mathbf{grad}\,p - \rho\mathbf{v}\cdot(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}.$$

In the last term we replace $\mathbf{v}\cdot(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ by $\frac{1}{2}\mathbf{v}\cdot\mathbf{grad}\,v^2$, and $\mathbf{grad}\,p$ by $\rho\,\mathbf{grad}\,w - \rho T\,\mathbf{grad}\,s$ (using the thermodynamic relation $\mathrm{d}w = T\mathrm{d}s + (1/\rho)\mathrm{d}p$), obtaining

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2) = -\tfrac{1}{2}v^2\,\mathrm{div}\,(\rho\mathbf{v}) - \rho\mathbf{v}\cdot\mathbf{grad}\,(\tfrac{1}{2}v^2 + w) + \rho T\mathbf{v}\cdot\mathbf{grad}\,s.$$

† It was derived for an incompressible fluid (§10) by D. Bernoulli in 1738.

In order to transform the derivative $\partial(\rho\varepsilon)/\partial t$, we use the thermodynamic relation

$$d\varepsilon = Tds - pdV = Tds + (p/\rho^2)d\rho.$$

Since $\varepsilon + p/\rho = \varepsilon + pV$ is simply the heat function w per unit mass, we find

$$d(\rho\varepsilon) = \varepsilon d\rho + \rho d\varepsilon = wd\rho + \rho Tds,$$

and so

$$\frac{\partial(\rho\varepsilon)}{\partial t} = w\frac{\partial\rho}{\partial t} + \rho T\frac{\partial s}{\partial t} = -w\operatorname{div}(\rho\mathbf{v}) - \rho T\mathbf{v}\cdot\mathbf{grad}\, s.$$

Here we have also used the general adiabatic equation (2.6).

Combining the above results, we find the change in the energy to be

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2 + \rho\varepsilon) = -(\tfrac{1}{2}v^2 + w)\operatorname{div}(\rho\mathbf{v}) - \rho\mathbf{v}\cdot\mathbf{grad}(\tfrac{1}{2}v^2 + w),$$

or, finally,

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2 + \rho\varepsilon) = -\operatorname{div}[\rho\mathbf{v}(\tfrac{1}{2}v^2 + w)]. \tag{6.1}$$

In order to see the meaning of this equation, let us integrate it over some volume:

$$\frac{\partial}{\partial t}\int(\tfrac{1}{2}\rho v^2 + \rho\varepsilon)dV = -\int\operatorname{div}[\rho\mathbf{v}(\tfrac{1}{2}v^2 + w)]dV,$$

or, converting the volume integral on the right into a surface integral,

$$\frac{\partial}{\partial t}\int(\tfrac{1}{2}\rho v^2 + \rho\varepsilon)dV = -\oint\rho\mathbf{v}(\tfrac{1}{2}v^2 + w)\cdot d\mathbf{f}. \tag{6.2}$$

The left-hand side is the rate of change of the energy of the fluid in some given volume. The right-hand side is therefore the amount of energy flowing out of this volume in unit time. Hence we see that the expression

$$\rho\mathbf{v}(\tfrac{1}{2}v^2 + w) \tag{6.3}$$

may be called the *energy flux density* vector. Its magnitude is the amount of energy passing in unit time through unit area perpendicular to the direction of the velocity.

The expression (6.3) shows that any unit mass of fluid carries with it during its motion an amount of energy $w + \frac{1}{2}v^2$. The fact that the heat function w appears here, and not the internal energy ε, has a simple physical significance. Putting $w = \varepsilon + p/\rho$, we can write the flux of energy through a closed surface in the form

$$-\oint\rho\mathbf{v}(\tfrac{1}{2}v^2 + \varepsilon)\cdot d\mathbf{f} - \oint p\mathbf{v}\cdot d\mathbf{f}.$$

The first term is the energy (kinetic and internal) transported through the surface in unit time by the mass of fluid. The second term is the work done by pressure forces on the fluid within the surface.

§7. The momentum flux

We shall now give a similar series of arguments for the momentum of the fluid. The momentum of unit volume is $\rho\mathbf{v}$. Let us determine its rate of change, $\partial(\rho\mathbf{v})/\partial t$. We shall use tensor notation. We have

$$\frac{\partial}{\partial t}(\rho v_i) = \rho\frac{\partial v_i}{\partial t} + \frac{\partial \rho}{\partial t}\, v_i.$$

Using the equation of continuity (1.2) in the form

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v_k)}{\partial x_k},$$

and Euler's equation (2.3) in the form

$$\frac{\partial v_i}{\partial t} = -v_k\frac{\partial v_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i},$$

we obtain

$$\begin{aligned}\frac{\partial}{\partial t}(\rho v_i) &= -\rho v_k\frac{\partial v_i}{\partial x_k} - \frac{\partial p}{\partial x_i} - v_i\frac{\partial(\rho v_k)}{\partial x_k}\\ &= -\frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_k}(\rho v_i v_k).\end{aligned}$$

We write the first term on the right in the form

$$\frac{\partial p}{\partial x_i} = \delta_{ik}\frac{\partial p}{\partial x_k},$$

and finally obtain

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial \Pi_{ik}}{\partial x_k}, \tag{7.1}$$

where the tensor Π_{ik} is defined as

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k. \tag{7.2}$$

This tensor is clearly symmetrical.

To see the meaning of the tensor Π_{ik}, we integrate equation (7.1) over some volume:

$$\frac{\partial}{\partial t}\int \rho v_i \mathrm{d}V = -\int \frac{\partial \Pi_{ik}}{\partial x_k}\mathrm{d}V.$$

The integral on the right is transformed into a surface integral by Green's formula:†

$$\frac{\partial}{\partial t}\int \rho v_i \mathrm{d}V = -\oint \Pi_{ik}\mathrm{d}f_k. \tag{7.3}$$

The left-hand side is the rate of change of the ith component of the momentum contained in the volume considered. The surface integral on the right is therefore the

† The rule for transforming an integral over a closed surface into one over the volume bounded by that surface can be formulated as follows: the surface element $\mathrm{d}f_i$ must be replaced by the operator $\mathrm{d}V\cdot\partial/\partial x_i$, which is to be applied to the whole of the integrand.

amount of momentum flowing out through the bounding surface in unit time. Consequently, $\Pi_{ik}\mathrm{d}f_k$ is the ith component of the momentum flowing through the surface element $\mathrm{d}\mathbf{f}$. If we write $\mathrm{d}f_k$ in the form $n_k\,\mathrm{d}f$, where $\mathrm{d}f$ is the area of the surface element, and $\mathbf{n}$ is a unit vector along the outward normal, we find that $\Pi_{ik}n_k$ is the flux of the ith component of momentum through unit surface area. We may notice that, according to (7.2), $\Pi_{ik}n_k = pn_i + \rho v_i v_k n_k$. This expression can be written in vector form

$$p\mathbf{n} + \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n}). \tag{7.4}$$

Thus Π_{ik} is the ith component of the amount of momentum flowing in unit time through unit area perpendicular to the x_k-axis. The tensor Π_{ik} is called the *momentum flux density tensor*. The energy flux is determined by a vector, energy being a scalar; the momentum flux, however, is determined by a tensor of rank two, the momentum itself being a vector.

The vector (7.4) gives the momentum flux in the direction of $\mathbf{n}$, i.e. through a surface perpendicular to $\mathbf{n}$. In particular, taking the unit vector $\mathbf{n}$ to be directed parallel to the fluid velocity, we find that only the longitudinal component of momentum is transported in this direction, and its flux density is $p + \rho v^2$. In a direction perpendicular to the velocity, only the transverse component (relative to $\mathbf{v}$) of momentum is transported, its flux density being just p.

§8. The conservation of circulation

The integral

$$\Gamma = \oint \mathbf{v}\cdot\mathrm{d}\mathbf{l},$$

taken along some closed contour, is called the *velocity circulation* round that contour.

Let us consider a closed contour drawn in the fluid at some instant. We suppose it to be a "fluid contour", i.e. composed of the fluid particles that lie on it. In the course of time these particles move about, and the contour moves with them. Let us investigate what happens to the velocity circulation. In other words, let us calculate the time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint \mathbf{v}\cdot\mathrm{d}\mathbf{l}.$$

We have written here the total derivative with respect to time, since we are seeking the change in the circulation round a "fluid contour" as it moves about, and not round a contour fixed in space.

To avoid confusion, we shall temporarily denote differentiation with respect to the coordinates by the symbol δ, retaining the symbol d for differentiation with respect to time. Next, we notice that an element $\mathrm{d}\mathbf{l}$ of the length of the contour can be written as the difference $\delta\mathbf{r}$ between the position vectors $\mathbf{r}$ of the points at the ends of the element. Thus we write the velocity circulation as $\oint \mathbf{v}\cdot\delta\mathbf{r}$. In differentiating this integral with respect to time, it must be borne in mind that not only the velocity but also the contour itself (i.e. its shape) changes. Hence, on taking the time differentiation under the integral sign, we must differentiate not only $\mathbf{v}$ but also $\delta\mathbf{r}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint \mathbf{v}\cdot\delta\mathbf{r} = \oint \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\cdot\delta\mathbf{r} + \oint \mathbf{v}\cdot\frac{\mathrm{d}\delta\mathbf{r}}{\mathrm{d}t}.$$

Since the velocity **v** is just the time derivative of the position vector **r**, we have

$$\mathbf{v}\cdot\frac{\mathrm{d}\delta\mathbf{r}}{\mathrm{d}t} = \mathbf{v}\cdot\delta\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v}\cdot\delta\mathbf{v} = \delta(\tfrac{1}{2}v^2).$$

The integral of a total differential along a closed contour, however, is zero. The second integral therefore vanishes, leaving

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint\mathbf{v}\cdot\delta\mathbf{r} = \oint\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\cdot\delta\mathbf{r}.$$

It now remains to substitute for the acceleration $\mathrm{d}\mathbf{v}/\mathrm{d}t$ its expression from (2.9):

$$\mathrm{d}\mathbf{v}/\mathrm{d}t = -\,\mathbf{grad}\,w.$$

Using Stokes' formula, we then have

$$\oint\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\cdot\delta\mathbf{r} = \oint\mathbf{curl}\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)\cdot\delta\mathbf{f} = 0,$$

since $\mathbf{curl}\,\mathbf{grad}\,w \equiv 0$. Thus, going back to our previous notation, we find†

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint\mathbf{v}\cdot\mathrm{d}\mathbf{l} = 0,$$

or

$$\oint\mathbf{v}\cdot\mathrm{d}\mathbf{l} = \text{constant}. \qquad (8.1)$$

We have therefore reached the conclusion that, in an ideal fluid, the velocity circulation round a closed "fluid" contour is constant in time (*Kelvin's theorem* (1869) or the *law of conservation of circulation*).

It should be emphasized that this result has been obtained by using Euler's equation in the form (2.9), and therefore involves the assumption that the flow is isentropic. The theorem does not hold for flows which are not isentropic.‡

By applying Kelvin's theorem to an infinitesimal closed contour δC and transforming the integral according to Stokes' theorem, we get

$$\oint\mathbf{v}\cdot\mathrm{d}\mathbf{l} = \int\mathbf{curl}\,\mathbf{v}\cdot\mathrm{d}\mathbf{f} \cong \delta\mathbf{f}\cdot\mathbf{curl}\,\mathbf{v} = \text{constant}, \qquad (8.2)$$

where $\mathrm{d}\mathbf{f}$ is a fluid surface element spanning the contour δC. The vector **curl v** is often called the *vorticity* of the fluid flow at a given point. The constancy of the product (8.2) can be intuitively interpreted as meaning that the vorticity moves with the fluid.

PROBLEM

Show that, in flow which is not isentropic, any moving particle carries with it a constant value of the product $(1/\rho)\,\mathbf{grad}\,s\cdot\mathbf{curl}\,\mathbf{v}$ (H. Ertel 1942).

† This result remains valid in a uniform gravitational field, since in that case $\mathbf{curl}\,\mathbf{g} \equiv 0$.

‡ Mathematically, it is necessary that there should be a one-to-one relation between p and ρ (which for isentropic flow is $s(p,\rho) = $ constant); then $-(1/\rho)\,\mathbf{grad}\,p$ can be written as the gradient of some function, a result which is needed in deriving Kelvin's theorem.

SOLUTION. When the flow is not isentropic, the right-hand side of Euler's equation (2.3) cannot be replaced by $-\mathbf{grad}\, w$, and (2.11) becomes

$$\partial \boldsymbol{\omega}/\partial t = \mathbf{curl}\,(\mathbf{v}\times\boldsymbol{\omega}) + (1/\rho^2)\,\mathbf{grad}\,\rho\times\mathbf{grad}\,p,$$

where for brevity $\boldsymbol{\omega} = \mathbf{curl}\,\mathbf{v}$. We multiply scalarly by $\mathbf{grad}\,s$; since $s = s(p, \rho)$, $\mathbf{grad}\,s$ is a linear function of $\mathbf{grad}\,p$ and $\mathbf{grad}\,\rho$, and $\mathbf{grad}\,s\cdot(\mathbf{grad}\,\rho\times\mathbf{grad}\,p) = 0$. The expression on the right-hand side can then be transformed as follows:

$$\begin{aligned}\mathbf{grad}\,s\cdot\partial\boldsymbol{\omega}/\partial t &= \mathbf{grad}\,s\cdot\mathbf{curl}\,(\mathbf{v}\times\boldsymbol{\omega})\\ &= -\operatorname{div}\,[\mathbf{grad}\,s\times(\mathbf{v}\times\boldsymbol{\omega})]\\ &= -\operatorname{div}\,[\mathbf{v}(\boldsymbol{\omega}\cdot\mathbf{grad}\,s)] + \operatorname{div}\,[\boldsymbol{\omega}(\mathbf{v}\cdot\mathbf{grad}\,s)]\\ &= -(\boldsymbol{\omega}\cdot\mathbf{grad}\,s)\operatorname{div}\mathbf{v} - \mathbf{v}\cdot\mathbf{grad}\,(\boldsymbol{\omega}\cdot\mathbf{grad}\,s) + \boldsymbol{\omega}\cdot\mathbf{grad}\,(\mathbf{v}\cdot\mathbf{grad}\,s).\end{aligned}$$

From (2.6), $\mathbf{v}\cdot\mathbf{grad}\,s = -\partial s/\partial t$, and therefore

$$\frac{\partial}{\partial t}(\boldsymbol{\omega}\cdot\mathbf{grad}\,s) + \mathbf{v}\cdot\mathbf{grad}\,(\boldsymbol{\omega}\cdot\mathbf{grad}\,s) + (\boldsymbol{\omega}\cdot\mathbf{grad}\,s)\operatorname{div}\mathbf{v} = 0.$$

The first two terms can be combined as $\mathrm{d}(\boldsymbol{\omega}\cdot\mathbf{grad}\,s)/\mathrm{d}t$, where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \mathbf{v}\cdot\mathbf{grad}$; in the last term, we put from (1.3) $\rho\operatorname{div}\mathbf{v} = -\mathrm{d}\rho/\mathrm{d}t$. The result is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\omega}\cdot\mathbf{grad}\,s}{\rho}\right) = 0,$$

which gives the required conservation law.

§9. Potential flow

From the law of conservation of circulation we can derive an important result. Let us at first suppose that the flow is steady, and consider a streamline of which we know that **curl v** is zero at some point. We draw an arbitrary infinitely small closed contour to encircle the streamline at that point. In the course of time, this contour moves with the fluid, but always encircles the same streamline. Since the product (8.2) must remain constant, it follows that **curl v** must be zero at every point on the streamline.

Thus we reach the conclusion that, if at any point on a streamline the vorticity is zero, the same is true at all other points on that streamline. If the flow is not steady, the same result holds, except that instead of a streamline we must consider the path described in the course of time by some particular fluid particle;† we recall that in non-steady flow these paths do not in general coincide with the streamlines.

At first sight it might seem possible to base on this result the following argument. Let us consider steady flow past some body. Let the incident flow be uniform at infinity; its velocity **v** is a constant, so that $\mathbf{curl}\,\mathbf{v} \equiv 0$ on all streamlines. Hence we conclude that **curl v** is zero along the whole of every streamline, i.e. in all space.

A flow for which $\mathbf{curl}\,\mathbf{v} = 0$ in all space is called a *potential flow* or *irrotational flow*, as opposed to *rotational flow*, in which the curl of the velocity is not everywhere zero. Thus we should conclude that steady flow past any body, with a uniform incident flow at infinity, must be potential flow.

Similarly, from the law of conservation of circulation, we might argue as follows. Let us suppose that at some instant we have potential flow throughout the volume of the fluid. Then the velocity circulation round any closed contour in the fluid is zero.‡ By Kelvin's

† To avoid misunderstanding, we may mention here that this result has no meaning in turbulent flow. We may also remark that a non-zero vorticity may occur on a streamline after the passage of a shock wave. We shall see that this is because the flow is no longer isentropic (§114).

‡ Here we suppose for simplicity that the fluid occupies a simply-connected region of space. The same final result would be obtained for a multiply-connected region, but restrictions on the choice of contours would have to be made in the derivation.

theorem, we could then conclude that this will hold at any future instant, i.e. we should find that, if there is potential flow at some instant, then there is potential flow at all subsequent instants (in particular, any flow for which the fluid is initially at rest must be a potential flow). This is in accordance with the fact that, if $\mathbf{curl}\,\mathbf{v} = 0$, equation (2.11) is satisfied identically.

In fact, however, all these conclusions are of only very limited validity. The reason is that the proof given above that $\mathbf{curl}\,\mathbf{v} = 0$ all along a streamline is, strictly speaking, invalid for a line which lies in the surface of a solid body past which the flow takes place, since the presence of this surface makes it impossible to draw a closed contour in the fluid encircling such a streamline. The equations of motion of an ideal fluid therefore admit solutions for which *separation* occurs at the surface of the body: the streamlines, having followed the surface for some distance, become separated from it at some point and continue into the fluid. The resulting flow pattern is characterized by the presence of a "surface of tangential discontinuity" proceeding from the body; on this surface the fluid velocity, which is everywhere tangential to the surface, has a discontinuity. In other words, at this surface one layer of fluid "slides" on another. Figure 1 shows a surface of discontinuity which separates moving fluid from a region of stationary fluid behind the body. From a mathematical point of view, the discontinuity in the tangential velocity component corresponds to a surface on which the curl of the velocity is non-zero.

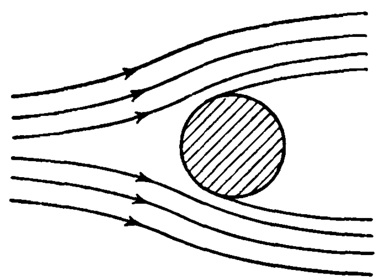

FIG. 1

When such discontinuous flows are included, the solution of the equations of motion for an ideal fluid is not unique: besides continuous flow, they admit also an infinite number of solutions possessing surfaces of tangential discontinuity starting from any prescribed line on the surface of the body past which the flow takes place. It should be emphasized, however, that none of these discontinuous solutions is physically significant, since tangential discontinuities are absolutely unstable, and therefore the flow would in fact become turbulent (see Chapter III).

The actual physical problem of flow past a given body has, of course, a unique solution. The reason is that ideal fluids do not really exist; any actual fluid has a certain viscosity, however small. This viscosity may have practically no effect on the motion of most of the fluid, but, no matter how small it is, it will be important in a thin layer of fluid adjoining the body. The properties of the flow in this *boundary layer* decide the choice of one out of the infinity of solutions of the equations of motion for an ideal fluid. It is found that, in the general case of flow past bodies of arbitrary form, solutions with separation must be taken, which in turn will result in turbulence.

In spite of what we have said above, the study of the solutions of the equations of motion for continuous steady potential flow past bodies is in some cases meaningful. Although, in the general case of flow past bodies of arbitrary form, the actual flow pattern bears almost no relation to the pattern of potential flow, for bodies of certain special ("streamlined"—§46) shapes the flow may differ very little from potential flow; more precisely, it will be potential flow except in a thin layer of fluid at the surface of the body and in a relatively narrow "wake" behind the body.

Another important case of potential flow occurs for small oscillations of a body immersed in fluid. It is easy to show that, if the amplitude a of the oscillations is small compared with the linear dimension l of the body ($a \ll l$), the flow past the body will be potential flow. To show this, we estimate the order of magnitude of the various terms in Euler's equation

$$\partial \mathbf{v}/\partial t + (\mathbf{v} \cdot \mathbf{grad})\mathbf{v} = -\mathbf{grad}\, w.$$

The velocity $\mathbf{v}$ changes markedly (by an amount of the same order as the velocity $\mathbf{u}$ of the oscillating body) over a distance of the order of the dimension l of the body. Hence the derivatives of $\mathbf{v}$ with respect to the coordinates are of the order of u/l. The order of magnitude of $\mathbf{v}$ itself (at fairly small distances from the body) is determined by the magnitude of $\mathbf{u}$. Thus we have $(\mathbf{v} \cdot \mathbf{grad})\mathbf{v} \sim u^2/l$. The derivative $\partial \mathbf{v}/\partial t$ is of the order of ωu, where ω is the frequency of the oscillations. Since $\omega \sim u/a$, we have $\partial \mathbf{v}/\partial t = u^2/a$. It now follows from the inequality $a \ll l$ that the term $(\mathbf{v} \cdot \mathbf{grad})\mathbf{v}$ is small compared with $\partial \mathbf{v}/\partial t$ and can be neglected, so that the equation of motion of the fluid becomes $\partial \mathbf{v}/\partial t = -\mathbf{grad}\, w$. Taking the curl of both sides, we obtain $\partial(\mathbf{curl}\, \mathbf{v})/\partial t = 0$, whence $\mathbf{curl}\, \mathbf{v} = \text{constant}$. In oscillatory motion, however, the time average of the velocity is zero, and therefore $\mathbf{curl}\, \mathbf{v} = \text{constant}$ implies that $\mathbf{curl}\, \mathbf{v} = 0$. Thus the motion of a fluid executing small oscillations is potential flow to a first approximation.

We shall now obtain some general properties of potential flow. We first recall that the derivation of the law of conservation of circulation, and therefore all its consequences, were based on the assumption that the flow is isentropic. If the flow is not isentropic, the law does not hold, and therefore, even if we have potential flow at some instant, the vorticity will in general be non-zero at subsequent instants. Thus only isentropic flow can in fact be potential flow.

In potential flow, the velocity circulation along any closed contour is zero:

$$\oint \mathbf{v} \cdot \mathbf{dl} = \int \mathbf{curl}\, \mathbf{v} \cdot \mathbf{df} = 0. \tag{9.1}$$

It follows from this that, in particular, closed streamlines cannot exist in potential flow.† For, since the direction of a streamline is at every point the direction of the velocity, the circulation along such a line can never be zero.

In rotational flow the velocity circulation is not in general zero. In this case there may be closed streamlines, but it must be emphasized that the presence of closed streamlines is not a necessary property of rotational flow.

Like any vector field having zero curl, the velocity in potential flow can be expressed as the gradient of some scalar. This scalar is called the *velocity potential*; we shall denote it by ϕ:

$$\mathbf{v} = \mathbf{grad}\, \phi. \tag{9.2}$$

† This result, like (9.1), may not be valid for motion in a multiply-connected region of space. In potential flow in such a region, the velocity circulation may be non-zero if the closed contour round which it is taken cannot be contracted to a point without crossing the boundaries of the region.

Writing Euler's equation in the form (2.10)

$$\partial \mathbf{v}/\partial t + \tfrac{1}{2}\,\mathbf{grad}\,v^2 - \mathbf{v}\times\mathbf{curl}\,\mathbf{v} = -\mathbf{grad}\,w$$

and substituting $\mathbf{v} = \mathbf{grad}\,\phi$, we have

$$\mathbf{grad}\left(\frac{\partial\phi}{\partial t} + \tfrac{1}{2}v^2 + w\right) = 0,$$

whence

$$\partial\phi/\partial t + \tfrac{1}{2}v^2 + w = f(t), \tag{9.3}$$

where $f(t)$ is an arbitrary function of time. This equation is a first integral of the equations of potential flow. The function $f(t)$ in equation (9.3) can be put equal to zero without loss of generality, because the potential is not uniquely defined: since the velocity is the space derivative of ϕ, we can add to ϕ any function of the time.

For steady flow we have (taking the potential ϕ to be independent of time) $\partial\phi/\partial t = 0$, $f(t) = \text{constant}$, and (9.3) becomes Bernoulli's equation:

$$\tfrac{1}{2}v^2 + w = \text{constant}. \tag{9.4}$$

It must be emphasized here that there is an important difference between the Bernoulli's equation for potential flow and that for other flows. In the general case, the "constant" on the right-hand side is a constant along any given streamline, but is different for different streamlines. In potential flow, however, it is constant throughout the fluid. This enhances the importance of Bernoulli's equation in the study of potential flow.

§10. Incompressible fluids

In a great many cases of the flow of liquids (and also of gases), their density may be supposed invariable, i.e. constant throughout the volume of the fluid and throughout its motion. In other words, there is no noticeable compression or expansion of the fluid in such cases. We then speak of *incompressible flow*.

The general equations of fluid dynamics are much simplified for an incompressible fluid. Euler's equation, it is true, is unchanged if we put $\rho = \text{constant}$, except that ρ can be taken under the gradient operator in equation (2.4):

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = -\mathbf{grad}\left(\frac{p}{\rho}\right) + \mathbf{g}. \tag{10.1}$$

The equation of continuity, on the other hand, takes for constant ρ the simple form

$$\operatorname{div}\mathbf{v} = 0. \tag{10.2}$$

Since the density is no longer an unknown function as it was in the general case, the fundamental system of equations in fluid dynamics for an incompressible fluid can be taken to be equations involving the velocity only. These may be the equation of continuity (10.2) and equation (2.11):

$$\frac{\partial}{\partial t}(\mathbf{curl}\,\mathbf{v}) = \mathbf{curl}(\mathbf{v}\times\mathbf{curl}\,\mathbf{v}). \tag{10.3}$$

Bernoulli's equation too can be written in a simpler form for an incompressible fluid. Equation (10.1) differs from the general Euler's equation (2.9) in that it has $\mathbf{grad}\,(p/\rho)$ in place of **grad** w. Hence we can write down Bernoulli's equation immediately by simply replacing the heat function in (5.4) by p/ρ:

$$\tfrac{1}{2}v^2 + p/\rho + gz = \text{constant}. \tag{10.4}$$

For an incompressible fluid, we can also write p/ρ in place of w in the expression (6.3) for the energy flux, which then becomes

$$\rho \mathbf{v}\left(\tfrac{1}{2}v^2 + \frac{p}{\rho}\right). \tag{10.5}$$

For we have, from a well-known thermodynamic relation, the expression $d\varepsilon = Tds - pdV$ for the change in internal energy; for $s = \text{constant}$ and $V = 1/\rho = \text{constant}$, $d\varepsilon = 0$, i.e. $\varepsilon = \text{constant}$. Since constant terms in the energy do not matter, we can omit ε in $w = \varepsilon + p/\rho$.

The equations are particularly simple for potential flow of an incompressible fluid. Equation (10.3) is satisfied identically if **curl v** $= 0$. Equation (10.2), with the substitution $\mathbf{v} = \mathbf{grad}\,\phi$, becomes

$$\triangle \phi = 0, \tag{10.6}$$

i.e. Laplace's equation† for the potential ϕ. This equation must be supplemented by boundary conditions at the surfaces where the fluid meets solid bodies. At fixed solid surfaces, the fluid velocity component v_n normal to the surface must be zero, whilst for moving surfaces it must be equal to the normal component of the velocity of the surface (a given function of time). The velocity v_n, however, is equal to the normal derivative of the potential ϕ: $v_n = \partial\phi/\partial n$. Thus the general boundary conditions are that $\partial\phi/\partial n$ is a given function of coordinates and time at the boundaries.

For potential flow, the velocity is related to the pressure by equation (9.3). In an incompressible fluid, we can replace w in this equation by p/ρ:

$$\partial\phi/\partial t + \tfrac{1}{2}v^2 + p/\rho = f(t). \tag{10.7}$$

We may notice here the following important property of potential flow of an incompressible fluid. Suppose that some solid body is moving through the fluid. If the result is potential flow, it depends at any instant only on the velocity of the moving body at that instant, and not, for example, on its acceleration. For equation (10.6) does not explicitly contain the time, which enters the solution only through the boundary conditions, and these contain only the velocity of the moving body.

From Bernoulli's equation, $\tfrac{1}{2}v^2 + p/\rho = \text{constant}$, we see that, in steady flow of an incompressible fluid (not in a gravitational field), the greatest pressure occurs at points where the velocity is zero. Such a point usually occurs on the surface of a body past which the fluid is moving (at the point O in Fig. 2), and is called a *stagnation point*. If $\mathbf{u}$ is the velocity of the incident current (i.e. the fluid velocity at infinity), and p_0 the pressure at infinity, the pressure at the stagnation point is

$$p_{\max} = p_0 + \tfrac{1}{2}\rho u^2. \tag{10.8}$$

If the velocity distribution in a moving fluid depends on only two coordinates (x and y, say), and the velocity is everywhere parallel to the xy-plane, the flow is said to be *two-*

† The velocity potential was first introduced by Euler, who obtained an equation of the form (10.6) for it; this form later became known as Laplace's equation.

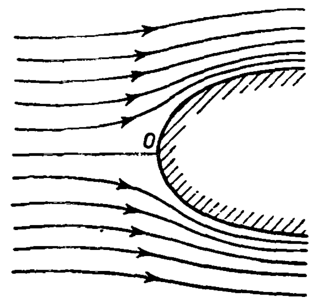

FIG. 2

dimensional or *plane flow*. To solve problems of two-dimensional flow of an incompressible fluid, it is sometimes convenient to express the velocity in terms of what is called the *stream function*. From the equation of continuity $\operatorname{div}\mathbf{v} \equiv \partial v_x/\partial x + \partial v_y/\partial y = 0$ we see that the velocity components can be written as the derivatives

$$v_x = \partial\psi/\partial y, \qquad v_y = -\partial\psi/\partial x \tag{10.9}$$

of some function $\psi(x, y)$, called the stream function. The equation of continuity is then satisfied automatically. The equation that must be satisfied by the stream function is obtained by substituting (10.9) in equation (10.3). We then obtain

$$\frac{\partial}{\partial t}\triangle\psi - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\triangle\psi + \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\triangle\psi = 0. \tag{10.10}$$

If we know the stream function we can immediately determine the form of the streamlines for steady flow. For the differential equation of the streamlines (in two-dimensional flow) is $\mathrm{d}x/v_x = \mathrm{d}y/v_y$ or $v_y\,\mathrm{d}x - v_x\,\mathrm{d}y = 0$; it expresses the fact that the direction of the tangent to a streamline is the direction of the velocity. Substituting (10.9), we have

$$\frac{\partial\psi}{\partial x}\mathrm{d}x + \frac{\partial\psi}{\partial y}\mathrm{d}y = \mathrm{d}\psi = 0,$$

whence ψ = constant. Thus the streamlines are the family of curves obtained by putting the stream function $\psi(x, y)$ equal to an arbitrary constant.

If we draw a curve between two points A and B in the xy-plane, the mass flux Q across this curve is given by the difference in the values of the stream function at these two points, regardless of the shape of the curve. For, if v_n is the component of the velocity normal to the curve at any point, we have

$$Q = \rho\oint_A^B v_n\,\mathrm{d}l = \rho\oint_A^B (-v_y\,\mathrm{d}x + v_x\,\mathrm{d}y) = \rho\int_A^B \mathrm{d}\psi,$$

or

$$Q = \rho(\psi_B - \psi_A). \tag{10.11}$$

There are powerful methods of solving problems of two-dimensional potential flow of an incompressible fluid past bodies of various profiles, involving the application of the theory of functions of a complex variable.† The basis of these methods is as follows. The potential and the stream function are related to the velocity components by‡

$$v_x = \partial\phi/\partial x = \partial\psi/\partial y, \qquad v_y = \partial\phi/\partial y = -\partial\psi/\partial x.$$

These relations between the derivatives of ϕ and ψ, however, are the same, mathematically, as the well-known Cauchy–Riemann conditions for a complex expression

$$w = \phi + i\psi \tag{10.12}$$

to be an analytic function of the complex argument $z = x + iy$. This means that the function $w(z)$ has at every point a well-defined derivative

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = v_x - iv_y. \tag{10.13}$$

The function w is called the *complex potential*, and $\mathrm{d}w/\mathrm{d}z$ the *complex velocity*. The modulus and argument of the latter give the magnitude v of the velocity and the angle θ between the direction of the velocity and that of the x-axis:

$$\mathrm{d}w/\mathrm{d}z = ve^{-i\theta}. \tag{10.14}$$

At a solid surface past which the flow takes place, the velocity must be along the tangent. That is, the profile contour of the surface must be a streamline, i.e. ψ = constant along it; the constant may be taken as zero, and then the problem of flow past a given contour reduces to the determination of an analytic function $w(z)$ which takes real values on the contour. The statement of the problem is more involved when the fluid has a free surface; an example is found in Problem 9.

The integral of an analytic function round any closed contour C is well known to be equal to $2\pi i$ times the sum of the residues of the function at its simple poles inside C; hence

$$\oint w'\,\mathrm{d}z = 2\pi i \sum_k A_k,$$

where A_k are the residues of the complex velocity. We also have

$$\begin{aligned}\oint w'\,\mathrm{d}z &= \oint (v_x - iv_y)(\mathrm{d}x + i\mathrm{d}y)\\ &= \oint (v_x\mathrm{d}x + v_y\mathrm{d}y) + i\oint (v_x\mathrm{d}y - v_y\mathrm{d}x).\end{aligned}$$

The real part of this expression is just the velocity circulation Γ round the contour C. The imaginary part, multiplied by ρ, is the mass flux across C; if there are no sources of fluid within the contour, this flux is zero and we then have simply

$$\Gamma = 2\pi i \sum_k A_k; \tag{10.15}$$

all the residues A_k are in this case purely imaginary.

† A more detailed account of these methods and their numerous applications may be found in many books which treat fluid dynamics from a more mathematical standpoint. Here, we shall describe only the basic idea.

‡ The existence of the stream function depends, however, only on the flow's being two-dimensional, not necessarily a potential flow.

Finally, let us consider the conditions under which the fluid may be regarded as incompressible. When the pressure changes adiabatically by Δp, the density changes by $\Delta\rho = (\partial\rho/\partial p)_s \Delta p$. According to Bernoulli's equation, however, Δp is of the order of ρv^2 in steady flow. We shall show in §64 that the derivative $(\partial p/\partial\rho)_s$ is the square of the velocity c of sound in the fluid, so that $\Delta\rho \sim \rho v^2/c^2$. The fluid may be regarded as incompressible if $\Delta\rho/\rho \ll 1$. We see that a necessary condition for this is that the fluid velocity be small compared with that of sound:

$$v \ll c. \tag{10.16}$$

However, this condition is sufficient only in steady flow. In non-steady flow, a further condition must be fulfilled. Let τ and l be a time and a length of the order of the times and distances over which the fluid velocity undergoes significant changes. If the terms $\partial\mathbf{v}/\partial t$ and $(1/\rho)\,\mathbf{grad}\,p$ in Euler's equation are comparable, we find, in order of magnitude, $v/\tau \sim \Delta p/l\rho$ or $\Delta p \sim l\rho v/\tau$, and the corresponding change in ρ is $\Delta\rho \sim l\rho v/\tau c^2$. Now comparing the terms $\partial\rho/\partial t$ and $\rho\,\mathrm{div}\,\mathbf{v}$ in the equation of continuity, we find that the derivative $\partial\rho/\partial t$ may be neglected (i.e. we may suppose ρ constant) if $\Delta\rho/\tau \ll \rho v/l$, or

$$\tau \gg l/c. \tag{10.17}$$

If the conditions (10.16) and (10.17) are both fulfilled, the fluid may be regarded as incompressible. The condition (10.17) has an obvious meaning: the time l/c taken by a sound signal to traverse the distance l must be small compared with the time τ during which the flow changes appreciably, so that the propagation of interactions in the fluid may be regarded as instantaneous.

PROBLEMS

PROBLEM 1. Determine the shape of the surface of an incompressible fluid subject to a gravitational field, contained in a cylindrical vessel which rotates about its (vertical) axis with a constant angular velocity Ω.

SOLUTION. Let us take the axis of the cylinder as the z-axis. Then $v_x = -y\Omega$, $v_y = x\Omega$, $v_z = 0$. The equation of continuity is satisfied identically, and Euler's equation (10.1) gives

$$x\Omega^2 = \frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad y\Omega^2 = \frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad \frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0.$$

The general integral of these equations is

$$p/\rho = \tfrac{1}{2}\Omega^2(x^2+y^2) - gz + \text{constant}.$$

At the free surface $p = $ constant, so that the surface is a paraboloid:

$$z = \tfrac{1}{2}\Omega^2(x^2+y^2)/g,$$

the origin being taken at the lowest point of the surface.

PROBLEM 2. A sphere, with radius R, moves with velocity $\mathbf{u}$ in an incompressible ideal fluid. Determine the potential flow of the fluid past the sphere.

SOLUTION. The fluid velocity must vanish at infinity. The solutions of Laplace's equation $\triangle\phi = 0$ which vanish at infinity are well known to be $1/r$ and the derivatives, of various orders, of $1/r$ with respect to the coordinates (the origin is taken at the centre of the sphere). On account of the complete symmetry of the sphere, only one constant vector, the velocity $\mathbf{u}$, can appear in the solution, and, on account of the linearity of both Laplace's equation and the boundary condition, ϕ must involve $\mathbf{u}$ linearly. The only scalar which can be formed from $\mathbf{u}$ and the derivatives of $1/r$ is the scalar product $\mathbf{u}\cdot\mathbf{grad}(1/r)$. We therefore seek ϕ in the form

$$\phi = \mathbf{A}\cdot\mathbf{grad}(1/r) = -(\mathbf{A}\cdot\mathbf{n})/r^2,$$

where $\mathbf{n}$ is a unit vector in the direction of $\mathbf{r}$. The constant $\mathbf{A}$ is determined from the condition that the normal components of the velocities $\mathbf{v}$ and $\mathbf{u}$ must be equal at the surface at the sphere, i.e. $\mathbf{v}\cdot\mathbf{n} = \mathbf{u}\cdot\mathbf{n}$ for $r = R$. This condition gives $\mathbf{A} = \frac{1}{2}\mathbf{u}R^3$, so that

$$\phi = -\frac{R^3}{2r^2}\mathbf{u}\cdot\mathbf{n}, \qquad \mathbf{v} = \frac{R^3}{2r^3}[3\mathbf{n}(\mathbf{u}\cdot\mathbf{n}) - \mathbf{u}].$$

The pressure distribution is given by equation (10.7):

$$p = p_0 - \tfrac{1}{2}\rho v^2 - \rho\partial\phi/\partial t,$$

where p_0 is the pressure at infinity. To calculate the derivative $\partial\phi/\partial t$, we must bear in mind that the origin (which we have taken at the centre of the sphere) moves with velocity $\mathbf{u}$. Hence

$$\partial\phi/\partial t = (\partial\phi/\partial\mathbf{u})\cdot\dot{\mathbf{u}} - \mathbf{u}\cdot\mathbf{grad}\,\phi.$$

The pressure distribution over the surface of the sphere is given by the formula

$$p = p_0 + \tfrac{1}{8}\rho u^2(9\cos^2\theta - 5) + \tfrac{1}{2}\rho R\mathbf{n}\cdot \mathrm{d}\mathbf{u}/\mathrm{d}t,$$

where θ is the angle between $\mathbf{n}$ and $\mathbf{u}$.

PROBLEM 3. The same as Problem 2, but for an infinite cylinder moving perpendicular to its axis.†

SOLUTION. The flow is independent of the axial coordinate, so that we have to solve Laplace's equation in two dimensions. The solutions which vanish at infinity are the first and higher derivatives of $\log r$ with respect to the coordinates, where $\mathbf{r}$ is the radius vector perpendicular to the axis of the cylinder. We seek a solution in the form

$$\phi = \mathbf{A}\cdot\mathbf{grad}\log r = \mathbf{A}\cdot\mathbf{n}/r,$$

and from the boundary conditions we obtain $\mathbf{A} = -R^2\mathbf{u}$, so that

$$\phi = -\frac{R^2}{r}\mathbf{u}\cdot\mathbf{n}, \qquad \mathbf{v} = \frac{R^2}{r^2}[2\mathbf{n}(\mathbf{u}\cdot\mathbf{n}) - \mathbf{u}].$$

The pressure at the surface of the cylinder is given by

$$p = p_0 + \tfrac{1}{2}\rho u^2(4\cos^2\theta - 3) + \rho R\mathbf{n}\cdot \mathrm{d}\mathbf{u}/\mathrm{d}t.$$

PROBLEM 4. Determine the potential flow of an incompressible ideal fluid in an ellipsoidal vessel rotating about a principal axis with angular velocity Ω, and determine the total angular momentum of the fluid.

SOLUTION. We take Cartesian coordinates x, y, z along the axes of the ellipsoid at a given instant, the z-axis being the axis of rotation. The velocity of points in the vessel wall is

$$\mathbf{u} = \mathbf{\Omega}\times\mathbf{r},$$

so that the boundary condition $v_n = \partial\phi/\partial n = u_n$ is

$$\partial\phi/\partial n = \Omega(xn_y - yn_x),$$

or, using the equation of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$,

$$\frac{x}{a^2}\frac{\partial\phi}{\partial x} + \frac{y}{b^2}\frac{\partial\phi}{\partial y} + \frac{z}{c^2}\frac{\partial\phi}{\partial z} = xy\Omega\left(\frac{1}{b^2} - \frac{1}{a^2}\right).$$

The solution of Laplace's equation which satisfies this boundary condition is

$$\phi = \Omega\frac{a^2 - b^2}{a^2 + b^2}xy. \qquad (1)$$

The angular momentum of the fluid in the vessel is

$$M = \rho\int(xv_y - yv_x)\,\mathrm{d}V.$$

† The solution of the more general problems of potential flow past an ellipsoid and an elliptical cylinder may be found in: N. E. Kochin, I. A. Kibel' and N. V. Roze, *Theoretical Hydromechanics* (*Teoreticheskaya gidromekhanika*), Part 1, chapter VII, Moscow 1963; H. Lamb, *Hydrodynamics*, 6th ed., §§103–116, Cambridge 1932.

Integrating over the volume V of the ellipsoid, we have

$$M = \frac{\Omega\rho V}{5}\frac{(a^2-b^2)^2}{a^2+b^2}.$$

Formula (1) gives the absolute motion of the fluid relative to the instantaneous position of the axes x, y, z which are fixed to the rotating vessel. The motion relative to the vessel (i.e. relative to a rotating system of coordinates x, y, z) is found by subtracting the velocity $\mathbf{\Omega}\times\mathbf{r}$ from the absolute velocity; denoting the relative velocity of the fluid by $\mathbf{v}'$, we have

$$v'_x = \frac{\partial\phi}{\partial x} + y\Omega = \frac{2\Omega a^2}{a^2+b^2}y, \qquad v'_y = -\frac{2\Omega b^2}{a^2+b^2}x, \qquad v'_z = 0.$$

The paths of the relative motion are found by integrating the equations $\dot{x} = v'_x$, $\dot{y} = v'_y$, and are the ellipses $x^2/a^2 + y^2/b^2 = \text{constant}$, which are similar to the boundary ellipse.

PROBLEM 5. Determine the flow near a stagnation point (Fig. 2).

SOLUTION. A small part of the surface of the body near the stagnation point may be regarded as plane. Let us take it as the xy-plane. Expanding ϕ for x, y, z small, we have as far as the second-order terms

$$\phi = ax + by + cz + Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx;$$

a constant term in ϕ is immaterial. The constant coefficients are determined so that ϕ satisfies the equation $\triangle\phi = 0$ and the boundary conditions $v_z = \partial\phi/\partial z = 0$ for $z = 0$ and all x, y, $\partial\phi/\partial x = \partial\phi/\partial y = 0$ for $x = y = z = 0$ (the stagnation point). This gives $a = b = c = 0$; $C = -A-B$, $E = F = 0$. The term Dxy can always be removed by an appropriate rotation of the x and y axes. We then have

$$\phi = Ax^2 + By^2 - (A+B)z^2. \tag{1}$$

If the flow is axially symmetrical about the z-axis (symmetrical flow past a solid of revolution), we must have $A = B$, so that

$$\phi = A(x^2 + y^2 - 2z^2).$$

The velocity components are $v_x = 2Ax$, $v_y = 2Ay$, $v_z = -4Az$. The streamlines are given by equations (5.2), from which we find $x^2z = c_1$, $y^2z = c_2$, i.e. the streamlines are cubical hyperbolae.

If the flow is uniform in the y-direction (e.g. flow in the z-direction past a cylinder with its axis in the y-direction), we must have $B = 0$ in (1), so that

$$\phi = A(x^2 - z^2).$$

The streamlines are the hyperbolae $xz = \text{constant}$.

PROBLEM 6. Determine the potential flow near an angle formed by two intersecting planes.

SOLUTION. Let us take polar coordinates r, θ in the cross-sectional plane (perpendicular to the line of intersection), with the origin at the vertex of the angle; θ is measured from one of the arms of the angle. Let the angle be α radians; for $\alpha < \pi$ the flow takes place within the angle, for $\alpha > \pi$ outside it. The boundary condition that the normal velocity component vanish means that $\partial\phi/\partial\theta = 0$ for $\theta = 0$ and $\theta = \alpha$. The solution of Laplace's equation satisfying these conditions can be written†

$$\phi = Ar^n \cos n\theta, \qquad n = \pi/\alpha,$$

so that

$$v_r = nAr^{n-1}\cos n\theta, \qquad v_\theta = -nAr^{n-1}\sin n\theta.$$

For $n < 1$ (flow outside an angle; Fig. 3), v_r becomes infinite as $1/r^{1-n}$ at the origin. For $n > 1$ (flow inside an angle; Fig. 4), v becomes zero for $r = 0$.

The stream function, which gives the form of the streamlines, is $\psi = Ar^n \sin n\theta$. The expressions obtained for ϕ and ψ are the real and imaginary parts of the complex potential $w = Az^n$.‡

PROBLEM 7. A spherical hole with radius a is suddenly formed in an incompressible fluid filling all space. Determine the time taken for the hole to be filled with fluid (Besant 1859; Rayleigh 1917).

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† We take the solution which involves the lowest positive power of r, since r is small.

‡ If the boundary planes are supposed infinite, Problems 5 and 6 involve degeneracy, in that the values of the constants A and B in the solutions are indeterminate. In actual cases of flow past finite bodies, they are determined by the general conditions of the problem.

SOLUTION. The flow after the formation of the hole will be spherically symmetrical, the velocity at every point being directed to the centre of the hole. For the radial velocity $v_r \equiv v < 0$ we have Euler's equation in spherical polar coordinates:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}. \tag{1}$$

The equation of continuity gives

$$r^2 v = F(t), \tag{2}$$

where $F(t)$ is an arbitrary function of time; this equation expresses the fact that, since the fluid is incompressible, the volume flowing through any spherical surface is independent of the radius of that surface.

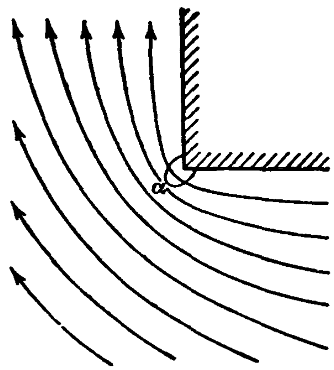

FIG. 3

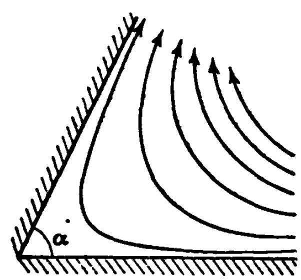

FIG. 4

Substituting v from (2) in (1), we have

$$\frac{F'(t)}{r^2} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}.$$

Integrating this equation over r from the instantaneous radius $R = R(t) \leqslant a$ of the hole to infinity, we obtain

$$-\frac{F'(t)}{R} + \tfrac{1}{2}V^2 = \frac{p_0}{\rho} \tag{3}$$

where $V = \mathrm{d}R(t)/\mathrm{d}t$ is the rate of change of the radius of the hole, and p_0 is the pressure at infinity; the fluid velocity at infinity is zero, and so is the pressure at the surface of the hole. From equation (2) for points on the surface of the hole we find

$$F(t) = R^2(t)\,V(t),$$

and, substituting this expression for $F(t)$ in (3), we obtain the equation

$$-\frac{3V^2}{2} - \tfrac{1}{2}R\frac{\mathrm{d}V^2}{\mathrm{d}R} = \frac{p_0}{\rho}. \tag{4}$$

The variables are separable; integrating with the boundary condition $V = 0$ for $R = a$ (the fluid being initially at rest), we have

$$V \equiv \frac{dR}{dt} = -\sqrt{\left[\frac{2p_0}{3\rho}\left(\frac{a^3}{R^3}-1\right)\right]}.$$

Hence we have for the required total time for the hole to be filled

$$\tau = \sqrt{\frac{3\rho}{2p_0}}\int_0^a \frac{dR}{\sqrt{[(a/R)^3-1]}}.$$

This integral reduces to a beta function, and we have finally

$$\tau = \sqrt{\frac{3a^2\rho\pi}{2p_0}}\frac{\Gamma(5/6)}{\Gamma(1/3)} = 0{\cdot}915a\sqrt{\frac{\rho}{p_0}}.$$

PROBLEM 8. A sphere immersed in an incompressible fluid expands according to a given law $R = R(t)$. Determine the fluid pressure at the surface of the sphere.

SOLUTION. Let the required pressure be $P(t)$. Calculations exactly similar to those of Problem 7, except that the pressure at $r = R$ is $P(t)$ and not zero, give instead of (3) the equation

$$-\frac{F'(t)}{R(t)} + \tfrac{1}{2}V^2 = \frac{p_0}{\rho} - \frac{P(t)}{\rho}$$

and accordingly instead of (4) the equation

$$\frac{p_0 - P(t)}{\rho} = -\frac{3V^2}{2} - RV\frac{dV}{dR}.$$

Bearing in mind the fact that $V = dR/dt$, we can write the expression for $P(t)$ in the form

$$P(t) = p_0 + \tfrac{1}{2}\rho\left[\frac{d^2(R^2)}{dt^2} + \left(\frac{dR}{dt}\right)^2\right].$$

PROBLEM 9. Determine the form of a jet emerging from an infinitely long slit in a plane wall.

SOLUTION. Let the wall be along the x-axis in the xy-plane, and the aperture be the segment $-\frac{1}{2}a \leqslant x \leqslant \frac{1}{2}a$ of that axis, the fluid occupying the half-plane $y > 0$. Far from the wall ($y \to \infty$) the fluid velocity is zero, and the pressure is p_0, say.

At the free surface of the jet (BC and $B'C'$ in Fig. 5a) the pressure $p = 0$, while the velocity takes the constant value $v_1 = \sqrt{(2p_0/\rho)}$, by Bernoulli's equation. The wall lines are streamlines, and continue into the free boundary of the jet. Let ψ be zero on the line ABC; then, on the line $A'B'C'$, $\psi = -Q/\rho$, where $Q = \rho a_1 v_1$ is the rate at which the fluid emerges in the jet (a_1, v_1 being the jet width and velocity at infinity). The potential ϕ varies from $-\infty$ to $+\infty$ both along ABC and along $A'B'C'$; let ϕ be zero at B and B'. Then, in the plane of the complex variable w, the region of flow is an infinite strip of width Q/ρ (Fig. 5b). (The points in Fig. 5b, c, d are named to correspond with those in Fig. 5a.)

We introduce a new complex variable, the logarithm of the complex velocity:

$$\zeta = -\log\left[\frac{1}{v_1 e^{\frac{1}{2}i\pi}}\frac{dw}{dz}\right] = \log\frac{v_1}{v} + i(\tfrac{1}{2}\pi + \theta); \tag{1}$$

here $v_1 e^{\frac{1}{2}i\pi}$ is the complex velocity of the jet at infinity. On $A'B'$ we have $\theta = 0$; on AB, $\theta = -\pi$; on BC and $B'C'$, $v = v_1$, while at infinity in the jet $\theta = -\frac{1}{2}\pi$. In the plane of the complex variable ζ, therefore, the region of flow is a semi-infinite strip of width π in the right half-plane (Fig. 5c). If we can now find a conformal transformation which carries the strip in the w-plane into the half-strip in the ζ-plane (with the points corresponding as in Fig. 5), we shall have determined w as a function of dw/dz, and w can then be found by a simple quadrature.

In order to find the desired transformation, we introduce one further auxiliary complex variable, u, such that the region of flow in the u-plane is the upper half-plane, the points B and B' corresponding to $u = \pm 1$, the points C and C' to $u = 0$, and the infinitely distant points A and A' to $u = \pm\infty$ (Fig. 5d). The dependence of w on this auxiliary variable is given by the conformal transformation which carries the upper half of the u-plane into the strip in the w-plane. With the above correspondence of points, this transformation is

$$w = -\frac{Q}{\rho\pi}\log u. \tag{2}$$

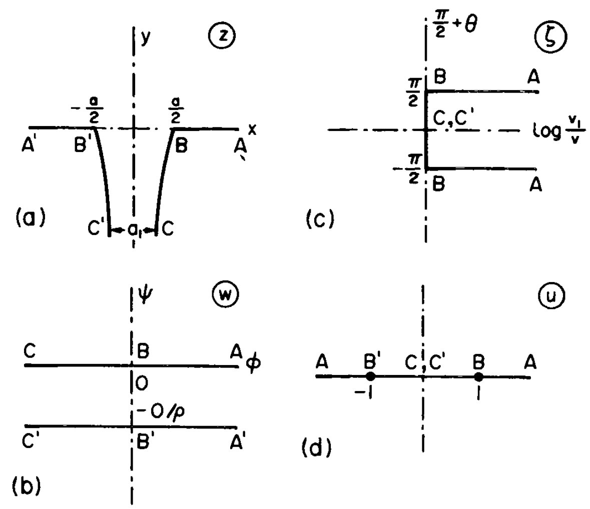

FIG. 5

In order to find the dependence of ζ on u, we have to find a conformal transformation of the half-strip in the ζ-plane into the upper half of the u-plane. Regarding this half-strip as a triangle with one vertex at infinity, we can find the desired transformation by means of the well-known Schwarz–Christoffel formula; it is

$$\zeta = -i \sin^{-1} u. \tag{3}$$

Formulae (2) and (3) give the solution of the problem, since they furnish the dependence of $\mathrm{d}w/\mathrm{d}z$ on w in parametric form.

Let us now determine the form of the jet. On BC we have $w = \phi$, $\zeta = i(\frac{1}{2}\pi + \theta)$, while u varies from 1 to 0. From (2) and (3) we obtain

$$\phi = -\frac{Q}{\rho\pi} \log(-\cos\theta), \tag{4}$$

and from (1) we have

$$\mathrm{d}\phi/\mathrm{d}z = v_1 e^{-i\theta},$$

or

$$\mathrm{d}z \equiv \mathrm{d}x + i\,\mathrm{d}y = \frac{1}{v_1} e^{i\theta}\,\mathrm{d}\phi = \frac{a_1}{\pi} e^{i\theta} \tan\theta\,\mathrm{d}\theta,$$

whence we find, by integration with the conditions $y = 0$, $x = \frac{1}{2}a$ for $\theta = -\pi$, the form of the jet, expressed parametrically. In particular, the compression of the jet is $a_1/a = \pi/(2+\pi) = 0{\cdot}61$.

§11. The drag force in potential flow past a body

Let us consider the problem of potential flow of an incompressible ideal fluid past some solid body. This problem is, of course, completely equivalent to that of the motion of a fluid when the same body moves through it. To obtain the latter case from the former, we need only change to a system of coordinates in which the fluid is at rest at infinity. We shall, in fact, say in what follows that the body is moving through the fluid.

Let us determine the nature of the fluid velocity distribution at great distances from the moving body. The potential flow of an incompressible fluid satisfies Laplace's equation, $\triangle\phi = 0$. We have to consider solutions of this equation which vanish at infinity, since the fluid is at rest there. We take the origin somewhere inside the moving body; the coordinate system moves with the body, but we shall consider the fluid velocity distribution at a particular instant. As we know, Laplace's equation has a solution $1/r$, where r is the distance from the origin. The gradient and higher space derivatives of $1/r$ are also solutions. All these solutions, and any linear combination of them, vanish at infinity. Hence the general form of the required solution of Laplace's equation at great distances from the body is

$$\phi = -\frac{a}{r} + \mathbf{A}\cdot\mathbf{grad}\frac{1}{r} + \ldots,$$

where a and $\mathbf{A}$ are independent of the coordinates; the omitted terms contain higher-order derivatives of $1/r$. It is easy to see that the constant a must be zero. For the potential $\phi = -a/r$ gives a velocity

$$\mathbf{v} = -\mathbf{grad}(a/r) = a\mathbf{r}/r^3.$$

Let us calculate the corresponding mass flux through some closed surface, say a sphere with radius R. On this surface the velocity is constant and equal to a/R^2; the total flux through it is therefore $\rho(a/R^2)4\pi R^2 = 4\pi\rho a$. But the flux of an incompressible fluid through any closed surface must, of course, be zero. Hence we conclude that $a = 0$.

Thus ϕ contains terms of order $1/r^2$ and higher. Since we are seeking the velocity at large distances, the terms of higher order may be neglected, and we have

$$\phi = \mathbf{A}\cdot\mathbf{grad}(1/r) = -\mathbf{A}\cdot\mathbf{n}/r^2, \tag{11.1}$$

and the velocity $\mathbf{v} = \mathbf{grad}\,\phi$ is

$$\mathbf{v} = (\mathbf{A}\cdot\mathbf{grad})\,\mathbf{grad}\frac{1}{r} = \frac{3(\mathbf{A}\cdot\mathbf{n})\mathbf{n} - \mathbf{A}}{r^3}, \tag{11.2}$$

where $\mathbf{n}$ is a unit vector in the direction of $\mathbf{r}$. We see that at large distances the velocity diminishes as $1/r^3$. The vector $\mathbf{A}$ depends on the actual shape and velocity of the body, and can be determined only by solving completely the equation $\triangle\phi = 0$ at all distances, taking into account the appropriate boundary conditions at the surface of the moving body.

The vector $\mathbf{A}$ which appears in (11.2) is related in a definite manner to the total momentum and energy of the fluid in its motion past the body. The total kinetic energy of the fluid (the internal energy of an incompressible fluid is constant) is $E = \frac{1}{2}\int \rho v^2 \,\mathrm{d}V$, where the integration is taken over all space outside the body. We take a region of space V bounded by a sphere with large radius R, whose centre is at the origin, and first integrate only over V, later letting R tend to infinity. We have identically

$$\int v^2 \,\mathrm{d}V = \int u^2 \,\mathrm{d}V + \int (\mathbf{v}+\mathbf{u})\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}V,$$

where $\mathbf{u}$ is the velocity of the body. Since $\mathbf{u}$ is independent of the coordinates, the first integral on the right is simply $u^2(V - V_0)$, where V_0 is the volume of the body. In the second integral, we write the sum $\mathbf{v}+\mathbf{u}$ as $\mathbf{grad}\,(\phi + \mathbf{u}\cdot\mathbf{r})$; using the facts that $\operatorname{div}\mathbf{v} = 0$ (equation of continuity) and $\operatorname{div}\mathbf{u} \equiv 0$, we have

$$\int v^2 \,\mathrm{d}V = u^2(V - V_0) + \int \operatorname{div}\left[(\phi + \mathbf{u}\cdot\mathbf{r})(\mathbf{v}-\mathbf{u})\right]\mathrm{d}V.$$

The second integral is now transformed into an integral over the surface S of the sphere and the surface S_0 of the body:

$$\int v^2\,\mathrm{d}V = u^2(V - V_0) + \oint\limits_{S+S_0} (\phi + \mathbf{u}\cdot\mathbf{r})(\mathbf{v}-\mathbf{u})\cdot\mathrm{d}\mathbf{f}.$$

On the surface of the body, the normal components of $\mathbf{v}$ and $\mathbf{u}$ are equal by virtue of the boundary conditions; since the vector $\mathrm{d}\mathbf{f}$ is along the normal to the surface, it is clear that the integral over S_0 vanishes identically. On the remote surface S we substitute the expressions (11.1), (11.2) for ϕ and $\mathbf{v}$, and neglect terms which vanish as $R \to \infty$. Writing the surface element on the sphere S in the form $\mathrm{d}\mathbf{f} = \mathbf{n}R^2\mathrm{d}o$, where $\mathrm{d}o$ is an element of solid angle, we obtain

$$\int v^2\,\mathrm{d}V = u^2(\tfrac{4}{3}\pi R^3 - V_0) + \int [3(\mathbf{A}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{n}) - (\mathbf{u}\cdot\mathbf{n})^2R^3]\,\mathrm{d}o.$$

Finally, effecting the integration† and multiplying by $\frac{1}{2}\rho$, we obtain the following expression for the total energy of the fluid:

$$E = \tfrac{1}{2}\rho(4\pi\mathbf{A}\cdot\mathbf{u} - V_0u^2). \tag{11.3}$$

As has been mentioned already, the exact calculation of the vector $\mathbf{A}$ requires a complete solution of the equation $\triangle\phi = 0$, taking into account the particular boundary conditions at the surface of the body. However, the general nature of the dependence of $\mathbf{A}$ on the velocity $\mathbf{u}$ of the body can be found directly from the facts that the equation is linear in ϕ, and the boundary conditions are linear in both ϕ and $\mathbf{u}$. It follows from this that $\mathbf{A}$ must be a linear function of the components of $\mathbf{u}$. The energy E given by formula (11.3) is therefore a quadratic function of the components of $\mathbf{u}$, and can be written in the form

$$E = \tfrac{1}{2}m_{ik}u_iu_k, \tag{11.4}$$

where m_{ik} is some constant symmetrical tensor, whose components can be calculated from those of $\mathbf{A}$; it is called the *induced-mass tensor.*

Knowing the energy E, we can obtain an expression for the total momentum $\mathbf{P}$ of the fluid. To do so, we notice that infinitesimal changes in E and $\mathbf{P}$ are related by‡ $\mathrm{d}E = \mathbf{u}\cdot\mathrm{d}\mathbf{P}$;

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† The integration over o is equivalent to averaging the integrand over all directions of the vector $\mathbf{n}$ and multiplying by 4π. To average expressions of the type $(\mathbf{A}\cdot\mathbf{n})(\mathbf{B}\cdot\mathbf{n}) \equiv A_in_iB_kn_k$, where $\mathbf{A}$, $\mathbf{B}$ are constant vectors, we notice that

$$\overline{(\mathbf{A}\cdot\mathbf{n})(\mathbf{B}\cdot\mathbf{n})} = A_iB_k\overline{n_in_k} = \tfrac{1}{3}\delta_{ik}A_iB_k = \tfrac{1}{3}\mathbf{A}\cdot\mathbf{B}.$$

‡ For, let the body be accelerated by some external force $\mathbf{F}$. The momentum of the fluid will thereby be increased; let it increase by $\mathrm{d}\mathbf{P}$ during a time $\mathrm{d}t$. This increase is related to the force by $\mathrm{d}\mathbf{P} = \mathbf{F}\,\mathrm{d}t$, and on scalar multiplication by the velocity $\mathbf{u}$ we have $\mathbf{u}\cdot\mathrm{d}\mathbf{P} = \mathbf{F}\cdot\mathbf{u}\,\mathrm{d}t$, i.e. the work done by the force $\mathbf{F}$ acting through the distance $\mathbf{u}\,\mathrm{d}t$, which in turn must be equal to the increase $\mathrm{d}E$ in the energy of the fluid.

It should be noticed that it would not be possible to calculate the momentum directly as the integral $\int\rho\mathbf{v}\,\mathrm{d}V$ over the whole volume of the fluid. The reason is that this integral, with the velocity $\mathbf{v}$ distributed in accordance with (11.2), diverges, in the sense that the result of the integration, though finite, depends on how the integral is taken: on effecting the integration over a large region, whose dimensions subsequently tend to infinity, we obtain a value depending on the shape of the region (sphere, cylinder, etc.). The method of calculating the momentum which we use here, starting from the relation $\mathbf{u}\cdot\mathrm{d}\mathbf{P} = \mathrm{d}E$, leads to a completely definite final result, given by formula (11.6), which certainly satisfies the physical relation between the rate of change of the momentum and the forces acting on the body.

it follows from this that, if E is expressed in the form (11.4), the components of $\mathbf{P}$ must be

$$P_i = m_{ik} u_k. \tag{11.5}$$

Finally, a comparison of formulae (11.3), (11.4) and (11.5) shows that $\mathbf{P}$ is given in terms of $\mathbf{A}$ by

$$\mathbf{P} = 4\pi\rho\mathbf{A} - \rho V_0 \mathbf{u}. \tag{11.6}$$

It must be noticed that the total momentum of the fluid is a perfectly definite finite quantity.

The momentum transmitted to the fluid by the body in unit time is $\mathrm{d}\mathbf{P}/\mathrm{d}t$. With the opposite sign it evidently gives the reaction $\mathbf{F}$ of the fluid, i.e. the force acting on the body:

$$\mathbf{F} = -\mathrm{d}\mathbf{P}/\mathrm{d}t. \tag{11.7}$$

The component of $\mathbf{F}$ parallel to the velocity of the body is called the *drag force*, and the perpendicular component is called the *lift force*.

If it were possible to have potential flow past a body moving uniformly in an ideal fluid, we should have $\mathbf{P}$ = constant, since $\mathbf{u}$ = constant, and so $\mathbf{F} = 0$. That is, there would be no drag and no lift; the pressure forces exerted on the body by the fluid would balance out (a result known as *d'Alembert's paradox*). The origin of this paradox is most clearly seen by considering the drag. The presence of a drag force in uniform motion of a body would mean that, to maintain the motion, work must be continually done by some external force, this work being either dissipated in the fluid or converted into kinetic energy of the fluid, and the result being a continual flow of energy to infinity in the fluid. There is, however, by definition no dissipation of energy in an ideal fluid, and the velocity of the fluid set in motion by the body diminishes so rapidly with increasing distance from the body that there can be no flow of energy to infinity.

However, it must be emphasized that all these arguments relate only to the motion of a body in an infinite volume of fluid. If, for example, the fluid has a free surface, a body moving uniformly parallel to this surface will experience a drag. The appearance of this force (called *wave drag*) is due to the occurrence of a system of waves propagated on the free surface, which continually remove energy to infinity.

Suppose that a body is executing an oscillatory motion under the action of an external force $\mathbf{f}$. When the conditions discussed in §10 are fulfilled, the fluid surrounding the body moves in a potential flow, and we can use the relations previously obtained to derive the equations of motion of the body. The force $\mathbf{f}$ must be equal to the time derivative of the total momentum of the system, and the total momentum is the sum of the momentum $M\mathbf{u}$ of the body (M being the mass of the body) and the momentum $\mathbf{P}$ of the fluid:

$$M\,\mathrm{d}\mathbf{u}/\mathrm{d}t + \mathrm{d}\mathbf{P}/\mathrm{d}t = \mathbf{f}.$$

Using (11.5), we then obtain

$$M\,\mathrm{d}u_i/\mathrm{d}t + m_{ik}\mathrm{d}u_k/\mathrm{d}t = f_i,$$

which can also be written

$$\frac{\mathrm{d}u_k}{\mathrm{d}t}(M\delta_{ik} + m_{ik}) = f_i. \tag{11.8}$$

This is the equation of motion of a body immersed in an ideal fluid.

Let us now consider what is in some ways the converse problem. Suppose that the fluid executes some oscillatory motion on account of some cause external to the body. This motion will set the body in motion also.† We shall derive the equation of motion of the body.

We assume that the velocity of the fluid varies only slightly over distances of the order of the dimension of the body. Let **v** be what the fluid velocity at the position of the body would be if the body were absent; that is, **v** is the velocity of the unperturbed flow. According to the above assumption, **v** may be supposed constant throughout the volume occupied by the body. We denote the velocity of the body by **u** as before.

The force which acts on the body and sets it in motion can be determined as follows. If the body were wholly carried along with the fluid (i.e. if $\mathbf{v} = \mathbf{u}$), the force acting on it would be the same as the force which would act on the liquid in the same volume if the body were absent. The momentum of this volume of fluid is $\rho V_0 \mathbf{v}$, and therefore the force on it is $\rho V_0 \, \mathrm{d}\mathbf{v}/\mathrm{d}t$. In reality, however, the body is not wholly carried along with the fluid; there is a motion of the body relative to the fluid, in consequence of which the fluid itself acquires some additional motion. The resulting additional momentum of the fluid is $m_{ik}(u_k - v_k)$, since in (11.5) we must now replace **u** by the velocity $\mathbf{u} - \mathbf{v}$ of the body relative to the fluid. The change in this momentum with time results in the appearance of an additional reaction force on the body of $-m_{ik}\,\mathrm{d}(u_k - v_k)/\mathrm{d}t$. Thus the total force on the body is

$$\rho V_0 \frac{\mathrm{d}v_i}{\mathrm{d}t} - m_{ik}\frac{\mathrm{d}}{\mathrm{d}t}(u_k - v_k).$$

This force is to be equated to the time derivative of the body momentum. Thus we obtain the following equation of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}(Mu_i) = \rho V_0 \frac{\mathrm{d}v_i}{\mathrm{d}t} - m_{ik}\frac{\mathrm{d}}{\mathrm{d}t}(u_k - v_k).$$

Integrating both sides with respect to time, we have

$$(M\delta_{ik} + m_{ik})u_k = (m_{ik} + \rho V_0 \delta_{ik})v_k. \qquad (11.9)$$

We put the constant of integration equal to zero, since the velocity **u** of the body in its motion caused by the fluid must vanish when **v** vanishes. The relation obtained determines the velocity of the body from that of the fluid. If the density of the body is equal to that of the fluid ($M = \rho V_0$), we have $\mathbf{u} = \mathbf{v}$, as we should expect.

PROBLEMS

PROBLEM 1. Obtain the equation of motion for a sphere executing an oscillatory motion in an ideal fluid, and for a sphere set in motion by an oscillating fluid.

SOLUTION. Comparing (11.1) with the expression for ϕ for flow past a sphere obtained in §10, Problem 2, we see that

$$\mathbf{A} = \tfrac{1}{2}R^3\mathbf{u},$$

where R is the radius of the sphere. The total momentum transmitted to the fluid by the sphere is, according to (11.6), $\mathbf{P} = \frac{2}{3}\pi\rho R^3\mathbf{u}$, so that the tensor m_{ik} is

$$m_{ik} = \tfrac{2}{3}\pi\rho R^3\delta_{ik}.$$

† For example, we may be considering the motion of a body in a fluid through which a sound wave is propagated, the wavelength being large compared with the dimension of the body.

The drag on the moving sphere is

$$\mathbf{F} = -\tfrac{2}{3}\pi\rho R^3 \mathrm{d}\mathbf{u}/\mathrm{d}t,$$

and the equation of motion of the sphere oscillating in the fluid is

$$\tfrac{4}{3}\pi R^3(\rho_0 + \tfrac{1}{2}\rho)\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{f},$$

where ρ_0 is the density of the sphere. The coefficient of $\mathrm{d}\mathbf{u}/\mathrm{d}t$ is the *virtual mass* of the sphere; it consists of the actual mass of the sphere and the induced mass, which in this case is half the mass of the fluid displaced by the sphere.

If the sphere is set in motion by the fluid, we have for its velocity, from (11.9),

$$\mathbf{u} = \frac{3\rho}{\rho + 2\rho_0}\mathbf{v}.$$

If the density of the sphere exceeds that of the fluid ($\rho_0 > \rho$), $u < v$, i.e. the sphere "lags behind" the fluid; if $\rho_0 < \rho$, on the other hand, the sphere "goes ahead".

PROBLEM 2. Express the moment of the forces acting on a body moving in a fluid in terms of the vector **A**.

SOLUTION. As we know from mechanics, the moment **M** of the forces acting on a body is determined from its Lagrangian function (in this case, the energy E) by the relation $\delta E = \mathbf{M}\cdot\delta\boldsymbol{\theta}$, where $\delta\boldsymbol{\theta}$ is the vector of an infinitesimal rotation of the body, and δE is the resulting change in E. Instead of rotating the body through an angle $\delta\theta$ (and correspondingly changing the components m_{ik}), we may rotate the fluid through an angle $-\delta\theta$ relative to the body (and correspondingly change the velocity $\mathbf{u}$). We have $\delta\mathbf{u} = -\delta\boldsymbol{\theta}\times\mathbf{u}$, so that

$$\delta E = \mathbf{P}\cdot\delta\mathbf{u} = -\delta\boldsymbol{\theta}\cdot\mathbf{u}\times\mathbf{P}.$$

Using the expression (11.6) for **P**, we then obtain the required formula:

$$\mathbf{M} = -\mathbf{u}\times\mathbf{P} = 4\pi\rho\mathbf{A}\times\mathbf{u}.$$

§12. Gravity waves

The free surface of a liquid in equilibrium in a gravitational field is a plane. If, under the action of some external perturbation, the surface is moved from its equilibrium position at some point, motion will occur in the liquid. This motion will be propagated over the whole surface in the form of waves, which are called *gravity waves*, since they are due to the action of the gravitational field. Gravity waves appear mainly on the surface of the liquid; they affect the interior also, but less and less at greater and greater depths.

We shall here consider gravity waves in which the velocity of the moving fluid particles is so small that we may neglect the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ in comparison with $\partial\mathbf{v}/\partial t$ in Euler's equation. The physical significance of this is easily seen. During a time interval of the order of the period τ of the oscillations of the fluid particles in the wave, these particles travel a distance of the order of the amplitude a of the wave. Their velocity v is therefore of the order of a/τ. It varies noticeably over time intervals of the order of τ and distances of the order of λ in the direction of propagation (where λ is the wavelength). Hence the time derivative of the velocity is of the order of v/τ, and the space derivatives are of the order of v/λ. Thus the condition $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} \ll \partial\mathbf{v}/\partial t$ is equivalent to

$$\frac{1}{\lambda}\left(\frac{a}{\tau}\right)^2 \ll \frac{a}{\tau}\cdot\frac{1}{\tau},$$

or

$$a \ll \lambda, \tag{12.1}$$

i.e. the amplitude of the oscillations in the wave must be small compared with the wavelength. We have seen in §9 that, if the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ in the equation of motion may be neglected, we have potential flow. Assuming the fluid incompressible, we can therefore use equations (10.6) and (10.7). The term $\frac{1}{2}v^2$ in the latter equation may be neglected, since it contains the square of the velocity; putting $f(t) = 0$ and including a term ρgz on account of the gravitational field, we obtain

$$p = -\rho gz - \rho\partial\phi/\partial t. \tag{12.2}$$

We take the z-axis vertically upwards, as usual, and the xy-plane in the equilibrium surface of the liquid.

Let us denote by ζ the z coordinate of a point on the surface; ζ is a function of x, y and t. In equilibrium $\zeta = 0$, so that ζ gives the vertical displacement of the surface in its oscillations. Let a constant pressure p_0 act on the surface. Then we have at the surface, by (12.2),

$$p_0 = -\rho g\zeta - \rho\partial\phi/\partial t.$$

The constant p_0 can be eliminated by redefining the potential ϕ, adding to it a quantity $p_0 t/\rho$ independent of the coordinates. We then obtain the condition at the surface as

$$g\zeta + (\partial\phi/\partial t)_{z=\zeta} = 0. \tag{12.3}$$

Since the amplitude of the wave oscillations is small, the displacement ζ is small. Hence we can suppose, to the same degree of approximation, that the vertical component of the velocity of points on the surface is simply the time derivative of ζ:

$$v_z = \partial\zeta/\partial t.$$

But $v_z = \partial\phi/\partial z$, so that

$$(\partial\phi/\partial z)_{z=\zeta} = \partial\zeta/\partial t = -\left(\frac{1}{g}\frac{\partial^2\phi}{\partial t^2}\right)_{z=\zeta}.$$

Since the oscillations are small, we can take the value of the derivatives at $z = 0$ instead of $z = \zeta$. Thus we have finally the following system of equations to determine the motion in a gravitational field:

$$\triangle\phi = 0, \tag{12.4}$$

$$\left(\frac{\partial\phi}{\partial z} + \frac{1}{g}\frac{\partial^2\phi}{\partial t^2}\right)_{z=0} = 0. \tag{12.5}$$

We shall here consider waves on the surface of a liquid whose area is unlimited, and we shall also suppose that the wavelength is small in comparison with the depth of the liquid; we can then regard the liquid as infinitely deep. We shall therefore omit the boundary conditions at the sides and bottom.

Let us consider a gravity wave propagated along the x-axis and uniform in the y-direction; in such a wave, all quantities are independent of y. We shall seek a solution which is a simple periodic function of time and of the coordinate x, i.e. we put

$$\phi = f(z)\cos(kx - \omega t).$$

Here ω is what is called the *circular frequency* (we shall say simply the *frequency*) of the wave; k is called the *wave number*; $\lambda = 2\pi/k$ is the *wavelength.*

Substituting in the equation $\triangle\phi = 0$, we have

$$d^2f/dz^2 - k^2f = 0.$$

The solution which decreases as we go into the interior of the liquid (i.e. as $z \to -\infty$) is

$$\phi = Ae^{kz}\cos(kx - \omega t). \tag{12.6}$$

We have also to satisfy the boundary condition (12.5). Substituting (12.6), we obtain

$$\omega^2 = kg \tag{12.7}$$

as the relation between the wave number and the frequency of a gravity wave (the *dispersion relation*).

The velocity distribution in the moving liquid is found by simply taking the space derivatives of ϕ:

$$v_x = -Ake^{kz}\sin(kx - \omega t), \qquad v_z = Ake^{kz}\cos(kx - \omega t). \tag{12.8}$$

We see that the velocity diminishes exponentially as we go into the liquid. At any given point in space (i.e. for given x, z) the velocity vector rotates uniformly in the xz-plane, its magnitude remaining constant.

Let us also determine the paths of fluid particles in the wave. We temporarily denote by x, z the coordinates of a moving fluid particle (and not of a point fixed in space), and by x_0, z_0 the values of x and z at the equilibrium position of the particle. Then $v_x = dx/dt$, $v_z = dz/dt$, and on the right-hand side of (12.8) we may approximate by writing x_0, z_0 in place of x, z, since the oscillations are small. An integration with respect to time then gives

$$\left.\begin{aligned} x - x_0 &= -A\frac{k}{\omega}e^{kz_0}\cos(kx_0 - \omega t),\\ z - z_0 &= -A\frac{k}{\omega}e^{kz_0}\sin(kx_0 - \omega t). \end{aligned}\right\} \tag{12.9}$$

Thus the fluid particles describe circles about the points (x_0, z_0) with a radius which diminishes exponentially with increasing depth.

The velocity of propagation U of the wave is, as we shall show in §67, $U = \partial\omega/\partial k$. Substituting here $\omega = \sqrt{(kg)}$, we find that the velocity of propagation of gravity waves on an unbounded surface of infinitely deep liquid is

$$U = \tfrac{1}{2}\sqrt{(g/k)} = \tfrac{1}{2}\sqrt{(g\lambda/2\pi)}. \tag{12.10}$$

It increases with the wavelength.

Long gravity waves

Having considered gravity waves whose length is small compared with the depth of the liquid, let us now discuss the opposite limiting case of waves whose length is large compared with the depth. These are called *long* waves.

Let us examine first the propagation of long waves in a channel. The channel is supposed to be along the x-axis, and of infinite length. The cross-section of the channel may have any shape, and may vary along its length. We denote the cross-sectional area of the liquid in the channel by $S = S(x, t)$. The depth and width of the channel are supposed small in comparison with the wavelength.

We shall here consider longitudinal waves, in which the liquid moves along the channel. In such waves the velocity component v_x along the channel is large compared with the components v_y, v_z.

We denote v_x by v simply, and omit small terms. The x-component of Euler's equation can then be written in the form

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

and the z-component in the form

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g;$$

we omit terms quadratic in the velocity, since the amplitude of the wave is again supposed small. From the second equation we have, since the pressure at the free surface ($z = \zeta$) must be p_0,

$$p = p_0 + g\rho(\zeta - z).$$

Substituting this expression in the first equation, we obtain

$$\partial v/\partial t = -g\partial\zeta/\partial x. \tag{12.11}$$

The second equation needed to determine the two unknowns v and ζ can be derived similarly to the equation of continuity; it is essentially the equation of continuity for the case in question. Let us consider a volume of liquid bounded by two plane cross-sections of the channel at a distance $\mathrm{d}x$ apart. In unit time a volume $(Sv)_x$ of liquid flows through one plane, and a volume $(Sv)_{x+\mathrm{d}x}$ through the other. Hence the volume of liquid between the two planes changes by

$$(Sv)_{x+\mathrm{d}x} - (Sv)_x = \frac{\partial(Sv)}{\partial x}\mathrm{d}x.$$

Since the liquid is incompressible, however, this change must be due simply to the change in the level of the liquid. The change per unit time in the volume of liquid between the two planes considered is $(\partial S/\partial t)\mathrm{d}x$. We can therefore write

$$\frac{\partial S}{\partial t}\mathrm{d}x = -\frac{\partial(Sv)}{\partial x}\mathrm{d}x,$$

or

$$\frac{\partial S}{\partial t} + \frac{\partial(Sv)}{\partial x} = 0. \tag{12.12}$$

This is the required equation of continuity.

Let S_0 be the equilibrium cross-sectional area of the liquid in the channel. Then $S = S_0 + S'$, where S' is the change in the cross-sectional area caused by the wave. Since the change in the liquid level is small, we can write S' in the form $b\zeta$, where b is the width of the channel at the surface of the liquid. Equation (12.12) then becomes

$$b\frac{\partial\zeta}{\partial t} + \frac{\partial(S_0 v)}{\partial x} = 0. \tag{12.13}$$

Differentiating (12.13) with respect to t and substituting $\partial v/\partial t$ from (12.11), we obtain

$$\frac{\partial^2\zeta}{\partial t^2} - \frac{g}{b}\frac{\partial}{\partial x}\left(S_0\frac{\partial\zeta}{\partial x}\right) = 0. \tag{12.14}$$

If the channel cross-section is the same at all points, then S_0 = constant and

$$\frac{\partial^2 \zeta}{\partial t^2} - \frac{g S_0}{b}\frac{\partial^2 \zeta}{\partial x^2} = 0. \tag{12.15}$$

This is called a *wave equation*: as we shall show in §64, it corresponds to the propagation of waves with a velocity U which is independent of frequency and is the square root of the coefficient of $\partial^2\zeta/\partial x^2$. Thus the velocity of propagation of long gravity waves in channels is

$$U = \sqrt{(gS_0/b)}. \tag{12.16}$$

In an entirely similar manner, we can consider long waves in a large tank, which we suppose infinite in two directions (those of x and y). The depth of liquid in the tank is denoted by h. The component v_z of the velocity is now small. Euler's equations take a form similar to (12.11):

$$\frac{\partial v_x}{\partial t} + g\frac{\partial \zeta}{\partial x} = 0, \qquad \frac{\partial v_y}{\partial t} + g\frac{\partial \zeta}{\partial y} = 0. \tag{12.17}$$

The equation of continuity is derived in the same way as (12.12) and is

$$\frac{\partial h}{\partial t} + \frac{\partial (h v_x)}{\partial x} + \frac{\partial (h v_y)}{\partial y} = 0.$$

We write the depth h as $h_0 + \zeta$, where h_0 is the equilibrium depth. Then

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (h_0 v_x)}{\partial x} + \frac{\partial (h_0 v_y)}{\partial y} = 0. \tag{12.18}$$

Let us assume that the tank has a horizontal bottom (h_0 = constant). Differentiating (12.18) with respect to t and substituting (12.17), we obtain

$$\frac{\partial^2 \zeta}{\partial t^2} - gh_0\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right) = 0. \tag{12.19}$$

This is again a (two-dimensional) wave equation; it corresponds to waves propagated with a velocity

$$U = \sqrt{(gh_0)}. \tag{12.20}$$

PROBLEMS

PROBLEM 1. Determine the velocity of propagation of gravity waves on an unbounded surface of liquid with depth h.

SOLUTION. At the bottom of the liquid, the normal velocity component must be zero, i.e. $v_z = \partial\phi/\partial z = 0$ for $z = -h$. From this condition we find the ratio of the constants A and B in the general solution

$$\phi = [Ae^{kz} + Be^{-kz}]\cos(kx - \omega t).$$

The result is

$$\phi = A\cos(kx - \omega t)\cosh k(z + h).$$

From the boundary condition (12.5) we find the relation between k and ω to be

$$\omega^2 = gk\tanh kh.$$

The velocity of propagation of the wave is

$$U = \frac{1}{2}\sqrt{\frac{g}{k\tanh kh}}\left[\tanh kh + \frac{kh}{\cosh^2 kh}\right].$$

For $kh \gg 1$ we have the result (12.10), and for $kh \ll 1$ the result (12.20).

PROBLEM 2. Determine the relation between frequency and wavelength for gravity waves on the surface separating two liquids, the upper liquid being bounded above by a fixed horizontal plane, and the lower liquid being similarly bounded below. The density and depth of the lower liquid are ρ and h, those of the upper liquid are ρ' and h', and $\rho > \rho'$.

SOLUTION. We take the xy-plane as the equilibrium plane of separation of the two liquids. Let us seek a solution having in the two liquids the forms

$$\left.\begin{aligned} \phi &= A\cosh k(z+h)\cos(kx-\omega t), \\ \phi' &= B\cosh k(z-h')\cos(kx-\omega t), \end{aligned}\right\} \tag{1}$$

so that the conditions at the upper and lower boundaries are satisfied; see the solution to Problem 1. At the surface of separation, the pressure must be continuous; by (12.2), this gives the condition

$$\rho g\zeta + \rho\frac{\partial\phi}{\partial t} = \rho' g\zeta + \rho'\frac{\partial\phi'}{\partial t} \quad \text{for} \quad z = \zeta,$$

or

$$\zeta = \frac{1}{g(\rho-\rho')}\left(\rho'\frac{\partial\phi'}{\partial t} - \rho\frac{\partial\phi}{\partial t}\right). \tag{2}$$

Moreover, the velocity component v_z must be the same for each liquid at the surface of separation. This gives the condition

$$\partial\phi/\partial z = \partial\phi'/\partial z \quad \text{for} \quad z = 0. \tag{3}$$

Now $v_z = \partial\phi/\partial z = \partial\zeta/\partial t$ and, substituting (2), we have

$$g(\rho-\rho')\frac{\partial\phi}{\partial z} = \rho'\frac{\partial^2\phi'}{\partial t^2} - \rho\frac{\partial^2\phi}{\partial t^2}. \tag{4}$$

Substituting (1) in (3) and (4) gives two homogeneous linear equations for A and B, and the condition of compatibility gives

$$\omega^2 = \frac{kg(\rho-\rho')}{\rho\coth kh + \rho'\coth kh'}.$$

For $kh \gg 1$, $kh' \gg 1$ (both liquids very deep),

$$\omega^2 = kg\frac{\rho-\rho'}{\rho+\rho'},$$

while for $kh \ll 1$, $kh' \ll 1$ (long waves),

$$\omega = k\sqrt{\frac{g(\rho-\rho')hh'}{\rho h' + \rho' h}}.$$

Lastly, if $kh \gtrsim 1$ and $kh' \ll 1$,

$$\omega^2 = k^2 gh'(\rho-\rho')/\rho.$$

PROBLEM 3. Determine the relation between frequency and wavelength for gravity waves propagated simultaneously on the surface of separation and on the upper surface of two liquid layers, the lower (density ρ) being infinitely deep, and the upper (density ρ') having depth h' and a free upper surface.

SOLUTION. We take the xy-plane as the equilibrium plane of separation of the two liquids. Let us seek a solution having in the two liquids the forms

$$\left.\begin{aligned} \phi &= Ae^{kz}\cos(kx-\omega t), \\ \phi' &= [Be^{-kz} + Ce^{kz}]\cos(kx-\omega t). \end{aligned}\right\} \tag{1}$$

At the surface of separation, i.e. for $z = 0$, we have the conditions (see Problem 2)

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi'}{\partial z}, \qquad g(\rho - \rho')\frac{\partial \phi}{\partial z} = \rho'\frac{\partial^2 \phi'}{\partial t^2} - \rho\frac{\partial^2 \phi}{\partial t^2}, \tag{2}$$

and at the upper surface, i.e. for $z = h'$, the condition

$$\frac{\partial \phi'}{\partial z} + \frac{1}{g}\frac{\partial^2 \phi'}{\partial t^2} = 0. \tag{3}$$

The first equation (2), on substitution of (1), gives $A = C - B$, and the remaining two conditions then give two equations for B and C; from the condition of compatibility we obtain a quadratic equation for ω^2, whose roots are

$$\omega^2 = kg\frac{(\rho - \rho')(1 - e^{-2kh'})}{\rho + \rho' + (\rho - \rho')e^{-2kh'}}, \qquad \omega^2 = kg.$$

For $h' \to \infty$ these roots correspond to waves propagated independently on the surface of separation and on the upper surface.

PROBLEM 4. Determine the characteristic frequencies of oscillation (see §69) of a liquid with depth h in a rectangular tank with width a and length b.

SOLUTION. We take the x and y axes along two sides of the tank. Let us seek a solution in the form of a stationary wave:

$$\phi = f(x, y)\cosh k(z + h)\cos \omega t.$$

We obtain for f the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + k^2 f = 0,$$

and the condition at the free surface gives, as in Problem 1, the relation

$$\omega^2 = gk \tanh kh.$$

We take the solution of the equation for f in the form

$$f = \cos px \cos qy, \qquad p^2 + q^2 = k^2.$$

At the sides of the tank we must have the conditions

$$v_x = \partial\phi/\partial x = 0 \quad \text{for} \quad x = 0, a;$$

$$v_y = \partial\phi/\partial y = 0 \quad \text{for} \quad y = 0, b.$$

Hence we find $p = m\pi/a$, $q = n\pi/b$, where m, n are integers. The possible values of k^2 are therefore

$$k^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right).$$

§13. Internal waves in an incompressible fluid

There is a kind of gravity wave which can be propagated inside an incompressible fluid. Such waves are due to an inhomogeneity of the fluid caused by the gravitational field. The pressure (and therefore the entropy s) necessarily varies with height; hence any displacement of a fluid particle in height destroys the mechanical equilibrium, and consequently causes an oscillatory motion. For, since the motion is adiabatic, the particle carries with it to its new position its old entropy s, which is not the same as the equilibrium value at the new position.

We shall suppose below that the wavelength is small in comparison with distances over which the gravitational field causes a marked change in density†; and we shall regard the fluid itself as incompressible. This means that we can neglect the change in its density caused by the pressure change in the wave. The change in density caused by thermal expansion cannot be neglected, since it is this that causes the phenomenon in question.

Let us write down a system of hydrodynamic equations for this motion. We shall use a suffix 0 to distinguish the values of quantities in mechanical equilibrium, and a prime to mark small deviations from those values. Then the equation of conservation of the entropy $s = s_0 + s'$ can be written, to the first order of smallness,

$$\partial s'/\partial t + \mathbf{v}\cdot\mathbf{grad}\, s_0 = 0, \tag{13.1}$$

where s_0, like the equilibrium values of other quantities, is a given function of the vertical coordinate z.

Next, in Euler's equation we again neglect the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ (since the oscillations are small); taking into account also the fact that the equilibrium pressure distribution is given by $\mathbf{grad}\, p_0 = \rho_0 \mathbf{g}$, we have to the same accuracy

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{\mathbf{grad}\, p}{\rho} + \mathbf{g} = -\frac{\mathbf{grad}\, p'}{\rho_0} + \frac{\mathbf{grad}\, p_0}{{\rho_0}^2}\rho'.$$

Since, from what has been said above, the change in density is due only to the change in entropy, and not to the change in pressure, we can put

$$\rho' = \left(\frac{\partial \rho_0}{\partial s_0}\right)_p s',$$

and we then obtain Euler's equation in the form

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\mathbf{g}}{\rho_0}\left(\frac{\partial \rho_0}{\partial s_0}\right)_p s' - \mathbf{grad}\frac{p'}{\rho_0}. \tag{13.2}$$

We can take ρ_0 under the gradient operator, since, as stated above, we always neglect the change in the equilibrium density over distances of the order of a wavelength. The density may likewise be supposed constant in the equation of continuity, which then becomes

$$\operatorname{div} \mathbf{v} = 0. \tag{13.3}$$

We shall seek a solution of equations (13.1)–(13.3) in the form of a plane wave:

$$\mathbf{v} = \text{constant} \times e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)},$$

and similarly for s' and p'. Substitution in the equation of continuity (13.3) gives

$$\mathbf{v}\cdot\mathbf{k} = 0, \tag{13.4}$$

† The density and pressure gradients are related by

$$\mathbf{grad}\, p = (\partial p/\partial \rho)_s\, \mathbf{grad}\, \rho = c^2\, \mathbf{grad}\, \rho,$$

where c is the speed of sound in the fluid. The hydrostatic equation $\mathbf{grad}\, p = \rho g$ thus gives $\mathbf{grad}\, \rho = (\rho/c^2)\mathbf{g}$. The density in the gravitational field therefore varies considerably over distances $l \cong c^2/g$. For air and water, $l \cong 10$ km and 200 km respectively.

i.e. the fluid velocity is everywhere perpendicular to the *wave vector* **k** (a transverse wave). Equations (13.1) and (13.2) give

$$i\omega s' = \mathbf{v}\cdot\mathbf{grad}\, s_0, \qquad -i\omega\mathbf{v} = \frac{1}{\rho_0}\left(\frac{\partial\rho_0}{\partial s_0}\right)_p s'\mathbf{g} - \frac{i\mathbf{k}}{\rho_0}p'.$$

The condition $\mathbf{v}\cdot\mathbf{k} = 0$ gives with the second of these equations

$$ik^2 p' = \left(\frac{\partial\rho_0}{\partial s_0}\right)_p s'\mathbf{g}\cdot\mathbf{k},$$

and, eliminating **v** and s' from the two equations, we obtain the desired dispersion relation,

$$\omega^2 = \omega_0{}^2 \sin^2\theta, \tag{13.5}$$

where

$$\omega_0{}^2 = -\frac{g}{\rho}\left(\frac{\partial\rho}{\partial s}\right)_p \frac{\mathrm{d}s}{\mathrm{d}z}. \tag{13.6}$$

Here and henceforward we omit the suffix zero to the equilibrium values of thermodynamic quantities; the z-axis is vertically upwards, and θ is the angle between this axis and the direction of **k**. If the expression on the right of (13.6) is positive, the condition for the stability of the equilibrium distribution $s(z)$ (the condition that convection be absent—see §4) is fulfilled.

We see that the frequency depends only on the direction of the wave vector, and not on its magnitude. For $\theta = 0$ we have $\omega = 0$; this means that waves of the type considered, with the wave vector vertical, cannot exist.

If the fluid is in both mechanical equilibrium and complete thermodynamic equilibrium, its temperature is constant and we can write

$$\frac{\mathrm{d}s}{\mathrm{d}z} = \left(\frac{\partial s}{\partial p}\right)_T \frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g\left(\frac{\partial s}{\partial p}\right)_T.$$

Finally, using the well-known thermodynamic relations

$$\left(\frac{\partial s}{\partial p}\right)_T = \frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial T}\right)_p, \qquad \left(\frac{\partial\rho}{\partial s}\right)_p = \frac{T}{c_p}\left(\frac{\partial\rho}{\partial T}\right)_p,$$

where c_p is the specific heat per unit mass, we find

$$\omega_0 = \sqrt{\frac{T g}{c_p \rho}}\left|\left(\frac{\partial\rho}{\partial T}\right)_p\right|. \tag{13.7}$$

In particular, for a perfect gas,

$$\omega_0 = \frac{g}{\sqrt{(c_p T)}}. \tag{13.8}$$

The dependence of the frequency on the direction of the wave vector has the result that the wave propagation velocity $\mathbf{U} = \partial\omega/\partial\mathbf{k}$ is not parallel to **k**. Representing $\omega(\mathbf{k})$ in the form

$$\omega = \omega_0\sqrt{[1 - (\mathbf{k}\cdot\mathbf{v}/k)^2]},$$

where $\mathbf{\nu}$ is a unit vector in the vertically upward direction, and differentiating, we find

$$\mathbf{U} = -(\omega_0{}^2/\omega k)(\mathbf{n}\cdot\boldsymbol{\nu})[\boldsymbol{\nu}-(\mathbf{n}\cdot\boldsymbol{\nu})\mathbf{n}] \tag{13.9}$$

(where $\mathbf{n} = \mathbf{k}/k$). This is perpendicular to $\mathbf{k}$, and its magnitude is

$$U = (\omega_0/k)\cos\theta.$$

Its vertical component is

$$\mathbf{U}\cdot\boldsymbol{\nu} = -(\omega_0/k)\cos\theta\sin\theta.$$

§14. Waves in a rotating fluid

Another kind of internal wave can be propagated in an incompressible fluid uniformly rotating as a whole. These waves are due to the Coriolis forces which occur in rotation.

We shall consider the fluid in coordinates rotating with it. With this treatment, the mechanical equations of motion must include additional (centrifugal and Coriolis) terms. Correspondingly, forces (per unit mass of fluid) must be added on the right of Euler's equation. The centrifugal force can be written as $\mathbf{grad}\,\frac{1}{2}(\mathbf{\Omega}\times\mathbf{r})^2$, where $\mathbf{\Omega}$ is the angular velocity vector of the fluid rotation. This term can be combined with the force $-(1/\rho)\,\mathbf{grad}\,p$ by using an effective pressure

$$P = p - \tfrac{1}{2}\rho(\mathbf{\Omega}\times\mathbf{r})^2. \tag{14.1}$$

The Coriolis force is $2\mathbf{v}\times\mathbf{\Omega}$, and occurs only when the fluid has a motion relative to the rotating coordinates, $\mathbf{v}$ being the velocity in those coordinates. We can transfer this term to the left-hand side of Euler's equation, writing the equation as

$$\partial\mathbf{v}/\partial t + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v} + 2\mathbf{\Omega}\times\mathbf{v} = -(1/\rho)\mathbf{grad}\,P. \tag{14.2}$$

The equation of continuity is unchanged; for an incompressible fluid, it is simply $\operatorname{div}\mathbf{v} = 0$.

We shall again assume the wave amplitude to be small, and neglect the term quadratic in the velocity in (14.2), which becomes

$$\partial\mathbf{v}/\partial t + 2\mathbf{\Omega}\times\mathbf{v} = -(1/\rho)\,\mathbf{grad}\,p', \tag{14.3}$$

where p' is the variable part of the pressure in the wave, and ρ is a constant. The pressure can be eliminated by taking the curl of both sides. The right-hand side gives zero, and on the left-hand side, since the fluid is incompressible,

$$\begin{aligned}\mathbf{curl}\,(\mathbf{\Omega}\times\mathbf{v}) &= \mathbf{\Omega}\operatorname{div}\mathbf{v} - (\mathbf{\Omega}\cdot\mathbf{grad})\mathbf{v}\\ &= -(\mathbf{\Omega}\cdot\mathbf{grad})\mathbf{v}.\end{aligned}$$

Taking the direction of $\mathbf{\Omega}$ as the z-axis, we write the resulting equation as

$$\frac{\partial}{\partial t}\mathbf{curl}\,\mathbf{v} = 2\Omega\frac{\partial\mathbf{v}}{\partial z}. \tag{14.4}$$

We seek the solution as a plane wave

$$\mathbf{v} = \mathbf{A}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{14.5}$$

which, since $\operatorname{div}\mathbf{v} = 0$, satisfies the transversality condition

$$\mathbf{k}\cdot\mathbf{A} = 0. \tag{14.6}$$

Substitution of (14.5) in (14.4) gives

$$\omega \mathbf{k} \times \mathbf{v} = 2i\Omega k_z \mathbf{v}. \tag{14.7}$$

The dispersion relation for these waves is found by eliminating $\mathbf{v}$ from this vector equation. Vector multiplication on both sides by $\mathbf{k}$ gives

$$-\omega k^2 \mathbf{v} = 2i\Omega k_z \mathbf{k} \times \mathbf{v}$$

and a comparison of the two equations yields the dependence of ω on $\mathbf{k}$:

$$\omega = 2\Omega k_z/k = 2\Omega \cos\theta, \tag{14.8}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{\Omega}$.

With (14.4), (14.7) takes the form

$$\mathbf{n} \times \mathbf{v} = i\mathbf{v},$$

where $n = \mathbf{k}/k$. If we use the complex wave amplitude in the form $\mathbf{A} = \mathbf{a} + i\mathbf{b}$ with real vectors $\mathbf{a}$ and $\mathbf{b}$, it follows that $\mathbf{n} \times \mathbf{b} = \mathbf{a}$: the vectors $\mathbf{a}$ and $\mathbf{b}$ (both lying in the plane perpendicular to $\mathbf{k}$) are at right angles and equal in magnitude. By taking their directions as the x and y axes, and separating real and imaginary parts in (14.5), we find

$$v_x = a\cos(\omega t - \mathbf{k}\cdot\mathbf{r}), \qquad v_y = -a\sin(\omega t - \mathbf{k}\cdot\mathbf{r}).$$

The wave is thus circularly polarized: at each point in space, the vector $\mathbf{v}$ rotates in the course of time, remaining constant in magnitude.†

The wave propagation velocity is

$$\mathbf{U} = \partial\omega/\partial\mathbf{k} = (2\Omega/k)[\boldsymbol{\nu} - \mathbf{n}(\mathbf{n}\cdot\boldsymbol{\nu})], \tag{14.9}$$

where $\boldsymbol{\nu}$ is a unit vector along $\mathbf{\Omega}$; as with internal gravity waves, it is perpendicular to the wave vector. Its magnitude and its component along $\mathbf{\Omega}$ are

$$U = (2\Omega/k)\sin\theta, \qquad \mathbf{U}\cdot\boldsymbol{\nu} = (2\Omega/k)\sin^2\theta = U\sin\theta.$$

These are called *inertial waves*. Since the Coriolis forces do no work on the moving fluid, the energy in the waves is entirely kinetic energy.

One particular form of axially symmetrical (not plane) inertial waves can be propagated along the axis of rotation of the fluid; see Problem 1.

There is one more comment to be made, regarding steady motions in a rotating fluid rather than wave propagation in it.

Let l be a characteristic length for such motion, and u a characteristic velocity. In order of magnitude, the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ in (14.2) is u^2/l, and $2\mathbf{\Omega}\times\mathbf{v}$ is Ωu. The former can be neglected in comparison with the latter if $u/l\Omega \ll 1$, and the equation of steady motion then reduces to

$$2\mathbf{\Omega}\times\mathbf{v} = -(1/\rho)\,\mathbf{grad}\,P \tag{14.10}$$

or

$$2\Omega v_y = (1/\rho)\partial P/\partial x, \qquad 2\Omega v_x = -(1/\rho)\partial P/\partial y, \qquad \partial P/\partial z = 0,$$

† This motion is relative to rotating coordinates. For fixed coordinates, it is combined with the rotation of the whole fluid.

where x and y are Cartesian coordinates in the plane perpendicular to the axis of rotation. Hence we see that P, and therefore v_x and v_y, are independent of the longitudinal coordinate z. Next, eliminating P from the first two equations, we get

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0,$$

and the equation div $\mathbf{v}=0$ then shows that $\partial v_z/\partial z=0$. Thus steady motion (in rotating coordinates) in a rapidly rotating fluid is a superposition of two independent motions: two-dimensional flow in the transverse plane and axial flow independent of z (J. Proudman 1916).

PROBLEMS

PROBLEM 1. Determine the motion in an axially symmetrical wave propagated along the axis of an incompressible fluid rotating as a whole (W. Thomson 1880).

SOLUTION. We take cylindrical polar coordinates r, ϕ, z, with the z-axis parallel to $\mathbf{\Omega}$. In an axially symmetrical wave, all quantities are independent of the angle variable ϕ. The dependence on time and on the coordinate z is given by a factor $\exp[i(kz-\omega t)]$. Taking components in (14.3), we get

$$-i\omega v_r-2\Omega v_\phi=-(1/\rho)\partial p'/\partial r, \tag{1}$$

$$-i\omega v_\phi+2\Omega v_r=0, \qquad -i\omega v_z=-(ik/\rho)p'. \tag{2}$$

These are to be combined with the equation of continuity

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+ikv_z=0. \tag{3}$$

Expressing v_ϕ and p' in terms of v_r by means of (2) and (3) and substituting in (1), we find the equation

$$\frac{\mathrm{d}^2F}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}F}{\mathrm{d}r}+\left[\frac{4\Omega^2k^2}{\omega^2}-k^2-\frac{1}{r^2}\right]F=0 \tag{4}$$

for the function $F(r)$ which determines the radial dependence of v_r:

$$v_r=F(r)e^{i(\omega t-kz)}.$$

The solution that vanishes for $r=0$ is

$$F=\text{constant}\times J_1[kr\sqrt{\{(4\Omega^2/\omega^2)-1\}}], \tag{5}$$

where J_1 is a Bessel function of order 1.

The motion comprises regions between coaxial cylinders with radius r_n such that

$$kr_n\sqrt{\{(4\Omega^2/\omega^2)-1\}}=x_n,$$

where $x_1, x_2, \ldots$ are the successive zeros of $J_1(x)$. On these cylindrical surfaces $v_r=0$, and the fluid therefore does not cross them.

For these waves in an infinite fluid, ω is independent of k. The possible values of the frequency are, however, restricted by the condition $\omega<2\Omega$; if this is not satisfied, (4) has no solution satisfying the necessary conditions of finiteness.

If the rotating fluid is bounded by a cylindrical wall with radius R, we have to use the condition $v_r=0$ at the wall. This gives the relation

$$ka\sqrt{\{(4\Omega^2/\omega^2)-1\}}=x_n$$

between ω and k for a wave with a given n (the number of coaxial regions in it).

PROBLEM 2. Derive an equation describing an arbitrary small perturbation of the pressure in a rotating fluid.

SOLUTION. Equation (14.3) in components is

$$\frac{\partial v_x}{\partial t}-2\Omega v_y=-\frac{1}{\rho}\frac{\partial p'}{\partial x}, \qquad \frac{\partial v_y}{\partial t}+2\Omega v_x=-\frac{1}{\rho}\frac{\partial p'}{\partial y}, \qquad \frac{\partial v_z}{\partial t}=-\frac{1}{\rho}\frac{\partial p'}{\partial z}. \tag{1}$$

Differentiating these with respect to x, y, and z, adding, and using div $\mathbf{v} = 0$, we find

$$\frac{1}{\rho}\triangle p' = 2\Omega\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right).$$

Differentiation with respect to t, again using equations (1), gives

$$\frac{1}{\rho}\frac{\partial}{\partial t}\triangle p' = 4\Omega^2\frac{\partial v_z}{\partial z},$$

and by a further differentiation with respect to t we arrive at the final equation

$$\frac{\partial^2}{\partial t^2}\triangle p' + 4\Omega^2\frac{\partial^2 p'}{\partial z^2} = 0. \tag{2}$$

For periodic perturbations with frequency ω, this becomes

$$\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} + \left(1 - \frac{4\Omega^2}{\omega^2}\right)\frac{\partial^2 p'}{\partial z^2} = 0. \tag{3}$$

For waves having the form (14.5), this of course gives the known dispersion relation (14.8), with $\omega < 2\Omega$ and a negative coefficient of $\partial^2 p'/\partial z^2$ in (3). Perturbations from a point source are propagated along generators of a cone whose axis is along $\mathbf{\Omega}$ and whose vertical angle is 2θ, where $\sin\theta = \omega/2\Omega$.

When $\omega > 2\Omega$, the coefficient of $\partial^2 p'/\partial z^2$ in (3) is positive, and this equation becomes Laplace's equation by an obvious change in the z scale. In this case, a point source of perturbation affects the whole volume of the fluid, to an extent that decreases away from the source according to a power law.