

HAMILTON'S ECCENTRICITY VECTOR
GENERALISED TO NEWTON WONDERS

By D. Lynden-Bell
Institute of Astronomy

The vectorial velocity is given as a function of the position of a particle in orbit when a Newtonian central force is supplemented by an inverse cubic force as in Newton's theorem on revolving orbits.

Such expressions are useful in fitting orbits to radial velocities of orbital streams. The Hamilton-Laplace-Runge-Lenz eccentricity vector is generalised to give a constant of the motion for these systems and an approximate constant for orbits in general central potentials. A related vector is found for Hooke's centred ellipse.

Introduction

In a central orbit a particle with position $(r, \tilde{\phi})$ has $r^2 d\tilde{\phi}/dt = \tilde{h}$ constant. Newton¹ pointed out that if $\phi = \tilde{\phi}/n$, with n any constant, then $r^2 d\phi/dt = h$ is also constant. He then enquired what extra radial force would be needed to make a new orbit with ϕ replacing $\tilde{\phi}$ but with the $r(t)$ unchanged. Evidently $h = \tilde{h}/n$ so the old radial equation of motion $d^2r/dt^2 - \tilde{h}^2 r^{-3} = \tilde{F}$ will change to $d^2r/dt^2 - h^2 r^{-3} = F$.

Thus the extra radial force required is $F - \tilde{F} = -(n^{-2} - 1)\tilde{h}^2/r^3$ which is outward or inward according as n is greater or less than one. If the new motion were observed from axes that rotate (non-uniformly) at the rate $\Omega = (n^{-1} - 1)\dot{\tilde{\phi}}$ then one would see the particle perform the original orbit unchanged. Thus from fixed axes the whole orbit may be thought of as revolving with angular velocity $(n^{-1} - 1)\dot{\tilde{\phi}} = (1 - n)\dot{\phi}$. This is Newton's theorem on revolving orbits. Newton used it to demonstrate the accuracy of the inverse square law in the solar system; notice however that the theorem holds for orbits subject

to any central force \tilde{F} . Chandrasekhar² gives an elegant discussion but see Lynden-Bell & Lynden-Bell³ for the true shapes of Newton's orbits in uniformly rotating axes.

Hereafter we specialise to an inverse square law supplemented by an inverse cubic force so the potentials considered take the form

$$\psi = \mu r^{-1} + \frac{1}{2} K r^{-2}$$

where μ may be thought of as GM and K is a constant which in galactic orbits is normally negative though in Hartree Fock atoms it is positive. The equation of motion in fixed axes is

$$d^2\mathbf{r}/dt^2 = -(\mu r^{-2} + K r^{-3})\hat{\mathbf{r}}$$

where hats denote unit vectors.

Comparison with Newton's theorem on revolving orbits suggests that we think of the angular momentum \mathbf{h} as $n^{-1}\tilde{\mathbf{h}}$ and K as $(n^{-2} - 1)\tilde{h}^2 = (1 - n^2)h^2$. If we then go to axes rotating non-uniformly with angular velocity $(n^{-1} - 1)\tilde{\mathbf{h}}/r^2$, the orbital equations relative to the rotating axes must reduce to those under the action of the inverse square law alone. In practice it is h and K that are known, so in terms of those $n^2 = 1 - Kh^{-2}$. Should K be greater than h^2 we would be in trouble, but under such circumstances the inverse cube attraction overcomes the centrifugal repulsion so orbits spiral into the origin. Otherwise in the non-uniformly rotating axes with $\boldsymbol{\Omega} = (1 - \sqrt{1 - Kh^{-2}})\mathbf{h}/r^2$ we recover writing a dot for time derivatives relative to the rotating axes

$$\ddot{\mathbf{r}} = -\mu r^{-2}\hat{\mathbf{r}}. \tag{1}$$

We have indeed laboriously checked that with this $\boldsymbol{\Omega}$ the $2\boldsymbol{\Omega} \times \dot{\mathbf{r}} + \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$ terms cancel out with the $-K\hat{\mathbf{r}}/r^3$ force term.

The orbit seen in the rotating axes

From (1)

$$\mathbf{r} \times \dot{\mathbf{r}} = \tilde{\mathbf{h}}$$

a constant, and

$$\ddot{\mathbf{r}} \times \tilde{\mathbf{h}} = -\mu \hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \frac{\dot{\mathbf{r}}}{r} \right) = \mu \dot{\mathbf{r}} ,$$

thus

$$\dot{\mathbf{r}} \times \tilde{\mathbf{h}} = \mu(\hat{\mathbf{r}} + \tilde{\mathbf{e}}) , \quad (2)$$

where $\tilde{\mathbf{e}}$ is a constant vector fixed in these rotating axes. Since both the other terms are perpendicular to $\tilde{\mathbf{h}}$, $\tilde{\mathbf{e}}$ must be too, so it lies in the plane of the orbit. Dotting with $\hat{\mathbf{r}}/\mu$ we find setting $l = \tilde{h}^2/\mu$,

$$l/r = (1 + \tilde{\mathbf{e}} \cdot \hat{\mathbf{r}}) = 1 + \tilde{e} \cos \tilde{\phi} . \quad (3)$$

This is the equation of a conic of eccentricity \tilde{e} and semi-latus-rectum l and $\tilde{\phi}$ is the angle in the rotating axes measured from perihelion. Thus $\tilde{\mathbf{e}}$ points toward perihelion in the rotating axes and it will inevitably rotate relative to fixed axes. Since the axes rotate at the rate $(n^{-1} - 1)\tilde{h}/r^2 = (1 - n)\dot{\phi}$ we find

$$\tilde{\mathbf{e}} = \mathbf{e} \cos[(1 - n)\phi] + \hat{\mathbf{h}} \times \mathbf{e} \sin[(1 - n)\phi] \quad (4)$$

where \mathbf{e} is in the fixed absolute direction to the perihelion at which $\phi = 0$ and its magnitude is the eccentricity $|\tilde{\mathbf{e}}|$. For fixed axes this comes from Hamilton⁴. For the contributions of Laplace, Runge & Lenz see Goldstein⁵.

The Orbit seen from fixed axes

The equation of the orbit (3) is put into fixed axes by writing $\tilde{\phi} = n\phi$, $l = n^2 h^2/\mu$

$$l/r = 1 + e \cos n\phi . \quad (5)$$

The velocity in fixed axes will be

$$\mathbf{v} = \dot{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r}$$

where from (2)

$$\dot{\mathbf{r}} = \frac{\mu}{nh} \hat{\mathbf{h}} \times (\hat{\mathbf{r}} + \tilde{\mathbf{e}}) .$$

Thus using (4) for $\tilde{\mathbf{e}}$ and $\boldsymbol{\Omega} = (1-n)\dot{\phi}\hat{\mathbf{h}} = (1-n)\mathbf{h}/r^2$ we find our expression for \mathbf{v}

$$\mathbf{v} = \frac{\mu}{nh} \left\{ \hat{\mathbf{h}} \times \hat{\mathbf{r}} \left[1 + \frac{l}{r}(n^{-1} - 1) \right] + \hat{\mathbf{h}} \times \mathbf{e} \cos[(1-n)\phi] - \mathbf{e} \sin[(1-n)\phi] \right\} \quad (6)$$

or using (5) to eliminate l/r in favour of ϕ

$$\mathbf{v} = \frac{\mu}{nh} \left\{ \hat{\mathbf{h}} \times \hat{\mathbf{r}} \left[n^{-1} + (n^{-1} - 1)e \cos(n\phi) \right] + \hat{\mathbf{h}} \times \mathbf{e} \cos[(1-n)\phi] - \mathbf{e} \sin[(1-n)\phi] \right\} . \quad (7)$$

To get the radial velocity along any line of sight $\hat{\mathbf{l}}$ one merely uses $\hat{\mathbf{l}} \cdot \mathbf{v}$ and then corrects for the Sun's motion.

The eccentricity vector constant of the motion

Equations (6) or (7) give us the velocity in terms of a conserved constant vector \mathbf{e} pointing toward the initial perihelion. To get \mathbf{e} itself we need to invert this equation so \mathbf{e} is expressed as a function of \mathbf{v} etc. To do this we note that the final two terms in the $\{\}$ in (6) yield \mathbf{e} if we first cross multiply them by $\times \hat{\mathbf{h}} \cos[(1-n)\phi]$ and add the result to the same two terms $\times (-\sin[(1-n)\phi])$. But from (6) those final two terms are equal to

$$\frac{nh}{\mu} \mathbf{v} + \hat{\mathbf{r}} \times \hat{\mathbf{h}} \left[1 + \frac{l}{r}(n^{-1} - 1) \right]$$

hence both e^2 is the square of this, which yields $e^2 = 1 + 2l\mu^{-1}\varepsilon$ where $\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} - \frac{K}{2r^2}$ and

$$\mathbf{e} = \left\{ \frac{nh}{\mu} \mathbf{v} \times \hat{\mathbf{h}} - \hat{\mathbf{r}} \left[1 + \frac{l}{r}(n^{-1} - 1) \right] \right\} \cos[(1-n)\phi] - \left\{ \frac{nh}{\mu} \mathbf{v} + \hat{\mathbf{r}} \times \hat{\mathbf{h}} \left[1 + \frac{l}{r}(n^{-1} - 1) \right] \right\} \sin[(1-n)\phi] \quad (8)$$

which gives the new vector constant of the motion. We remind the reader that $\phi = \cos^{-1}(\hat{\mathbf{e}} \cdot \hat{\mathbf{r}})$ and $n = \sqrt{1 - Kh^{-2}}$. We may eliminate l/r in terms of ϕ and e by using (5). Notice that (8) is actually an implicit equation because ϕ is not known until $\hat{\mathbf{e}}$ is known. However by taking (x, y) coordinates in the plane of the motion and measuring a new ϕ' from the x axis we may write $\phi = \phi' - \phi_0$ where ϕ_0 is the azimuth of $\hat{\mathbf{e}}$. With ϕ' \mathbf{v} \mathbf{h} \mathbf{r} n l all known it is then possible to find a ϕ_0 from the components of (8)/ e .

So the implicit equation may be solved for $\hat{\mathbf{e}}$. There will be very many solutions unless we restrict ϕ_0 to be in the range $-\pi/n$ to $+\pi/n$ and ϕ' in the range $-\pi$ to $+\pi$. I gave a discussion earlier⁶ of the type of forces that leave the magnitude of the eccentricity unchanged but slew its direction.

Approximating orbits in other potentials

Suppose we are given a potential $\psi(r)$ and an orbit within it is defined by its pericentre at r_p and its apocentre at r_a . Then since \dot{r} vanishes at these points the angular momentum of the orbit is given by

$$h^2 = 2 \frac{\psi(r_p) - \psi(r_a)}{r_p^{-2} - r_a^{-2}}$$

and the energy of the orbit is given by

$$\varepsilon = \frac{h^2}{2r_p^2} - \psi(r_p) .$$

The angle between perihelion and aphelion is then writing $r = u^{-1}$

$$\Phi = \int_{r_a^{-1}}^{r_p^{-1}} \left\{ 2h^{-2} [\varepsilon + \psi(u^{-1})] - u^2 \right\}^{-1/2} du .$$

This may be compared with the angle given by the orbit (5) which is π/n .

Hence we may define the n of the approximating orbit by $n = \pi/\Phi$.

We shall make the angular momenta of the two orbits equal and define the eccentricity by

$$e = \frac{r_a - r_p}{r_a + r_p}$$

which is also in conformity with (5).

Evidently the K of our approximating potential is already known via n since $K = h^2(1 - n^2)$. To specify the approximating potential we still need μ which we fix by making the perihelion distances equal, which, via the eccentricity, implies the aphelion distances are equal and hence

$$\mu = (h^2 - K)(r_p^{-1} + r_a^{-1})/2 .$$

Thus we have an approximation scheme giving n , K , h , e , μ for any orbit specified by r_p and r_a in any known potential $\psi(r)$. We expect the \mathbf{e} defined by (8) to be approximately constant for such orbits.

The Eccentricity Vector for the Harmonic Oscillator

Newton⁷ explained the connection between the Keplerian ellipse and that generated by the two dimensional harmonic oscillator. Here we follow Chandrasekhar's preferred path². Set $x + iy = z$ in the plane of the orbit. The equation of the harmonic oscillator is then $\frac{d^2z}{dt^2} + \omega^2z = 0$ and its energy is $E = \frac{1}{2}(\frac{d\bar{z}}{dt}\frac{dz}{dt} + \omega^2\bar{z}z)$. Its angular momentum is $h = \frac{1}{2i}(\bar{z}\frac{dz}{dt} - \frac{d\bar{z}}{dt}z)$. Now consider the mapping of the complex plane $Z = z^2$. Following Newton we ask whether the mapped path considered with a new time $\tau(t)$ can be an orbit under a new central force. Evidently the angular momentum of the Z orbit is

$$\frac{1}{2i} \left(\bar{Z} \frac{dZ}{d\tau} - \frac{d\bar{Z}}{d\tau} Z \right) = \frac{1}{i} |z|^2 \left(\bar{z} \frac{dz}{d\tau} - \frac{d\bar{z}}{d\tau} z \right) = 2 \frac{dt}{d\tau} |z|^2 h$$

so if the angular momenta are to be equal then $\frac{d}{d\tau} = \frac{1}{2|z|^2} \frac{d}{dt}$. Chandrasekar (in error!) omits the factor 2. Now using the z equation of motion

$$\frac{d^2Z}{d\tau^2} = \frac{1}{4|z|^2} \frac{d}{dt} \left(\frac{2 dz}{\bar{z} dt} \right) = \frac{1}{2z\bar{z}^3} \left(\frac{d\bar{z}}{dt} \frac{dz}{dt} - \omega^2 \bar{z} z \right) = \frac{-\varepsilon}{z\bar{z}^3} = -\frac{\varepsilon Z}{|Z|^3}$$

but the final expression shows us Z is a motion under an inverse square law with a force constant $\mu = GM = \varepsilon$ where ε is the energy of the simple harmonic orbit. Thus under the mapping the simple harmonic centred ellipse becomes the Kepler eccentric ellipse. However the latter has a conserved Hamilton eccentricity vector so what does that vector become under the inverse transformation from Kepler's to Hooke's ellipse? In the notation of this section vectors in the plane of the motion are complex numbers. Since $d^2Z/d\tau^2 = -\mu Z/|Z|^3$ we follow our well trodden path and multiply by h and integrate to find

$$h dZ/d\tau = i\mu(Z/|Z| + e)$$

where e is the (complex) constant of integration. To transform this into the z plane we

write $d/d\tau = \frac{1}{2|z|^2} \frac{d}{dt}$, $Z = z^2$ and $\mu = \varepsilon$. So

$$-\left(\frac{ih}{\varepsilon} \frac{1}{\bar{z}} \frac{dz}{dt} + \frac{z}{\bar{z}}\right) = e. \quad (9)$$

I had to differentiate this extraordinary expression and show the result to be zero before I believed that it was indeed a constant of the motion! In the original Z space e pointed toward perihelion. After the transformation $Z = z^2$, e is still the same complex number so is unchanged but now the perihelion in z space will be at half the angle to the real axis. This suggests that we should be considering a new vector \tilde{e}_p pointing to perihelion and with the property that $e = \tilde{e}_p^2$. This \tilde{e}_p will then be a constant of the motion too and furthermore the \pm ambiguity in its definition reflects the fact that the Hooke ellipse has two perihelia in opposite directions. There is an intrinsic difficulty in the transformation that we have to place a cut in the complex Z plane to define \sqrt{Z} properly. If we place this cut arbitrarily then the direction of that cut intrudes into the resultant formulae. It is much more sensible to take the cut to be defined physically. Taking the cut along the real axis and that toward perihelion along e has advantages. Then both e and \tilde{e}_p are real. Rewriting our equation to give us the velocity we have

$$\frac{dz}{dt} = \frac{i\varepsilon}{h}(z + e\bar{z})$$

we may now rewrite this in vector form

$$\frac{d\mathbf{r}}{dt} = \frac{\varepsilon}{h} \left\{ \hat{\mathbf{h}} \times [\mathbf{r} + \tilde{\mathbf{e}}_p \cdot \mathbf{r} \tilde{\mathbf{e}}_p + \tilde{\mathbf{e}}_p \times (\tilde{\mathbf{e}}_p \times \mathbf{r})] \right\}$$

which gives the velocity in terms of the vector $\tilde{\mathbf{e}}_p$ that points to perihelion and the position vector \mathbf{r} in the orbital plane orthogonal to $\hat{\mathbf{h}}$. To find the magnitude of e and \tilde{e}_p we return to equation (9). The general solution to the harmonic equation is $z = pe^{i\omega t} + qe^{-i\omega t}$ where p and q are complex numbers. In terms of p and q , $h = \omega(p\bar{p} - q\bar{q})$ and $\varepsilon = \omega^2(p\bar{p} + q\bar{q})$. Putting these expressions into equation (9) we find

$$e = \frac{-2pq}{p\bar{p} + q\bar{q}}$$

choosing the real axis along e means that $p = Pe^{i\chi}$ and $q = -Qe^{-i\chi}$ with P and Q real and positive, then $z = (P - Q) \cos(\chi + \omega t) + i(P + Q) \sin(\chi + \omega t)$

$$e = \frac{2PQ}{P^2 + Q^2} = \frac{a^2 - b^2}{a^2 + b^2} = \frac{e_\star^2}{2 - e_\star^2},$$

where a , b are the semi axes and e_\star is the ‘eccentricity’ of the centred Hooke ellipse.

Thus the magnitude of \tilde{e}_p is given by

$$\tilde{e}_p = \frac{e_\star}{\sqrt{2 - e_\star^2}}.$$

Of course \tilde{e}_p could be generalised to cases in which the linear Hooke law is supplemented by an inverse cube repulsion, by following the method given in this paper.

References

- (1) I. Newton, *Principia* (Royal Society, London) Proposition 44, 1687.
- (2) S. Chandrasekhar, *Newton's Principia for the Common Reader*, (Clarendon Press, Oxford), pp. 184–191 & p. 123, 1995.
- (3) D. Lynden-Bell & R. M. Lynden-Bell, *Notes & Recds. R. Soc.* **51**, 195, 1997.
- (4) W. R. Hamilton, *Proc. R. Irish Acad.*, **III**, Appendix p. 36, 1845.
- (5) H. Goldstein, *Am. J. Phys.* **44**, 1123, 1976.
- (6) D. Lynden-Bell, *The Observatory*, **120**, No. **1155**, 131, 2000.
- (7) I. Newton, *Principia* (Royal Society, London) 2nd Edition 1713 Proposition VII Corollary III, Proposition XI The same otherwise. See (2) pp. 96 & 119