ON INSTABILITY OF PARALLEL FLOW OF INVISCID FLUID IN A ROTATING SYSTEM WITH VARIABLE CORIOLIS PARAMETER*

By Louis N. Howard and Philip G. Drazin

1. Introduction. In this paper we consider the stability of basic steady parallel flows of incompressible inviscid homogeneous fluid, with velocity $w_*(y_*)\mathbf{i}$ varying with the (northwards) cartesian coordinate y_*, where $\mathbf{i}$ is the unit vector in the (eastwards) x_*-direction. (We use asterisks as subscripts to denote dimensional quantities, but shall soon drop them to use dimensionless quantities.) We suppose each flow is bounded by two rigid parallel planes, $y_* = y_{*1}, y_{*2}$, either of which may be at infinity. The whole system rotates with variable Coriolis parameter, governed by the β-plane approximation of dynamic oceanography and meteorology. This is an approximation to the primary effects of the roundness and angular speed Ω of the earth, whereby the curvature $1/R$ of the earth is neglected but the variation of the Coriolis parameter $f = 2\Omega \sin \lambda$ with latitude λ is retained (Rossby 1939, Haurwitz 1940, cf. Phillips 1963) Then it follows (Kuo 1949) by the usual methods of linear hydrodynamic stability that the vorticity-perturbation equation is

$$(w - c)(D^2\phi - \alpha^2\phi) - \{D^2w - \beta(y)\}\phi = 0, \tag{1.1}$$

when it has been assumed that the dimensionless stream function of the two-dimensional perturbation is of the form

$$\psi' = \phi(y) \exp\{i\alpha(x - ct)\}$$

for positive wave-number α and complex velocity $c = c_r + ic_i$, where D denotes differentiation with respect to $y \equiv y_*/L \equiv \lambda R/L$; where $\beta_*(y_*) \equiv 2\Omega \cos \lambda/R \equiv VL^{-2}\beta(y)$ is the derivative of the Coriolis parameter f with respect to y_*; and where the variables have been made dimensionless in the usual way with some velocity scale V of $w_*(y_*)$ and some length scale L of the variations of $w_*(y_*)$. Note that when $\beta = 0$ equation (1.1) becomes the Rayleigh stability equation, i.e. the inviscid form of the Orr-Sommerfeld equation.

In future we shall assume that $\beta(y)$ is constant, though many of our results have simple extensions for more general functions $\beta(y)$. These extensions are of little physical value because the β-plane approximation itself is no more accurate than a linear approximation to f in mid-latitudes (cf. Phillips 1963).

At a rigid plane the normal velocity of the fluid is zero. This can be shown to imply that

$$\phi = 0 \qquad (y = y_1, y_2). \tag{1.2}$$

If the flow extends to infinity, then, by physical requirements, ϕ must be bounded there. (This may be more rigorously deduced from the associated

* This work was supported in part by the Office of Naval Research.

initial-value problem.) This condition can be made more specific for basic flows such that $w(y)$ tends smoothly to a limit, w_∞ say, as $y \to \infty$. For such unbounded flows, which include all those of physical interest, two independent solutions of the Kuo equation (1.1) are asymptotically $\exp(\pm \alpha l_+ y)$ as $y \to \infty$ for fixed $c \neq w_\infty$, where

$$l_+ \equiv +\{1 + \beta/\alpha^2(c - w_\infty)\}^{\frac{1}{2}} \tag{1.3}$$

is a root with non-negative real part. In order that ϕ is not exponentially increasing at infinity, and therefore is not unbounded,

$$\phi \sim \exp(-\alpha l_+ y) \qquad \text{as} \qquad y \to \infty,$$

when l_+ has non-zero real part. Thus boundedness of ϕ in general implies that ϕ tends to zero at infinity, and we may use condition (1.2) at an infinite as well as a finite boundary.

However, if l_+ is pure imaginary, we must specify its branch further. In this case both independent solutions ϕ are bounded at infinity, where they oscillate in y without damping or amplification. To find the correct form of ϕ at infinity we must use the initial-value problem whereby a bounded disturbance moves out towards infinity, or equivalently use a Sommerfeld radiation condition of outward flow of energy. This implies that the y-component of the group velocity is outward. Thus $\phi \sim \exp(i\alpha k_+ y)$ as $y \to \infty$, where $k_+ \equiv \{\beta/\alpha^2(w_\infty - c) - 1\}^{\frac{1}{2}}$ is the root such that $d(\alpha c)/d(\alpha k_+) > 0$. Note that c is always real in this case, which is consequently of little importance in seeking instability, because the disturbance oscillates like $\exp(-i\alpha ct)$ in time without amplification.

Thus, in general, stability is governed by equation (1.1) and boundary conditions (1.2). This is the eigenvalue problem posed by Kuo in 1949. It determines the eigenvalue $c(\alpha, \beta)$ and hence instability or stability according as c_i is positive or non-positive respectively Because the problem is invariant under complex conjugation, a damped wave with $c_i < 0$ implies the existence of an amplified conjugate wave with $c_i > 0$ (and *vice versa*). Thus stability of all disturbances of given wave-number corresponds to real c and instability to complex c.

We add further boundary conditions to allow for a discontinuity of w or Dw at any point, y_0 say. Continuity of pressure of the fluid at the material interface with mean position $y = y_0$ implies that

$$[(w - c)D\phi - (Dw)\phi] = 0, \tag{1.4}$$

where square brackets denote the 'jump' of their contents at $y = y_0$. Similarly continuity of normal velocity at the material interface implies that

$$[\phi/(w - c)] = 0. \tag{1.5}$$

In the next section we present solutions of the Kuo stability equation and the above boundary conditions for two important basic flows—those of no motion and of a vortex sheet. In §3 we give some general stability characteristics. In §4 a method of solution of the eigenvalue problem for long-wave disturbances of unbounded flows is derived. The results of §§3–4 are used with some further exact solutions in §5 for several basic flows Finally the mechanics of instability, and its occurrence as barotropic instability of the oceans and atmosphere, are discussed

2. Rossby waves and instability of a vortex sheet. (*a*) *No basic flow.* For a basic state of rest with

$$w = 0 \qquad (-\infty < y < \infty), \quad (2.1)$$

the Kuo stability equation (1.1) gives

$$D^2\phi = (\alpha^2 + \beta/c)\phi.$$

The only admissible solution which is bounded and for which energy is not radiated inwards at $y = \pm\infty$ is

$$\phi = \text{constant}, \qquad c = -\beta/\alpha^2 \equiv -a, \text{ say.} \quad (2.2)$$

This represents a *Rossby wave* with phase velocity a in the negative x-direction (Rossby 1939) Rossby waves are dispersive with group velocity $d(\alpha c)/d\alpha = -c = a$ Thus energy is propagated with speed a eastwards, in the opposite direction to the phase velocity.

(*b*) *Vortex sheet* For the basic velocity profile

$$w = \operatorname{sgn} y \qquad (-\infty < y < \infty), \quad (2.3)$$

we can solve the stability equation (1.1) in each of the regions $y < 0$, $y > 0$, and join the solutions up by use of conditions (1.4), (1.5) at $y = 0$. Thus we find the eigenfunction

$$\phi = \begin{Bmatrix} (c-1)\exp(-\{1 + a/(c-1)\}^{\frac{1}{2}}\alpha y)(y > 0) \\ (c+1)\exp(\{1 + a/(c+1)\}^{\frac{1}{2}}\alpha y)(y < 0) \end{Bmatrix}, \quad (2.4)$$

and the eigenvalue relation

$$(c-1)^2\{1 + a/(c-1)\}^{\frac{1}{2}} + (c+1)^2\{1 + a/(c+1)\}^{\frac{1}{2}} = 0 \quad (2.5)$$

for the vortex sheet (2.3). The square-roots are chosen to have non-negative real parts in order that the eigenfunction (2.4) is bounded at infinity; if a square-root is pure imaginary its sign must be such that there is outward radiation at infinity, as discussed in §1. Remembering this choice of branches of the square-roots, we square up relation (2.5) to get

$$f(c) \equiv c(c^2 + 1) + \tfrac{1}{4}a(3c^2 + 1) = 0. \quad (2.6)$$

(This cubic does not give the Rossby wave $c = -a$, $a = \infty$, which appears as an isolated root of relation (2 5) in the limit as $V \to 0$ for fixed $a_* = \beta_*/\alpha_*^2$).

The discriminant of the cubic $f(c)$ is

$$\Delta \equiv (1/27)(1 - 9a^2/16 + 27a^4/256) > 0$$

for all real a. Therefore two roots c of equation (2.6) are complex conjugate, and the third is real. This third root is in general inadmissible, it being a root of the unsquared relation (2.5) only when $c = \pm 1$, $a = \mp 2$; these roots represent Rossby waves relative to the main streams $w = \mp 1$ at $y = \mp\infty$, with $\phi = 0$ where $\pm y > 0$ respectively. Except for these special cases, disturbances to the vortex sheet are unstable, with conjugate eigenvalues

$$c = -\tfrac{1}{4}a - \omega(a^3/64 + \Delta^{\frac{1}{2}})^{\frac{1}{3}} - \omega^2(a^3/64 - \Delta^{\frac{1}{2}})^{\frac{1}{3}},$$
$$\text{or} \quad -\tfrac{1}{4}a - \omega^2(a^3/64 + \Delta^{\frac{1}{2}})^{\frac{1}{3}} - \omega(a^3/64 - \Delta^{\frac{1}{2}})^{\frac{1}{3}}; \tag{2.7}$$

where these cube- and square-roots are positive, and ω is a complex cube-root of one.

As $|a|$ increases from zero, the growth rate of the instability decreases, but the flow remains unstable however large $|a|$ becomes, with

$$c \to \pm 3^{-\frac{1}{2}} i \quad \text{as} \quad a \to \pm\infty.$$

Also, in the limit as $a \to 0$, $c = \pm i$, the eigenvalues for Helmholtz instability of a vortex sheet with $\beta = 0$.

It can be seen that c_r is an odd and c_i an even function of a. There is not exchange of stabilities, i.e. $c_r \neq 0$ when $c_i \neq 0$, because the Coriolis parameter introduces time asymmetry into this periodic perturbation of the steady basic flow of inviscid fluid.

3. General stability characteristics. Kuo (1949) noted that his stability equation (1.1) was merely the Rayleigh stability equation with basic absolute vorticity gradient $(D^2w - \beta)$ instead of the vorticity gradient D^2w relative to the earth. Accordingly, he generalised Rayleigh's theorem, deducing that a necessary condition for instability is that there is some point y_s in the field of flow where

$$D^2w = \beta. \tag{3.1}$$

Kuo (1949, p. 112) claimed further that if $(D^2w - \beta)$ changes sign at y_s then the condition is also sufficient, but this claim seems unsubstantiated. His argument sought to prove the existence of a neutral eigensolution $c = w_s \equiv w(y_s)$, $\alpha = \alpha_s \neq 0$, $\phi = \phi_s$ by Sturm-Liouville theory; and then to prove that neighbouring eigensolutions are unstable as $\alpha \to \alpha_s$ from above or from below.

The incompleteness of the first part of Kuo's argument can be shown by the following counter-example When

$$w = \sin y \qquad (y_1 \leqq y \leqq y_2), \tag{3.2}$$

$D^2w = -w$ and $w_s = -\beta$. Thence the eigensolution can be shown to be

$$c = w_s = -\beta, \qquad \alpha = \alpha_s = \{1 - n^2\pi^2/(y_2 - y_1)^2\}^{\frac{1}{2}},$$
$$\phi = \phi_s = \sin\{n\pi(y - y_1)/(y_2 - y_1)\}$$

for each integer n from one to the integral part of $(y_2 - y_1)/\pi$. However, if $(y_2 - y_1) < \pi$, then there is no neutral eigensolution with $c = -\beta$, and the flow can be shown to be stable, as it is when $\beta = 0$; this situation may occur whenever β is such that $w = -\beta$ in the domain ($y_r \leqq y \leqq y_2$) of flow, and it is a counter-example to Kuo's claim. However his proof seems sound elsewhere, and his claim of sufficiency is no doubt true for many commonly-used profiles of velocity.

The second part of Kuo's argument was to show that unstable eigensolutions exist near the neutral one discussed above. The argument essentially gives

$$(dc/d\alpha^2)_{\alpha\to\alpha_s} = -\left\{\int_{y_1}^{y_2} \phi_s^2\, dy\right\} \Big/ \left\{\mathcal{P}\int_{y_1}^{y_2} (D^2w - \beta)(w - w_s)^{-2}\phi_s^2\, dy + i\pi(D^3w\phi_s^2/(Dw)^2)_{y=y_s}\right\}, \tag{3.3}$$

on the assumption that $c_i > 0$ as $\alpha \to \alpha_s$ from one side, as in the method of Lin (cf. Lin 1955) for the case $\beta = 0$.

Further it can be proved by use of the variational principle of Sturm-Liouville theory that there is (neutral) stability for all $\alpha > \alpha_s$ for given β and w such that $(D^2w - \beta)/(w - w_s) \leqq 0$ everywhere in the field of flow. Unfortunately the condition $(D^2w - \beta)/(w - w_s) \leqq 0$ needed for application of classical Sturm-Liouville theory is satisfied by few velocity profiles for $\beta \neq 0$. This limits the practical value of some of the above results

The mean Reynolds stress over a wavelength is

$$\tau = \frac{\alpha}{2\pi}\int_0^{2\pi/\alpha} \rho u'v'\, dx = \frac{1}{4} i\alpha\rho(\phi D\phi^* - \phi^* D\phi)\exp(2\alpha c_i t), \tag{3.4}$$

where ρ is the density of the fluid, $u' = \partial\psi'/\partial y$ the longitudinal (eastwards) velocity perturbation, $v' = -\partial\psi'/\partial x$ the latitudinal (northwards) velocity, and asterisks denote complex conjugates. Then the stability equation (1.1) yields

$$D\tau = \tfrac{1}{2}\alpha\rho c_i(D^2w - \beta)|w - c|^{-2}|\phi|^2 \exp(2\alpha c_i t), \tag{3.5}$$

it being assumed that $D^2\phi$ exists everywhere. If $c_i = 0$ it follows that τ is constant where $w \neq c$. By taking the limit $c_i \to 0+$, it can be shown (cf. Lin 1955) that τ is discontinuous where $w = c$, having 'jump'

$$[\tau] = \tfrac{1}{2}\pi\alpha\rho\{(D^2w - \beta)|\phi|^2/Dw\}_{w=c}\,. \tag{3.6}$$

Now τ vanishes at the boundaries. Therefore the algebraic sum of the jumps $[\tau]$ is zero. In particular, if $w = c$ at only one point, the jump must be zero, i.e. $D^2w = \beta$ where $w = c$; and therefore w_s is the only possible value of c for a neutral eigensolution which is twice differentiable everywhere

Modification of the proof of the semi-circle theorem (Howard 1961) for equation (1.1) shows that instability ($c_i > 0$) implies that

$$c_i^2 + \{c_r - \tfrac{1}{2}(w_{\min} + w_{\max})\}^2 \leqq \{\tfrac{1}{2}(w_{\max} - w_{\min}) + \tfrac{1}{2}|a|\}^2, \tag{3.7}$$

i.e. it implies that c lies in the upper semi-circle with centre $\frac{1}{2}(w_{\min} + w_{\max})$ and radius $\frac{1}{2}\{w_{\max} - w_{\min} + |a|\}$ in the complex plane. It can further be shown that $c_r < w_{\max}$ when $a > 0$ and that $c_r > w_{\min}$ when $a < 0$. These modifications of Howard's semi-circle theorem are essentially due to Pedlosky (1963), who treated a two-layer model of the basic flow for $a \geqq 0$.

In general there is a continuous spectrum of real eigenvalues c in the range of w, as well as a discrete spectrum. For example if $w - c = D^2w - \beta = 0$ at any point of the flow, say y_0, then

$$D^2\phi - \alpha^2\phi - (D^2w - \beta)\phi/(w - c) = 0 \quad \text{or} \quad \delta(y - y_0)$$

for that value of c. The possibility of the Dirac δ-function leads to a continuous spectrum of stable eigenvalues, as will be illustrated in some examples of §5. Because these eigenvalues are not associated with instability, we shall go no further than refer the reader to the discussion by Case (1961) of the continuous spectrum when $\beta = 0$.

This mathematics illustrates the effects of the three basic mechanisms of the instability:

(1) inertial instability of the basic flow $w_*(y_*)\mathbf{i}$ relative to the rotating system whereby the balance of vorticity is upset;

(2) the addition of the vorticity gradient β_* due to the earth's rotation, it should be emphasised that β_* is just part of the absolute vorticity gradient, Rossby waves, for example, being no more than a form of neutrally-stable inertial oscillation in a non-rotating frame;

(3) the kinematic constraint of the boundaries, which by and large reduces instability.

The interplay of these mechanisms can be represented by dimensional analysis. The dimensional complex velocity must have functional form

$$c_* = c_*(\alpha_*, \beta_*, L, V)$$

for velocity profiles $w_*(y_*)$ corresponding to given $w(y)$ and varying scales L, V. By dimensional analysis we deduce

$$c = c(\alpha, a),$$

where the dimensionless Rossby wave-speed $a \equiv \beta_*/\alpha_*^2V$ etc.

Consider the limit $a \to 0$, which may be viewed as the limit $\beta_* \to 0$ for fixed V, α_*. Then we expect

$$c_* \to c_*(\alpha_*, 0, L, V) \equiv Vc(\alpha),$$

say, the eigenvalue for the flow in the absence of rotation of the system. In this limit mechanism (2) is absent, and the problem is reduced to a simpler and better known one.

Next let $a \to \infty$, which may be viewed as either the limit $\beta_* \to \infty$ for fixed V, α_* or the limit $V \to 0$ for fixed α_*, β_*. In this case we expect that c_* tends to its value for the limiting case $V = 0$. In this limit mechanism (1) is absent. When the flow is unbounded, mechanism (3) is also absent, and we expect that (one of the eigenvalues)

$$c \sim -a \quad \text{as} \quad a \to \infty,$$

because a Rossby wave is known to occur in the limiting case of example §2(a).

Now consider unbounded flows only. This removes mechanism (3) and allows us to let $L \to 0$ for fixed α_*, β_*, $V\, w(y)$ without the field of flow shrinking to zero. As $L \to 0$ for fixed α_*, $\alpha \to 0$, so an equivalent limit is $\alpha \to 0$ for fixed α_*. As $L \to 0$ for fixed y_*, $y \equiv y_*/L \to +\infty, 0$ or $-\infty$ according as $y_* >, =$ or < 0 respectively. Therefore, in the limit,

$$w_*(y_*) = \begin{cases} Vw(\infty) & (y_* > 0) \\ Vw(0) & (y_* = 0) \\ Vw(-\infty) & (y_* < 0) \end{cases}$$

It is here convenient to take $w(-\infty) = -w(\infty)$, which can be effected without loss of generality by a Galilean transformation if necessary. Then we classify each flow to be of *jet type* if $w(\infty) = 0$ or of *shear-layer type* if $w(\infty) \neq 0$, in which case we choose $V \equiv w_*(\infty)$. Therefore

$$w_* \to \begin{cases} 0 & (\text{jet}) \\ V \operatorname{sgn} y_* & (\text{shear layer}) \end{cases} \tag{3.8}$$

as $L \to 0$ for fixed y_*. (We neglect the special point $y_* = 0$, because translation of the origin cannot affect the stability characteristics, and we are assuming that all the limiting processes are physically reasonable.) For $w_* = 0$ we know the eigenvalue $c_* = -a_*$ from equation (2 2) for the Rossby wave. For $w_* = V \operatorname{sgn} y_*$, we know the eigenvalues from equation (2.7) for the vortex sheet We deduce that all flows of jet type have (at least one mode with) eigenvalue

$$c(\alpha, a) \to -a,$$

as $\alpha \to 0$ for fixed a, which gives stability. Similarly, flows of shear-layer type are unstable in the limit, having the same limiting eigenvalues as those (2.7) of a vortex sheet.

4. Long-wave approximation to the stability characteristics of unbounded flows. Taking the limit $L \to 0$ in our dimensional analysis pointed the need for a better approximation to the stability characteristics of unbounded flow as $\alpha \to 0$. Such a method has been developed by Drazin & Howard (1962) for the special case $\beta = 0$. To generalise that method for $\beta \neq 0$, first note that the Kuo stability equation (1.1) can be written

$$D(W^2DF) = \alpha^2W(W - a)F, \tag{4.1}$$

where $W \equiv w - c$, $F \equiv \phi/W$. Taking the limit as $\alpha \to 0$ for fixed a, $w(y)$, we deduce that F is constant except (possibly) at the critical latitude or latitudes y_c where $w = c$. The boundary condition (1.2) at infinity shows that $F = 0$ down from $y = \infty$ to the most northerly critical latitude $y = y_c$. On the other hand, if we take instead the limit $y \to \infty$, we find

$$F \sim \text{constant} \times \exp(-\alpha l_+ y),$$

as in §1. This asymptotic form is valid for all positive α, however small, provided $c \neq w_\infty$ Thus the order of the limits $\alpha \to 0$ and $y \to \infty$ cannot be changed without changing the resultant limit of the eigenfunction F. In spite of this non-uniformity, we can find an expansion of c in powers of α as follows.

Two independent solutions F_+, f_+ of the stability equation (4.1) can be defined by their asymptotic forms

$$\begin{aligned} F_+ &\sim \exp(-\alpha l_+ y), \qquad DF_+ \sim -\alpha l_+ \exp(-\alpha l_+ y), \\ f_+ &\sim \exp(\alpha l_+ y), \qquad Df_+ \sim \alpha l_+ \exp(\alpha l_+ y); \end{aligned} \tag{4.2$_+$}$$

provided that $\int^{\infty} |D^2 w/(w - c)|\, dy < \infty$ (cf. Coddington & Levinson, 1955). Therefore the eigenfunction satisfying both equation (4.1) and boundary condition (1 2) at $y = +\infty$ is

$$F = K_+ F_+ \tag{4.3$_+$}$$

for some complex constant K_+. Similarly we find

$$F = K_- F_-, \tag{4.3$_-$}$$

where

$$F_- \sim \exp(\alpha l_- y), DF_- \sim \alpha l_- \exp(\alpha l_- y) \text{ as } y \to -\infty \tag{4.2$_-$}$$

But eigenfunctions (4.3$_\pm$) are the same. Therefore their Wronskian vanishes at each and every point y, i e.

$$F_+ DF_- - F_- DF_+ = 0. \tag{4 4}$$

Equation (4.4) is the eigenvalue relation between a, c, α, because $F_\pm$ are definite functions specified by these parameters alone for given $w(y)$.

To expand eigenvalue relation (4.4) as a power series in α, we divide out the essential singularity of F_+ at $y = +\infty$ and expand the quotient as a series uniformly valid over a semiinfinite domain, $0 \leqq y < \infty$, say. Thus we put

$$\chi_+ \equiv F_+ \exp(\alpha l_+ y) \tag{4.5$_+$}$$

$$= \sum_{n=0}^{\infty} \alpha^n \chi_{+n}(y; a, c) \quad (0 \leqq y < \infty). \tag{4 6$_+$}$$

Stability equation (4.1) now becomes

$$D(W^2 D\chi_+) = \alpha l_+ \{D(W^2 \chi_+) + W^2 D\chi_+\} + \alpha^2 (1 - l_+^2) W(W - W_\infty)\chi_+ \tag{4.7$_+$}$$

Equating coefficients of successive powers of α, we now find

$$D(W^2 D\chi_{+0}) = 0, \qquad D(W^2 D\chi_{+1}) = l_+ \{D(W^2 \chi_{+0}) + W^2 D\chi_{+0}\},$$

$$D(W^2 D\chi_{+n}) = l_+ \{D(W^2 \chi_{+n-1}) + W^2 D\chi_{+n-1}\} + (1 - l_+^2) W(W - W_\infty)\chi_{+n-1}, \quad n = 2, 3, \cdots. \tag{4.8$_+$}$$

Boundary condition (4.2$_+$) at infinity gives

$$\chi_{+0} = 1, \quad D\chi_{+0} = 0, \quad \chi_{+n} = D\chi_{+n} = 0, \quad n = 1, 2, \cdots. \tag{4.9$_+$}$$

Whence it can be shown that

$$\chi_{+0} = 1 \tag{4.10$_{+0}$}$$

$$\chi_{+1} = l_+ \int_\infty^y dy (1 - W_\infty^2 / W_1^2), \tag{4.10$_{+1}$}$$

$$\chi_{+2} = \int_\infty^y dy_1 \int_\infty^y dy_2 \{l_+^2(W^2 - W_\infty^2)(W_1^{-2} + W_2^{-2}) + (1 - l_+^2)W_1^{-2}W_2(W_2 - W_\infty)\} \qquad (4\,10_{+2})$$

$$\chi_{+3} = \int_\infty^y dy_1 \int_\infty^{y_1} dy_2 \int_\infty^{y_2} dy_3 \{l_+^2(W_1^{-2} + W_2^{-2})(1 - W_\infty^2/W_3^2)(W_2^2 + W_3^2) + (1 - l_+^2)\{W_3(W_3 - W_\infty)(W_1^{-2} + W_2^{-2}) + W_1^{-2}W_2(W_2 - W_\infty)(1 - W^2/W_3^2)\}\}, \qquad (4.10_{+3})$$

etc., where subscripts 1, 2, ·· denote evaluations at y_1, y_2, ··· respectively.

There are similar expressions for

$$\chi_- \equiv F_- \exp(-\alpha l_- y) \qquad (4.5_-)$$

$$= \sum_{n=0}^\infty (-\alpha)^n \chi_{-n}(y, a, c), \qquad (4.6_-)$$

on replacing α by $(-\alpha)$, ∞ by $(-\infty)$, and l_+ by l_-.

We can now evaluate eigenvalue relation (4.4) at $y = 0$ as a power series in α. (We have chosen the point $y = 0$ merely as a convenience; this choice does not affect the function $c(\alpha, a)$ as an explicit expansion for small α.) Thus

$$[\{\chi_+(D\chi_- + \alpha l_- \chi_-) - \chi_-(D\chi_+ - \alpha l_+ \chi_+)\} \exp\{\alpha(l_- - l_+)y\}]_{y=0} = 0, \qquad (4.11)$$

and thence it can be shown that

$$\begin{aligned}
0 = {} & l_+ W_\infty^2 + l_- W_{-\infty}^2 + \alpha\Big\{l_+^2 \int_0^\infty (W^2 - W_\infty^2)\,dy + l_-^2 \int_{-\infty}^0 (W^2 - W_{-\infty}^2)\,dy \\
& - l_+ l_- W_{-\infty}^2 \int_0^\infty (1 - W_\infty^2/W^2)\,dy - l_+ l_- W_\infty^2 \int_{-\infty}^0 (1 - W_{-\infty}^2/W^2)\,dy \\
& + (1 - l_+^2) \int_0^\infty W(W - W_\infty)\,dy + (1 - l_-^2) \int_\infty^0 W(W - W_{-\infty})\,dy\Big\} \\
& + \alpha^2\Big\{-l_- \int_{-\infty}^0 (1 - W_{-\infty}^2/W^2)\,dy \\
& \qquad \cdot \int_0^\infty \{l_+^2(W^2 - W_\infty^2) + (1 - l_+^2)W(W - W_\infty)\}\,dy \\
& - l_+ \int_0^\infty (1 - W_\infty^2/W^2)\,dy \\
& \qquad \cdot \int_{-\infty}^0 \{l_-^2(W^2 - W_{-\infty}^2) + (1 - l_-^2)W(W - W_{-\infty})\}\,dy \\
& + \int_0^\infty dy \int_y^\infty dy_1\, l_+^2(W^2 - W_\infty^2)(W_1^{-2} + W^{-2})(l_- W_{-\infty}^2 - l_+ W^2) \\
& \qquad + (1 - l_+^2)\{(l - W_{-\infty}^2 - l_+ W^2)W^2 W_1(W_1 - W_\infty) \\
& \qquad - l_+ W(W - W_\infty)(1 - W_\infty^2/W_1^2)\} \\
& + \int_{-\infty}^0 dy \int_{-\infty}^y dy_1\, l_-^2(W_1^2 - W_{-\infty}^2)(W_1^{-2} + W^{-2})(l_+ W_\infty^2 - l_- W^2) \\
& \qquad + (1 - l_-^2)\{(l_+ W_\infty^2 - l_- W^2)W^2 W_1(W_1 - W_{-\infty}) \\
& \qquad - l_- W(W - W_{-\infty})(1 - W_{-\infty}^2/W^2)\}\Big\} + \cdots.
\end{aligned} \qquad (4.12)$$

For profiles of jet type, $W_{-\infty} = W_\infty = -c$, $l_- = l_+$, and this eigenvalue relation reduces to

$$0 = 2l_+c^2 + \left\{l_+^2 \int_{-\infty}^{\infty} (W^2 - c^2)^2 W^{-2}\, dy + (1 - l_+^2) \int_{-\infty}^{\infty} wW\, dy\right\} \alpha$$
$$- \alpha^2 \left\{l_+^2 \int_{-\infty}^{\infty} (W^2 - c^2)\, dy + (1 - l_+^2) \int_{-\infty}^{\infty} wW\, dy\right\} \qquad (4.12_J)$$
$$\cdot \int_{-\infty}^{\infty} (W^2 - c^2) W^{-2}\, dy + \cdots .$$

Some of the more rigorous work of Drazin & Howard (1962) on these series for small α in the case $\beta = 0$ can be readily generalised for the present case $\beta \neq 0$ (i.e. $a = \beta/\alpha^2 \neq 0$). For example, it can be shown that all the series used above converge absolutely for sufficiently small α and fixed $c_i \neq 0$ when $(w - w_{\pm\infty})$ tends to zero exponentially as $y \to \pm\infty$. Again, it can be shown that there is a mode with

$$c - w_m \sim e^{\pm 2\pi i/3} \left\{ \frac{\pi\alpha}{(-2D^2w)_m^{\frac{1}{2}}} \frac{l_+ l_- W_\infty^2 W_{-\infty}^2}{l_+ W_\infty^2 + l_- W_{-\infty}^2} \right\}^{\frac{1}{3}} \quad \text{as} \quad \alpha \to 0, \qquad (4.14)$$

where the subscript m denotes evaluation at any point y_m where w has a simple minimum (There is a similar result where w has a maximum.) In fact we shall see that eigenvalue relations (4.12) seem to give valid approximations to c except for modes of instability of shear layers in which $c \to w_{\pm\infty}$ as $\alpha \to 0$. In these cases the singularity of the stability equation (4.1) at $w = c$ tends to the singularity at $y = \pm\infty$ as $\alpha \to 0$.

Equation (4.12_J) has one root

$$c = -a - \frac{1}{4} \alpha^2 a^{-3} \left(\int_{-\infty}^{\infty} w(w + a)\, dy \right)^2$$
$$+ \cdots \left(a < -\int_{-\infty}^{\infty} w^2\, dy \Big/ \int_{-\infty}^{\infty} w\, dy \right). \qquad (4.13)$$

This represents a stable mode, for small α at any rate, which is a modified Rossby wave, as anticipated by the dimensional arguments of §3. Note that relation (4.13) gives relation (2.2) for a simple Rossby wave exactly when $w = 0$. (Similarly relation (4.12) reduces exactly to relation (2.5) for the vortex sheet when $w = \operatorname{sgn} y$.)

5. Further examples. (*a*) *Rectangular jet.* For the jet with basic velocity

$$w = \begin{cases} 0 & (|y| > 1) \\ 1 & (|y| < 1), \end{cases} \qquad (5.1)$$

we can find the eigenvalue relation by piecewise solution of the Kuo stability equation (1.1) and use of boundary conditions (1.2) at infinity and (1.4), (1.5) at $y = \pm 1$. The resulting relation has two factors, one, corresponding to a sinuous mode of instability (with even eigenfunction ϕ) gives

$$c^2(1 + a/c)^{\frac{1}{2}} + (1 - c)^2\{1 + a/(c - 1)\}^{\frac{1}{2}} \tanh\,(\alpha\{1 + a/(c - 1)\}^{\frac{1}{2}}) = 0; \quad (5.2)$$

the other, corresponding to a varicose mode (with odd eigenfunction) gives

$$c^2(1 + a/c)^{\frac{1}{2}} \tanh\,(\alpha\{1 + a/(c - 1)\}^{\frac{1}{2}}) + (1 - c)^2\{1 + a/(c - 1)\}^{\frac{1}{2}}) = 0. \quad (5.3)$$

The square-root $(1 + a/c)^{\frac{1}{2}}$ is to be taken with non-negative real part as usual. The root $c = 1 - a$ of relation (5.3) is inadmissible because in this case the eigenfunction ϕ is linear where $|y| < 1$ instead of exponential, as has been assumed in the derivation of relation (5.3).

When $a = 0$ we recover Rayleigh's classic results (cf. Rayleigh 1945) whereby the sinuous and varicose modes are each unstable for all α, however large. In the limit as $a \to \pm\infty$ for fixed $\alpha \neq 0$, $c = \frac{1}{2}(1 \pm i3^{-\frac{1}{2}})$ for both modes, which are therefore unstable.

In the limit as $\alpha \to 0$ for fixed $a \neq 0$, relation (5 2) for the sinuous mode gives two submodes. One is stable, representing a modified Rossby wave in the ambient fluid, with

$$c = -a - \alpha^2(1 + a)^2/a^3 \cdot \cdot \qquad (a < -1).$$

It should be noted that this is an admissible solution of (5.2) only when $a < -1$. These results can be verified from equation (4.13) for small α.

The other sinuous submode gives

$$c \sim \begin{cases} \frac{1}{2}(1 \pm i3^{\frac{1}{2}})\{-(1 - a)^2/a\}^{\frac{1}{3}}\alpha^{\frac{2}{3}} & (a < 0) \\ \pm i\alpha^{\frac{1}{2}} & (a = 0) \\ -\frac{1}{2}(1 \pm i3^{\frac{1}{2}})\{(1 - a)^2/a\}^{\frac{1}{3}}\alpha^{\frac{2}{3}} & (0 < a < 1) \\ \alpha^2 & (a = 1) \\ \{(a - 1)^2/a\}^{\frac{1}{3}}\alpha^{\frac{2}{3}} & (a > 1) \end{cases}$$

as $\alpha \to 0$. These results may be verified from equation (4.12_J) for small α It can be seen that this sinuous submode for small α is stable when $a \geqq 1$.

As $\alpha \to 0$ for fixed $a \neq 0$, relation (5.3) for the varicose mode gives

$$c = \begin{cases} 1 \pm i\alpha^{\frac{1}{2}}(1 + a)^{\frac{1}{4}} + \cdots & (a > -1) \\ 1 - \frac{1}{2}\alpha^{\frac{2}{3}}(1 \pm i3^{\frac{1}{2}}) + \cdots & (a = -1) \\ 1 + 2^{\frac{1}{2}}\alpha^{\frac{1}{2}}(1 \pm i)(-1 - a)^{\frac{1}{4}} + \cdots & (a < -1). \end{cases}$$

This can be verified from equation (4 12_J). It shows that the varicose mode is always unstable for sufficiently small α

As $\alpha \to \infty$ for fixed a, both relations (5.2), (5.3) give instability.

In summary, the varicose mode is unstable wherever we have investigated it. The sinuous submode like a Rossby wave seems stable where it exists. The other sinuous submode is unstable near the α^2-axis in the (α^2, β)-plane but stable near the β-axis, the stability boundary leaving the origin with tangent $\beta = \alpha^2$ and going off to infinity.

(*b*) *Bickley jet.* For the jet with profile

$$w = \operatorname{sech}^2 y \quad (-\infty < y < \infty) \qquad (5.4)$$

Lipps (1962) put $c = w_s$ and sought neutral parts of the eigensolution. He found part of the sinuous mode with

$$c = \tfrac{1}{6}\alpha^2, \quad \beta = \tfrac{1}{6}\alpha^2(4 - \alpha^2), \quad \phi = \operatorname{sech}^2 y, \tag{5.5}$$

and the varicose mode with

$$c = \tfrac{1}{6}(3 + \alpha^2), \quad \beta = \tfrac{1}{6}(1 - \alpha^2)(3 + \alpha^2), \quad \phi = \operatorname{sech} y \tanh y. \tag{5.6}$$

He used the perturbation (3.3) to show that these solutions are stability boundaries for $\beta > 0$. They are also stability boundaries for $\beta > -2$, as shown in

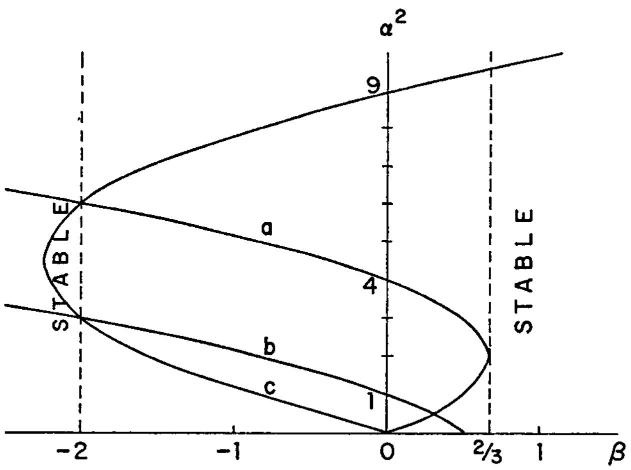

FIG 1. Stability characteristics of the jet $w = \operatorname{sech}^2 y$ (a) Stability boundary $\beta = \frac{1}{6}\alpha^2(4 - \alpha^2)$ of sinuous submode (b) Stability boundary $\beta = \frac{1}{6}(1 - \alpha^2)(3 + \alpha^2)$ of varicose mode (c) Neutral part $\beta = -\frac{1}{9}\alpha^2(9 - \alpha^2)$ of sinuous submode like Rossby wave

Figure 1. Note that condition (3.1) implies that the flow is stable when $\beta > \frac{2}{3}$ or $\beta < -2$.

We add a further neutral part of the sinuous mode, with

$$c = 1, \quad \beta = -\tfrac{1}{9}\alpha^2(9 - \alpha^2), \quad \phi = (\operatorname{sech} y)^{\alpha^2/3}(\tanh y)^{2-\alpha^2/3}. \tag{5.7}$$

This part, not having $c = w_s$, is not suitable for perturbation by formula (3.3). In fact it is part of the stable submode representing the modified Rossby wave, and is not a stability boundary in general

Formula (4.12_J) for small α gives

$$0 = 2c^2(1 + a/c)^{\frac{1}{2}} + \alpha\{\tfrac{4}{3} - 4c - c^2 - a(2 + cJ)\} - \alpha^2(\tfrac{4}{3} - 4c - 2a)J + \cdots, \tag{5.8}$$

where

$$J \equiv \int_{-\infty}^{\infty} (W^2 - c^2)W^{-2}\,dy$$
$$= (1 - c)^{-1} + \tfrac{1}{2}(3c - 2)(1 - c)^{-\frac{3}{2}} \log\left(\{(1 - c)^{\frac{1}{2}} + 1\}/\{(1 - c)^{\frac{1}{2}} - 1\}\right).$$

This gives the sinuous submode like a Rossby wave, with

$$c = -a - (a + \tfrac{2}{3})^2\alpha^2/a^3 - \cdots \qquad (a < -\tfrac{2}{3});$$

this submode is in agreement with the exact solution (5 7). Formula (5 8) also gives the varicose mode with

$$c - 1 \sim \begin{cases} e^{\pm 2\pi i/3}\{\tfrac{1}{4}\pi\alpha(1 + a)^{\frac{1}{2}}\}^{\frac{2}{3}} & (a > -1) \\ e^{\pm \pi i/3}\{\tfrac{1}{4}\pi\alpha(-1 - a)^{\frac{1}{2}}\}^{\frac{2}{3}} & (a < -1) \end{cases}$$

as $\alpha \to 0$, in agreement with (4.14). This gives instability of the varicose mode for small α and fixed a and is consistent with the exact result (5.6). Finally, relation (5.8) gives for the other sinuous submode

$$c \sim \begin{cases} \{\alpha^2(a - \tfrac{2}{3})^2/a\}^{\frac{1}{3}} & (a > \tfrac{2}{3}) \\ \tfrac{1}{2}(-1 \pm \sqrt{3}\, i)\{\alpha^2(\tfrac{2}{3} - a)^2/a\}^{\frac{1}{3}} & (\tfrac{2}{3} > a > 0) \\ \pm i(2\alpha/3)^{\frac{1}{2}} & (a = 0) \\ \tfrac{1}{2}(1 \pm \sqrt{3}i)\{\alpha^2(\tfrac{2}{3} - a)^2/(-a)\}^{\frac{1}{3}} & (a < 0) \end{cases}$$

as $\alpha \to 0$ for fixed a. This gives instability as $\alpha \to 0$ for $a > \frac{2}{3}$, in agreement with the known neutral part (5.5) of the sinuous mode. Further, (5.8) gives $c = O(\alpha^2)$ when $a = \frac{2}{3} + O(\alpha^2)$, though a further term in (5.8) is needed to give the eigensolution (5.5) numerically to order α^2.

In summary of the stability characteristics of the Bickley jet, we refer to Figure 1 and the analysis above Near $\alpha^2 = 0 = a$, the characteristics are qualitatively those of the rectangular jet (5.1). However all modes of the Bickley jet are stable for $\beta > \frac{2}{3}$ or $\beta < -2$ by condition (3.1). The varicose mode is more unstable than the sinuous mode to each long wave, though the sinuous mode is unstable to some waves for a larger range of β. The jet is more easily stabilized by positive than negative values of β. This means that eastward jets ($w > 0$) are more stable than westward jets ($w < 0$) in geophysics, because change of sign of β is *mathematically* equivalent to a change of sign of w, although β is always positive in practice Numerical work is needed to give the missing stability boundaries from $(-2, 6)$ and $(-2, 3)$ to the negative β-axis in the (β, α^2)-plane of Figure 1.

(*c*) *Piecewise-linear shear layer.* When

$$w = \begin{cases} \operatorname{sgn} y & (|y| > 1) \\ y & (|y| < 1) \end{cases} \tag{5.9}$$

the eigenvalue relation can readily be shown to be

$$\begin{vmatrix} [D\phi_1 + \{(c-1)^{-1} + \alpha l_+\}\phi_1]_{y=1} & [D\phi_2 + \{(c-1)^{-1} + \alpha l_+\}\phi_2]_{y=1} \\ [D\phi_1 + \{(c+1)^{-1} - \alpha l_-\}\phi_1]_{y=-1} & [D\phi_2 + \{(c+1)^{-1} - \alpha l_-\}\phi_2]_{y=-1} \end{vmatrix} = 0,$$

where ϕ_1, ϕ_2 are any two independent solutions of

$$(y - c)D^2\phi - \{\beta - \alpha^2(y - c)\}\phi = 0.$$

It can be seen that solutions of this equation are related to the confluent hypergeometric function, with

$$\phi_1, \phi_2 = e^{\pm\alpha y}\mathfrak{F}(\pm\beta/2\alpha, 0, \mp 2\alpha(y - c))$$

in the notation of Erdélyi *et al.* (1953, p. 250). It can thence be shown that the eigenvalue relation is

$$0 = (c-1)^2 l_+ + (c+1)^2 l_- + 2\alpha\{(c^2-1)l_+l_- - ac + \tfrac{1}{3} + c^2\} + \cdots$$

for small α and fixed $a \neq 0$. Equation (4.12) gives an equivalent result. In the case $\alpha = 0$, ϕ_1, ϕ_2 are related to a Bessel function, and it can be shown that, for small β,

$$0 = (c-1)^{\frac{3}{2}} + (c+1)^{\frac{3}{2}} + 2\beta^{\frac{1}{2}}\{c - (c^2-1)^{\frac{1}{2}}\} + \cdots .$$

These results show that for long waves the stability characteristics of this shear layer are similar to those of the vortex sheet §2(b), there being instability when either α or β is small enough. There is just one mode of instability for each of these two shear layers. However the shear layer (5.4) is sometimes stable, as it is well known to be for $\beta = 0$ and large enough α.

(*d*) *Hyperbolic-tangent shear layer.* When

$$w = \tanh y \qquad (-\infty < y < \infty) \tag{5.10}$$

we may put $c = w_s$ where $2w_s(1-w_s)^2 = -\beta$ to find three real values of y_s when $-4/3\sqrt{3} < \beta < 4/3\sqrt{3}$. These values are associated with the neutral eigensolution

$$\begin{aligned} \phi &= (1-\tanh y)^{\frac{1}{2}\{1\pm(1-\alpha^2)^{\frac{1}{2}}\}}(1+\tanh y)^{\frac{1}{2}\{1\mp(1-\alpha^2)^{\frac{1}{2}}\}}, \\ c &= \pm(1-\alpha^2)^{\frac{1}{2}},\ \beta = \mp 2\alpha^2(1-\alpha^2)^{\frac{1}{2}}\ (0 \leqq \alpha^2 \leqq 1). \end{aligned} \tag{5.11}$$

For each value of β there are only two proper values of c in this solution, the third corresponding to $\alpha^2 < 0$. The neutral curves in the (β, α^2)-plane are shown in Figure 2. It can be shown by the perturbation (3.3) that each curve is a stability boundary, the interior corresponding to instability.

Eigenvalue relation (4.12) for small α gives

$$0 = (1-c)^2 l_+ + (1+c)^2 l_- - 2\alpha\{1 + l_+l_-(1 + c\log\{(c+1)/(c-1)\})\} + \cdots . \tag{5.12}$$

This gives two complex conjugate roots c similar to those of the vortex sheet and shear layer (5.9); this mode is unstable for all values of α^2, β sufficiently small.

Also relation (5.12) suggests the existence of a mode in which $c \to \pm 1$ as $\alpha \to 0$; for example, we might expect $c \to 1$ by balance of the terms $(1+c)l_-$ and $2\alpha l_+l_-\ c\log(c-1)$ above. However, scrutiny of equation (4.12) shows that the next term (in α^2) of equation (5.12) has asymptotic form $\alpha^2 l_- k_1(1+ak_2)/(c-1)^2$ as $c \to 1$ for some numerical constants k_1, k_2 which can be found in principle. If this extra term is included in (5.12), balance of the most significant terms as $c \to 1$ gives

$$(1+c)^2 l_- \sim -\alpha^2 k_1 l_-(1+ak_2)/(c-1)^2.$$

Therefore

$$c - 1 \sim \tfrac{1}{2}\alpha\{-k_1(1+ak_2)\}^{\frac{1}{2}} \qquad \text{as } \alpha \to 0 \text{ for fixed } a.$$

There is stability if $k_1(1 + ak_2) < 0$ and instability if $k_1(1 + ak_2) > 0$. If, however, $a = -1/k_2$, the term of order α^2 in equation (5.12) is asymptotically $\alpha^2 l_- k_3/(c - 1)$ as $c \to 1$, where k_3 is another numerical constant Then

$$c - 1 \sim -\tfrac{1}{4}\alpha^2 k_3$$

as $\alpha \to 0$. The exact solution (5 11) suggests that $k_1 > 0$, $k_2 = \frac{1}{2}$, $k_3 = 2$. It should be noted that the results of this paragraph are conjectured. Later terms in equation (5.12) may be dominant, and we do not know which later terms, if any, are the most significant, because we have no proofs of convergence of the series in this case for which c tends to a value of $w(y)$ in the flow as $\alpha \to 0$.

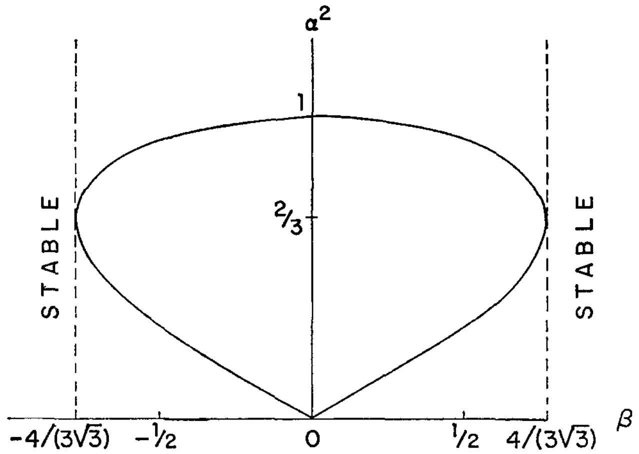

FIG 2 Stability characteristics of the shear layer $w = \tanh y$ Stability boundary $\beta = \mp 2\alpha^2(1 - \alpha^2)^{\frac{1}{2}}$

Similar remarks may be applied to the root for which $c \to -1$ as $\alpha \to 0$, by symmetry

Some of the above results may seem at first to be incompatible. The exact solution (5.11) indicates that there is a stability boundary touching $\beta = \mp 2\alpha^2$ near the origin in the (β, α^2)-plane. Expansion (5 12) for small α shows that there is also a mode, like that of a vortex sheet, which is unstable on and on both sides of this stability boundary. The convergence of our expansions may be proved in this case Therefore there must be at least two modes of instability, one like the instability of the vortex sheet and one like the exact solution (5 11), near the origin. An extra term of expansion (5 12) suggests the latter mode, but the expansion may diverge. This latter mode represents Rossby waves relative to the streams of $y = \mp\infty$ as $\alpha \to 0$, this mode does not exist for the vortex sheet or the broken-line shear layer (5.9) except as an isolated neutral mode with $c = \pm 1$, $a = \mp 2$ The stability boundaries of the former mode have yet to be found for the hyperbolic-tangent profile Nonetheless, condition (3 1) implies stability when $|\beta| > 4/3\sqrt{3}$.

(*e*) *Sinusoidal flow.* We finally give two examples of the continuous c-spectrum discussed in the middle of §3. Kuo (1949) has given one of the discrete eigensolutions for

$$w = \sin y \qquad (y_1 \leqq y \leqq y_2), \tag{5.13}$$

and we have mentioned them briefly in §3. However, when $c = -\beta$, $y_s = \sin^{-1}\beta$, it can easily be verified that an additional solution is

$$\phi = \begin{cases} \sin\{(1-\alpha^2)^{\frac{1}{2}}(y_2 - y_s)\} \sin\{(1-\alpha^2)^{\frac{1}{2}}(y - y_1)\} & (y_1 \leqq y < y_s) \\ \sin\{(1-\alpha^2)^{\frac{1}{2}}(y_s - y_1)\} \sin\{(1-\alpha^2)^{\frac{1}{2}}(y_2 - y)\} & (y_s < y \leqq y_2) \end{cases}$$

for each $\alpha^2 < 1$, and similar solutions can be found for $\alpha^2 = 1$, $\alpha^2 > 1$.

(*f*) *Plane Poiseuille flow.* When

$$w = 1 - y^2 \qquad (-1 \leqq y \leqq 1), \tag{5.14}$$

the stability equation becomes

$$(1 - c - y^2)(D^2\phi - \alpha^2\phi) - (\beta + 2)\phi = 0.$$

When $\beta = -2$ it follows that

$$D^2\phi - \alpha^2\phi = 0 \qquad \text{or } \delta(\pm(1-c)^{\frac{1}{2}}).$$

In the former case, of the discrete spectrum, there is no solution. In the latter case, of the continuous spectrum, an eigensolution can easily be found for each α and each c in the range [0, 1] of w.

6. Discussion. To apply the above results to motions of the atmosphere and oceans one must bear in mind the idealizations of the model of barotropic stability that we have used. For example the vertical structure (i.e. variation of the velocity, density and temperature with height) is not represented in the two-dimensional model. Again, the β-plane approximation is invalid for disturbances whose wavelengths are comparable to the radius of the earth. Notwithstanding these and other limitations of the model, the stability characteristics illustrate qualitatively some geophysical phenomena.

The absolute vorticity has components due to the basic flow relative to the earth and due to the rotation of the earth. It has been shown how the northerly vorticity gradient β due to the rotation of the earth displaces the latitude y_s where the absolute vorticity gradient vanishes, and thereby acts by and large as a stabilizing influence. In particular, a westerly jet has been found to be more easily stabilized than an easterly jet, which is in fact more unstable for small $\beta > 0$ than for $\beta = 0$. Thus barotropic instability should be considered as a cause of the infrequent occurrence of easterly jets in the atmosphere and oceans.

The semi-circle theorem (3.7) shows that unstable waves travel at a phase speed within the range $[w_{\min} - \frac{1}{2}a, w_{\max} + \frac{1}{2}a]$, a being the Rossby wave speed. In so far as one can approximate to the mean motion of the atmosphere by a steady basic flow, westerly winds are observed in mid-latitudes below the top of the stratosphere. Thus $w_{\min} > 0$ at each height, and the semi-circle theorem implies that no unstable disturbance moves in an easterly direction with speed greater than $\frac{1}{2}a$. This agrees, to order of magnitude, with observed speeds of propagation of large-scale wave-like disturbances on weather maps.

REFERENCES

CASE, K M , 1961, J Fluid Mech **10,** 420

CODDINGTON, E A , AND LEVINSON, N , 1955, Theory of Ordinary Differential Equations, New York, McGraw-Hill

DRAZIN, P. G , AND HOWARD, L N , 1962, J Fluid Mech **14,** 257

ERDELYI, A , MAGNUS, W , OBERHETTINGER, F , AND TRICOMI, F. G , 1953, Higher Transcendental Functions, Vol I, New York, McGraw-Hill.

HAURWITZ, B , 1940, J Marine Res **3,** 35

HOWARD, L N , 1961, J Fluid Mech **10,** 509

KUO, H L , 1949, J Met **6,** 105

LIN, C C , 1955, The Theory of Hydrodynamic Stability, Cambridge University Press

LIPPS, F B , 1962, J Fluid Mech **12,** 397

PEDLOSKY, J , 1963, Tellus **15,** 20

PHILLIPS, N A , 1963, Rev Geophys **1,** 123

RAYLEIGH, J W S , 1945, The Theory of Sound, Vol II, Chap XXI (reprint of 2nd ed of 1894), New York, Dover

ROSSBY, C G , 1939, J Marine Res **2,** 33

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
UNIVERSITY OF BRISTOL

(Received December 19, 1963)