BOUNDS ON FLOW QUANTITIES

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Introduction

Ideas from variational calculus are relevant and helpful in fluid mechanics in diverse ways. The connection is of course particularly close in ideal irrotational flow because of the manifold variational aspects of the Laplace equation, but it is familiar also, for example, in Stokes flow. One sometimes gets the impression, especially from reading general books about physics, that many people regard a variational formulation as an essential component of a true and deep understanding of the fundamental character of almost anything. This idealistic but rather narrow-minded attitude, a bit akin to the once-popular view that planetary orbits must obviously be circles, probably limits scientific progress somewhat, but when variational relationships *are* present or can be introduced it is prudent to be aware of the fact and look for ways in which they can be put to use. It is an interesting fact that the lowest frequency of a natural oscillation in a shallow lake is the minimum of a certain functional, as well as being the eigenvalue of a certain partial differential equation, but probably of greater importance to the applied mathematician is that this fact suggests valuable ways of estimating this frequency (by trial functions or comparison theorems) when the partial differential equation cannot readily be solved.

This estimation of the seiche period of a lake, or to cite another example, the estimates of the virtual mass of a solid moving in an ideal irrotational flow at rest at infinity, which can be based on the extremal properties of virtual mass (it is the minimum energy of incompressible flows and the maximum energy of irrotational flows satisfying the boundary conditions, and has certain monotonicity properties with respect to variation of the shape of the solid—see for instance Bergman & Shiffer 1953), are examples of the use of variational properties possessed by the mathematical model under consideration. Similar ideas can sometimes be usefully employed even when the model does not itself have such a variational character, by the introduction of other variational problems related to but not identical with the actual problem. This is the case, for instance, in the conditions for the stability of various steady flows that have been obtained by use of the *energy method*. There have been a number of important results of this kind in recent

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years, notably those of Joseph (1966), Serrin (1959), Velte (1962), and others. The basic idea of this approach to stability questions is indicated by the following: suppose we have a steady solution $\mathbf{V}$, P of (say) the incompressible Navier–Stokes equations

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla\varpi = R^{-1}\nabla^2\mathbf{v} \qquad \nabla\cdot\mathbf{v} = 0 \tag{1}$$

in some region R, satisfying certain boundary conditions, say $\mathbf{v}$ given on the boundary B of R. If $\mathbf{v}$, ϖ is *any* solution of the equations satisfying the boundary conditions, we write $\mathbf{v}=\mathbf{V}+\mathbf{u}$, $\varpi=P+p$ so that $\mathbf{u}$, p satisfy

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{V} + \mathbf{V}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \frac{1}{R}\nabla^2\mathbf{u} \qquad \nabla\cdot\mathbf{u} = 0 \tag{2}$$

and $\mathbf{u}$ vanishes on the boundary. From this we find the *perturbation energy equation*

$$\frac{1}{2}\frac{\partial}{\partial t}\langle|\mathbf{u}|^2\rangle = -\frac{1}{2}\langle\mathbf{u}\cdot D\cdot\mathbf{u}\rangle - \frac{1}{R}\langle|\nabla\mathbf{u}|^2\rangle \tag{3}$$

where $\langle\ \rangle$ denotes the integral over R (or in case of an unbounded region a suitable average over R, whose precise specification as well as that of boundary conditions at infinity need not concern us at the moment) and D is the deformation tensor of the basic flow $\mathbf{V}$: $D_{ij}=V_{i,j}+V_{j,i}$ ($|\nabla\mathbf{u}|^2 \equiv\Sigma|\nabla u_i|^2$). Assuming that enough is known about $\mathbf{V}$ that we can compute a lower bound $-\lambda$ for the most negative eigenvalue of D, we then have

$$\frac{\partial}{\partial t}\langle|\mathbf{u}|^2\rangle \le \lambda\langle|\mathbf{u}|^2\rangle - \frac{2}{R}\langle|\nabla\mathbf{u}|^2\rangle \tag{4}$$

We now consider the variational problem: find the minimum m of the functional $\mathfrak{F}=\langle|\nabla\mathbf{u}|^2\rangle/\langle|\mathbf{u}|^2\rangle$ among all $\mathbf{u}$ satisfying the boundary conditions (and possibly some other conditions such as $\nabla\cdot\mathbf{u}=0$, which the $\mathbf{u}$ of the flow problem is known to satisfy). If this problem has a positive solution m that we can calculate or adequately estimate we then obtain from (4)

$$\frac{\partial}{\partial t}\langle|\mathbf{u}|^2\rangle \le \left(\lambda - \frac{2m}{R}\right)\langle|\mathbf{u}|^2\rangle \tag{5}$$

from which we see in the usual way that $\langle|\mathbf{u}|^2\rangle$ must tend exponentially to zero if $R<2m/\lambda$. We thus obtain a condition insuring *stability* (in the sense of the decay toward zero of the deviation, as measured by the perturbation energy $\frac{1}{2}\langle|\mathbf{u}|^2\rangle$, of any solution $\mathbf{v}$ from the basic flow $\mathbf{V}$) of the flow $\mathbf{V}$—an estimate from below of its "critical Reynolds number." The variational problem for m is not necessarily easy, but it is anyway much more tractable than the initial-value problem for the Navier-Stokes equation, and has been successfully solved for some geometries. The above gives a condition for absolute stability—somewhat more precise results have been obtained for *linear* stability problems by making use of more detailed properties of the linear stability equations rather than simply the energy integral as above, but the general idea is similar (see for instance Joseph 1969).

This basic idea, to use variational results to estimate certain terms in one or more of the consequences (like the perturbation energy equation) of the underlying mathematical model, and thereby obtain some information about solutions to the real problem without explicitly knowing these solutions themselves, can be applied in other ways. In particular, it is sometimes possible to obtain estimates for flow quantities of physical interest (like momentum or heat transports) applicable also to turbulent flows. The present article is about some results of this sort.

Plane Couette Flow

Consider an incompressible viscous flow [satisfying Equation (1)] in the slab $|z| < 1$, with boundary conditions $\mathbf{v} = \mp\mathbf{i}$ on $z = \pm 1$. A *steady* flow of this kind is $\mathbf{v} = \mathbf{V} \equiv -z\mathbf{i}$, the plane Couette flow, whose pressure field may be taken to be zero. In general we suppose that there is no *overall* pressure gradient and that ϖ (as well as $\mathbf{v}$ of course) remain bounded at infinity. The linear stability theory indicates that $\mathbf{V}$ is always stable, but there is no doubt that this is not true when finite disturbances are present. Other motions with the same boundary conditions surely exist mathematically and occur in experiments approximately described by this model, if the Reynolds number R is large enough. There is some experimental evidence that a steady flow with a "cat's eye" pattern occurs for a certain range of Reynolds numbers, and the flow is turbulent at large enough values of R. We shall consider flows $\mathbf{v}$ for which horizontal averages (over planes z = constant) always exist and are independent of time; such averages will be denoted by an overbar. Probably all such flows have the property that $\bar{\mathbf{v}}$ is in the direction $\mathbf{i}$; in any case we consider only such, writing $\mathbf{v} = U(z)\mathbf{i} + \mathbf{u}$ and $\varpi = P(z) + p$, where $\bar{\mathbf{u}}$ and $\bar{p}$ are zero. Averaging the momentum equation and using $\nabla \cdot \mathbf{u} = 0$ to rewrite $\overline{\mathbf{u} \cdot \nabla \mathbf{u}}$ as the divergence of the Reynolds-stress tensor, we obtain the mean-flow equations

$$\begin{aligned}
(\overline{uw})_z &= \frac{1}{R} U''(z) \\
(\overline{vw})_z &= 0 \\
(\overline{ww})_z + P'(z) &= 0
\end{aligned}$$

(u, v, w are the components of $\mathbf{u}$).

Thus $U(z)$ can be explicitly obtained in terms of $\overline{uw}$: $U' - R\overline{uw}$ must be a constant whose value is obtainable by averaging in z and using the boundary conditions, namely

$$U'(z) = -1 + R(\overline{uw} - \langle uw \rangle) \tag{6}$$

where the average over the whole layer is denoted by $\langle\ \rangle$ ($\langle f \rangle = \frac{1}{2}\int_{-1}^{1} \bar{f}\, dz$); U

itself follows immediately from (6) by integrating from $z=-1$. The perturbation energy equation (3) becomes, in the present case, since horizontal averages are time independent,

$$-\langle \overline{uw}U' \rangle = \frac{\langle | \nabla \mathbf{u} |^2 \rangle}{R} \tag{7}$$

which in view of (6) can be written

$$\langle uw \rangle - R\langle (\overline{uw} - \langle uw \rangle)^2 \rangle = \frac{\langle | \nabla \mathbf{u} |^2 \rangle}{R} \tag{8}$$

The total mean rate of energy dissipation is proportional to

$$F \equiv \langle | \nabla \mathbf{v} |^2 \rangle = \langle | \nabla \mathbf{u} |^2 \rangle + \langle U'^2 \rangle \tag{9}$$

Since this power is put into the fluid by the mean stress at the boundaries, whose velocities are fixed, F is also proportional to this mean stress, or to the momentum transfer across the layer. In the case of the plane Couette flow, $\mathbf{u}=0$ and $U'=-1$, so that $F \equiv F_0 = 1$. Thus F may be regarded as a measure of energy dissipation, stress, or momentum transfer of the flow $\mathbf{v}$, compared with the corresponding values for plane Couette flow as a reference. Using (6), we may also express F entirely in terms of $\mathbf{u}$:

$$F = 1 + R^2\langle (\overline{uw} - \langle uw \rangle)^2 \rangle + \langle | \nabla \mathbf{u} |^2 \rangle \tag{10}$$

which, on using (8), can also be written

$$F = 1 + R\langle uw \rangle \tag{11}$$

It is clear from (10) that F cannot be less than F_0 and is equal to F_0 only when $\mathbf{u}=0$, which implies that $U=-z$ and we have the plane Couette flow—turbulence or any deviation from plane Couette flow always increases the friction. Of course, this fact that F is minimized by plane Couette flow ($\mathbf{V}$) among the set of all solutions $\mathbf{v}$ is actually a consequence also of the fact that $\mathbf{V}$ is not only a solution of the Navier–Stokes equations, but also of the Stokes equations $\nabla P = \nabla^2 \mathbf{V}/R$, $\nabla \cdot \mathbf{V} = 0$, and we know that $\langle |\nabla \mathbf{v}|^2 \rangle$ is minimized by the *Stokes* flow among *all* solenoidal fields $\mathbf{v}$ satisfying the boundary conditions.

It is clear from (8) that $\langle uw \rangle - \langle |\nabla \mathbf{u}|^2 \rangle / R$ is positive unless $\mathbf{u} \equiv 0$; however, if R is sufficiently small this quantity *cannot* be positive. This can be seen, for instance, from the facts that $|\langle uw \rangle| \leq \frac{1}{2}\langle |\mathbf{u}|^2 \rangle$ and

$$\langle | \nabla \mathbf{u} |^2 \rangle \geq \langle u_z^2 + v_z^2 + w_z^2 \rangle \geq (\pi/2)^2 \langle | \mathbf{u} |^2 \rangle$$

[since $\mathbf{u}$ vanishes at $z = \pm 1$ and $\int_{-1}^{1} \psi_z^2 dz / \int_{-1}^{1} \psi^2 dz$ is minimized among ψ's vanishing at ± 1 by $\psi = \sin \frac{1}{2}\pi(z+1)$]. Thus

$$\langle uw \rangle - \langle | \nabla \mathbf{u} |^2 \rangle / R \leq \tfrac{1}{2} \langle | \mathbf{u} |^2 \rangle (1 - \pi^2/2R),$$

which is ≤ 0 if $R \leq \pi^2/2$. Thus the plane Couette flow is the only Navier–Stokes flow with time-independent averages if R is small enough, certainly for $R \leq \pi^2/2$. This result can be strengthened a little—if R_c is the minimum of $\langle |\nabla \mathbf{u}|^2 \rangle$ among all $\mathbf{u}$'s that vanish on $|z| = 1$ and satisfy $\langle uw \rangle - 1 = \nabla \cdot \mathbf{u} = 0$, then plane Couette flow is the only possibility if $R \leq R_c$. In fact, R_c can be computed: the variational problem defining it is essentially equivalent to finding the minimum critical Rayleigh number for thermal convection in a layer of Boussinesq fluid heated from below, and $R_c \cong \frac{1}{2}(1707.77)^{1/2} \cong 20.663$. This fact is of course very closely related to the condition for absolute stability of $\mathbf{V}$ given by the energy method, which shows that $\mathbf{V}$ is stable if $R \leq R_c$.

Since plane Couette flow has the least energy dissipation among all solenoidal fields satisfying the boundary conditions and is in fact the motion that will occur when R is small, one might be tempted to consider as a sort of metaphysical principle the statement that "nature chooses that motion which minimizes energy dissipation." Such a statement, while true for $R < R_c$, could not always be true since plane Couette flow does not in fact occur if R is large. But even for small R such a statement is misleading, for one should not compare the flow that occurs with all solenoidal vector fields, but only with those motions that are *possible*, i.e. the real question is: "Among all solutions (with steady averages) of the Navier–Stokes equations, which one (or ones) actually occur under the given boundary conditions?" Since when $R \leq R_c$ there is actually only one competitor, the exactly opposite metaphysical principle "Nature chooses (from among the possibilities available) that motion which *maximizes* the energy dissipation" is equally true, Any selection principle at all will be "correct" when there is no choice. When R is large enough that solutions other than $\mathbf{V}$ exist, we do not at present know how nature decides—about the only thing we know for sure is that for large enough R the *minimum* energy dissipation principle is certainly wrong. If one feels it would be nice to have a general principle, hopefully of an extremal sort, for selecting out of the possibilities the flows that *do* take place, the hypothesis of maximum F is more promising—at least we do not know it is wrong! Actually it is almost certainly the case that such a selection principle does not exist at all. One indication of this is already apparent in the present case of the plane Couette boundary conditions. It seems to be a reasonable conjecture that for any finite R, $\mathbf{V}$ is stable not only to infinitesimal perturbations but also to finite perturbations whose amplitude is in some sense below a certain critical level—though the critical level probably goes to zero as $R \rightarrow \infty$. If this is true, then for some large but finite R the plane Couette flow would actually occur if the motion were set up properly (and if external disturbances were kept small enough), but some other motion, also having steady average properties, would be the realized one under other initial circumstances even though the boundary conditions are the same. A still better indication is provided by the well-known experiments of Coles on circular Couette flow in which two different motions (characterized by different combinations of azimuthal and vertical wavenumbers and almost certainly having somewhat different mean properties) were achieved at the *same* set of boundary conditions. Both motions were stable under the perturbations present in the experiment, and the choice between them depended on the method of setting up the flow. Nature's selection rule must thus in general be expected to depend on initial conditions and the like, and cannot be expected to be so simple a thing as a "principle of maximum dissipation," for instance.

On the other hand, there is a wide-spread popular superstition to the effect that at high enough Reynolds number there *is* a well-defined "turbulent state" which essentially always occurs—it is not that there is only one possible motion under given steady boundary conditions, but that the motions actually realized, those that do occur and persist, are thought to have the same mean properties. There is probably something in this, and it suggests the possibility that a selection principle of a simpler character might have a sort of asymptotic validity. To show from the Navier–Stokes equations that this is so (supposing that it *is*) seems at present rather out of sight, but perhaps a hypothesis of a particular "principle," if its consequences could be sufficiently well understood, might be tested (and perhaps ruled out) by comparison with physical or numerical experiments—indeed, the observed occurrence of turbulence does rule out the hypothesis of minimum energy dissipation. But to investigate, say, the conjecture that the observed mean properties of turbulent Couette flow resemble those of the solutions of the Navier–Stokes equations that maximize F appears to require that we know not only what F is but also what the class of competitors for maximizing it is—all solutions of the flow equations with the given boundary conditions. Since there is not much hope of describing all such solutions in a useful form, it will probably be necessary to be satisfied with less.

As a first step we might ask simply for an existence theorem—is there, in fact, an upper bound on the values of the functional $\langle|\nabla\mathbf{v}|^2\rangle$ for solutions of the equation, so that the conjecture at any rate is meaningful? This is not immediately clear, but it can be demonstrated, and without having to know much about the set of solutions, by showing that there is an upper bound for a larger but more simply described class of vector fields. There evidently is no upper bound for, say, all solenoidal $\mathbf{v}$'s satisfying the boundary conditions (a class that does contain all solutions but also too many other things), but we can obtain one as follows: let $\mathbf{u}$ be a vector field satisfying the boundary conditions (and having any needed horizontal averages) that satisfies Equation (8). From it we can construct a $U(z)$ using (6), and so get a $\mathbf{v}$ and a value for $F=\langle|\nabla\mathbf{v}|^2\rangle$, evidently also given by (10) or (11). Although the continuity equation has been used—for example, in showing that (8) is a property of solutions of the flow equations—we shall not for the moment insist on $\nabla\cdot\mathbf{u}=0$. Of course, if R is too small ($<\pi^2/2$), the only $\mathbf{u}$ satisfying (8) and the boundary conditions may be 0, but for larger R there are admissible $\mathbf{u}$'s that give values of F exceeding 1. To see that nevertheless F is bounded, use (11) and (8) to express F in the form

$$F = 1 + \frac{\langle uw\rangle^2 - R^{-1}\langle |\nabla \mathbf{u}|^2\rangle\langle uw\rangle}{\langle(\overline{uw} - \langle uw\rangle)^2\rangle} \tag{12}$$

(Here we assume that $\langle uw\rangle > 0$ so that F does exceed 1, evidently the only interesting case, and thus R is supposed to be large enough, say $>\pi^2/2$.) Equation (12) expresses F as a homogeneous functional of $\mathbf{u}$, and we may temporarily forget about the constraint (8)—any $\mathbf{u}$ that, on being inserted in (12), gives a value of $F>1$ can be multiplied by a suitable constant, irrelevant in (12), so as to make (8) hold. For the same reason we may normalize $\mathbf{u}$ so that $\langle uw\rangle = 1$, and rewrite (12) as

$$F - 1 = \frac{1 - R^{-1}\langle |\nabla \mathbf{u}|^2\rangle}{\langle(1 - \overline{uw})^2\rangle} \tag{13}$$

for any $\mathbf{u}$ satisfying the boundary conditions and $\langle uw\rangle = 1$.

We now call upon the following elementary result:

Lemma 1: If $f(z)$ is continuous, $f(\pm 1) = 0$, and f' is in $L_2(-1, 1)$, then $f^2(z) \leq (1 - z^2)\langle f'^2\rangle$.

Proof:

$$f^2(z) = \left(\int_{-1}^{z} f'(t)dt\right)^2 \leq \int_{-1}^{z} f'^2(t)dt \cdot \int_{-1}^{z} dt = (1 + z)\int_{-1}^{z} f'^2 dt$$

and similarly

$$f^2(z) = \left(\int_{z}^{1} f' dt\right)^2 \leq (1 - z)\int_{z}^{1} f'^2 dt$$

Therefore

$$\frac{2f^2(z)}{1 - z^2} \equiv f^2(z)\left[\frac{1}{1 + z} + \frac{1}{1 - z}\right] \leq \int_{-1}^{1} f'^2(t)dt = 2\langle f'^2\rangle$$

Since u and w vanish on $z = \pm 1$, we have

$$\begin{aligned} |\overline{uw}| &\leq \overline{(u^2w^2)^{1/2}} \leq (1 - z^2)\langle u_z^2\rangle^{1/2}\langle w_z^2\rangle^{1/2} \leq \tfrac{1}{2}(1 - z^2)\langle u_z^2 + w_z^2\rangle \\ &\leq \tfrac{1}{2}(1 - z^2)\langle |\nabla \mathbf{u}|^2\rangle \end{aligned}$$

Thus $|\overline{uw}|$ is certainly less than 1 if z is within a distance $\delta = \langle |\nabla \mathbf{u}|^2\rangle^{-1}$ of either boundary, and so also

$$\begin{aligned} \langle(1 - \overline{uw})^2\rangle \geq \frac{1}{2}\Bigg\{\int_{-1}^{-1+\delta} &[1 - (1 + z)\delta^{-1}]^2 dz \\ &+ \int_{1-\delta}^{1} [1 - (1 - z)\delta^{-1}]^2 dz\Bigg\} = \delta/3 \end{aligned}$$

(This assumes $\delta < 1$, evidently the only interesting case if F is to be large; the only way to make F large is to make $\overline{uw}$ nearly equal to 1 nearly everywhere, and since it is zero at the boundaries this can only be done at the expense of large gradients; such gradients only begin to become harmful to F when $\langle |\nabla \mathbf{u}|^2 \rangle$ begins to approach R.) If we use this inequality, equation (13) gives us

$$F - 1 \leq \frac{1 - (R\delta)^{-1}}{(\delta/3)} = 3R(1 - (R\delta)^{-1})(R\delta)^{-1}$$

and since no matter what a is, $(1-a)a \leq 1/4$, we see that for all $\mathbf{u}$ satisfying the boundary conditions and (8), and so in particular for all flows with steady mean properties, we must have

$$1 \leq F \leq 1 + \frac{3R}{4} \tag{14}$$

Thus F is indeed bounded for all flows and the question of whether or not the realized flows dissipate as much energy as the maximum of F over all flows is at least sensible. Of course we do not know what this maximum is—Equation (14) simply gives an upper bound on it, and one may well suspect that since only boundary conditions and the single relation (8) were used in obtaining (14) this may not be a very close upper bound. It can be improved by investigating more closely the variational problem lying behind the estimate just obtained, and perhaps further strengthened by subjecting the class of competitors to more constraints satisfied by all flows than those used above. One may hope that even relatively few such constraints will restrict the competitors sufficiently that the maximum of F over them is fairly close to the maximum over all flows—certainly this would be true if an upper bound found this way is nearly reached by *experimental* values of F. Thus, at least during periods of speculative optimism, one may regard the study of such variational problems as a contribution to the search for nature's asymptotic selection principle, and perhaps hope for a close relationship between the maximizing solution and the realized flow in other respects than simply the value of F, such as the mean profile $U(z)$. Such a viewpoint is essentially the suggestion that a "theory of turbulence," in almost the same sense that this term might be used in connection with a model of the statistical type using some kind of closure hypothesis, could be based on the hypothesis of an asymptotic selection principle of, say, maximum rate of energy dissipation.

A number of analogies between these two rather different-seeming approaches can in fact be drawn, and the parallelism between them made to look remarkably close. However, perhaps the most obvious common feature of such "turbulence theories" is that they rest on a shaky foundation of hypothesis—one talks in physical terms, but about quantities that are to be calculated from a mathematical structure whose relationship to what we think is a good model of reality—the Navier–Stokes equations—is only conjectured. It seems, to me at least, that "theories" in this sense cannot in the long run be regarded as physically satisfactory, however well they might be able to predict the values of observed quantities; in addition, some kind of much more detailed and mechanistic description would appear to be required for a real understanding of turbulence. It is perhaps, then, a fortunate aspect of investigations of the kind we are discussing here that a mood of speculative optimism is not really required at all times—even if turbulence is fundamentally incomprehensible, we can at least get upper bounds whose physical meaning is perfectly clear, and which seem to be not without interest.

An improvement of the upper bound (14) can be obtained by explicitly solving the variational problem of maximizing the functional (13), with $\langle uw\rangle = 1$ and the boundary conditions. This has been carried out by Howard (1963) in the different but closely related context of thermal convection, and the relationship between R and the maximum F can be written down explicitly in terms of elliptic integrals; for *large* R one finds

$$F \leq 1 + \frac{9R}{32} \tag{15}$$

This asymptotic bound is actually valid for all R, and can be obtained directly in the following rather simple way: one first shows (as in Howard 1963) that the maximum of (13) is obtained with $\mathbf{u} = u(z)(\mathbf{i}+\mathbf{k})$, so that we have $F-1 \leq (1-2R^{-1}\langle u'^2\rangle)/\langle(1-u^2)^2\rangle$, where $\langle u^2\rangle = 1$ and $u(\pm 1) = 0$. (It is immediately apparent from (13) that the optimum $\mathbf{u}$ is independent of x and y and has zero y component; that also $u = w$ follows from a slightly tricky but simple argument using the Euler equations.) One then applies the following:

Lemma 2: If $f(z)$ is continuous, $f(\pm 1) = 0$, f' is in $L_2(-1, 1)$, and $\langle f^2\rangle = 1$, then

$$\langle f'^2\rangle\langle(1 - f^2)^2\rangle \geq \frac{4}{9}$$

Using this, we have

$$F - 1 \leq (1 - 2R^{-1}\langle f'^2\rangle)\,\frac{9}{4}\,\langle f'^2\rangle = 2R^{-1}\langle f'^2\rangle(1 - 2R^{-1}\langle f'^2\rangle)\cdot\frac{9R}{8} \leq \frac{9R}{32}\,.$$

Proof of Lemma 2: For any constant A and any t in $(-1, 1)$, we have

$$0 \leq \frac{1}{2}\int_{-1}^{t} (f' - A(1 - f^2))^2 dz + \frac{1}{2}\int_{t}^{1} (f' + A(1 - f^2))^2 dz$$
$$= \langle f'^2\rangle + A^2\langle(1 - f^2)^2\rangle - 2A\left[f(t) - \tfrac{1}{3}f^3(t)\right]$$

Choose A so as to minimize this expression, namely set

$$A = (f(t) - f^3(t)/3)\langle(1 - f^2)^2\rangle^{-1},$$

which then gives $0 \leq \langle f'^2\rangle - f^2(t)(1 - f^2(t)/3)^2\langle(1-f^2)^2\rangle^{-1}$. Since $\langle f^2\rangle = 1$ and f is continuous, there is a value of t in $(-1, 1)$ such that $f^2(t) = \langle f^2\rangle = 1$; choosing this t gives the result asserted in the lemma.

[Remark: it is clear from the derivation that the minimum 4/9 is not attained; however, it can be arbitrarily approximated, for instance by an f whose graph is nearly a symmetrical trapezoid based on $[-1, 1]$ having very steep sides and an altitude just slightly over 1. On the other hand, the maximum of the functional (13) is actually achieved for any R—it is only asymptotically the same as (15), when in fact the maximizing function does resemble the above-mentioned near-trapezoid.]

In the range where experimental information is available, $R \sim 10^4$ (Reichardt 1959, Busse 1970—note that Busse's Reynolds number is four times the one used here), the upper bound (15) is about two orders of magnitude above what actually happens. The most obvious way to obtain a better bound is to subject the class of competitors for maximizing (13) to the continuity equation $\nabla \cdot \mathbf{u} = 0$, which is not in fact satisfied by the maximizing solution that leads (asymptotically) to (15). Since $\nabla \cdot \mathbf{u} = 0$ is a linear homogeneous constraint, the equivalence of (13) to the original problem is not upset thereby, but it is no longer true that the maximizing $\mathbf{u}$ has so simple a form. This problem has been attacked by Busse (1970) in a very interesting paper. He gives plausibility arguments, but unfortunately no proof (and I have not found one either), which suggests that the maximum is to be found among fields independent of x, the coordinate along the mean flow direction. If we assume that this is correct, the continuity constraint links v and w alone; if we replace v and w by kv and kw, and u by u/k, where $k^4 = \langle|\nabla u|^2\rangle / \langle|\nabla v|^2 + |\nabla w|^2\rangle$, the functional (13) becomes

$$F - 1 = \frac{1 - 2R^{-1}\langle|\nabla u|^2\rangle^{1/2}\langle|\nabla v|^2 + |\nabla w|^2\rangle^{1/2}}{\langle(1 - uw)^2\rangle} \tag{16}$$

which is to be maximized subject to $\langle uw\rangle = 1$ and $v_y + w_z = 0$, everything being independent of x. This problem is then found to be essentially equivalent to the corresponding upper-bound problem for thermal convection (see below), and Busse was able to take over with little change the results that had been found previously for that problem by Howard (1963) and especially in his own remarkable extension thereof (Busse 1969).

If one restricts the competitors for maximizing (16) to functions that depend on the y coordinate through a *single* wave number α, setting say $u = U(z)2^{1/2}\sin\alpha y$, $v = W'(z)\alpha^{-1}\, 2^{1/2}\cos\alpha y$, $w = W(z)\, 2^{1/2}\sin\alpha y$ (it is easy to see that u and w should be in phase for the optimal case), the functional becomes

$$F - 1 = \frac{(1 - 2R^{-1}\langle U'^2 + \alpha^2 U^2\rangle^{1/2}\langle \alpha^{-2}W''^2 + 2W'^2 + \alpha^2 W^2\rangle^{1/2})}{\langle (1 - UW)^2\rangle} \quad (17)$$

which is to be maximized with $\langle UW\rangle = 1$, $U(\pm 1) = W(\pm 1) = W'(\pm 1) = 0$. [Here of course the averages are simply over $(-1, 1)$.] It was this restricted form of the problem that (in the convection context) was studied by Howard (1963). The technique employed was to solve the Euler equations for the variational problem asymptotically for large R, using a "boundary-layer" approach; of course the wavenumber α must also be adjusted properly to maximize F. The problem does not seem to be altogether trivial, and for details we must here simply refer to the original paper; the result found for the maximum F among the single wavenumber functions is, translated into the present notation, $F-1 \leq KR^{3/4}$, where $K \sim 0.15$. Since this is claimed only to be an asymptotic bound, and is obtained only after a somewhat difficult and not completely rigorous calculation, it is perhaps of interest to give here a direct estimate that simply and rigorously provides a bound of this same form with a constant only about 50 percent larger. This can be obtained with the aid of the following analogue of Lemma 1:

Lemma 3: If $f(z)$ and $f'(z)$ are continuous, and both vanish at $z = \pm 1$, and if f'' is in $L_2(-1, 1)$, then

$$f^2(z) \leq \frac{1}{12}(1 - z^2)^3\langle f''^2\rangle$$

Proof: For z in $(-1, 1)$ let

$$\psi(z, t) = \begin{cases} A(z)(1 - b(z)(z - t)) & t < z \\ A(z)(1 + b(-z)(z - t)) & t > z \end{cases}$$

where $A(z) = -6f(z)/(1 - z^2)$ and $b(z) = (z+2)/(1+z)^2$. Then

$$0 \leq \int_{-1}^{1} [f''(t) - \psi(z, t)]^2 dt$$

$$= 2\langle f''^2\rangle + 2\langle \psi^2\rangle - 2\int_{-1}^{1} f''(t)\psi(z, t)dt.$$

Now, since $\psi(z, t)$ is continuous in t at $t = z$ we have

$$\int_{-1}^{1} f''(t)\psi(z, t)dt = -\int_{-1}^{1} f'(t)\psi_t(z, t)dt$$

$$= -Ab(z)\int_{-1}^{z} f'dt + Ab(-z)\int_{z}^{1} f'dt$$

$$= -A(b(z) + b(-z))f(z) = 24f^2(z)(1 - z^2)^{-3}.$$

One also finds $\int_{-1}^{1}\psi^2 dt = 24f^2(z)(1-z^2)^{-3}$, so the original inequality becomes $0 \le 2\langle f''^2\rangle - 24f^2(z)(1-z^2)^{-3}$, which is the result claimed. For any fixed z it is possible to construct an admissible f with $f''(t) = \psi(z, t)$, so the constant 12 is optimal.

By combining Lemmas 1 and 3 we can obtain a result somewhat analogous to Lemma 2; however, in this case the constant is probably not quite the best possible—an analogue of the trick used in the proof of Lemma 2 does not seem to be immediately apparent.

Lemma 4: If U and W' are continuous, $U(\pm 1) = W(\pm 1) = W'(\pm 1) = 0$, $\langle UW\rangle = 1$, and U' and W'' are in $L_2(-1, 1)$, then

$$\langle (1 - UW)^2\rangle \langle U'^2\rangle^{1/4} \langle W''^2\rangle^{1/4} \ge \frac{8}{15}\left(\frac{3}{4}\right)^{1/4} \qquad (\simeq 0.4963)$$

Proof: Since $\langle UW\rangle = 1$, $\langle U^2\rangle\langle W^2\rangle \ge 1$ and thus $\langle U'^2\rangle\langle W''^2\rangle \ge (\pi/2)^6$, by a crude estimate. From Lemmas 1 and 3,

$$\begin{aligned} |U(z)W(z)| &\le 12^{-1/2}(1 - z^2)^2 \langle U'^2\rangle^{1/2}\langle W''^2\rangle^{1/2} \\ &\le 2\cdot 3^{-1/2}\langle U'^2\rangle^{1/2}\langle W''^2\rangle^{1/2}\zeta^2, \end{aligned}$$

where ζ is either $1-z$ or $1+z$, the distance from the boundary. Let $\zeta_1^2 = (3/4)^{1/2}\langle U'^2\rangle^{-1/2}\langle W''^2\rangle^{-1/2}$; from the above rough estimate $\zeta_1^2 \le (3/4)^{1/2} \cdot (2/\pi)^3 < 1$. Thus

$$\langle (1 - UW)^2\rangle \ge \frac{1}{2}\int_{-1}^{-1+\zeta_1} + \frac{1}{2}\int_{1-\zeta_1}^{1} \ge \int_0^{\zeta_1} (1 - \zeta^2/\zeta_1^2)^2 d\zeta = 8\zeta_1/15$$

Therefore

$$\begin{aligned} \langle (1 - UW)^2\rangle\langle U'^2\rangle^{1/4}\langle W''^2\rangle^{1/4} &\ge (8\zeta_1/15)(3/4)^{1/2}\zeta_1^{-2})^{1/2} \\ &= (8/15)(3/4)^{1/4}. \end{aligned}$$

(Remark: evidently $\langle UW\rangle = 1$ is stronger than necessary to assure as that $\zeta_1 < 1$, but in our application we do have $\langle UW\rangle = 1$.)

To apply this result to estimate the functional (17) we note first that since $\langle U^2\rangle\langle W^2\rangle \ge 1$ we have

$$\begin{aligned} &\langle U'^2 + \alpha^2 U^2\rangle\langle \alpha^{-2}W''^2 + 2W'^2 + \alpha^2 W^2\rangle \\ &\qquad \ge \langle U'^2 + \alpha^2 U^2\rangle\left(\alpha^{-2}\langle W''^2\rangle + \frac{\alpha^2}{\langle U^2\rangle}\right) \\ &\qquad = \langle U'^2\rangle\langle W''^2\rangle\alpha^{-2} + \alpha^4 + \langle W''^2\rangle\langle U^2\rangle + \frac{\alpha^2\langle U'^2\rangle}{\langle U^2\rangle} \\ &\qquad \ge \langle U'^2\rangle\langle W''^2\rangle\alpha^{-2} + \alpha^4 + 2\alpha\langle W''^2\rangle^{1/2}\langle U'^2\rangle^{1/2} \\ &\qquad = (\langle U'^2\rangle^{1/2}\langle W''^2\rangle^{1/2}\alpha^{-1} + \alpha^2)^2 \end{aligned}$$

Minimizing this with respect to α, we find that the optimal α is $(\langle U'^2\rangle^{1/2}\langle W''^2\rangle^{1/2}/2)^{1/3}$, and thus

$$\begin{aligned}&\langle U'^2 + \alpha^2 U^2\rangle^{1/2}\langle \alpha^{-2}W''^2 + 2W'^2 + \alpha^2 W^2\rangle^{1/2}\\ &\geq 3(\tfrac{1}{2}\langle U'^2\rangle^{1/2}\langle W''^2\rangle^{1/2})^{2/3}\end{aligned} \tag{18}$$

Using this and Lemma 4 in (17) we get

$$\begin{aligned}F - 1 &\leq (1 - 3R^{-1}2^{1/3}\langle U'^2\rangle^{1/3}\langle W''^2\rangle^{1/3})\\ &\quad\cdot\frac{15}{8}\left(\frac{4}{3}\right)^{1/4}\langle U'^2\rangle^{1/4}\langle W''^2\rangle^{1/4}\\ &= (1 - 3R^{-1}2^{1/3}A^{4/3})\cdot\frac{15}{8}\left(\frac{4}{3}\right)^{1/4}A\end{aligned}$$

Maximizing this with respect to A, we finally obtain

$$F - 1 \leq \frac{5}{7}\left(\frac{3R}{14}\right)^{3/4} \qquad (\cong 0.225\, R^{3/4}) \tag{19}$$

Where experimental information in the turbulent range is available, this bound is about one order of magnitude too high; with the value 0.15 $R^{3/4}$ obtained from the numerical results of Howard (1963) it is about a factor of 5 over the observations. However, this estimate assumes that only one horizontal wavenumber is needed for the optimum, a conjecture that I made in the convection case (Howard 1963). In the summer of 1967 Busse discovered that this conjecture is incorrect, at least for sufficiently large R; a higher bound, varying like $R^{15/16}$ for large R, is obtained with two wavenumbers allowed, and with N wavenumbers Busse (1969, 1970) found by asymptotic methods that the exponent is $1-4^{-N}$. The coefficients obtained are such that if the asymptotic results are taken literally at any fixed finite R then the optimum is given by a certain finite number of wavenumbers, the number increasing with R—however, it is not *certain* that this finiteness is correct, since the results are only asymptotic. In any case, however, Busse's results indicate that the true bound implied by the energy integral and the continuity equation is about ten times the experimental value where that is available.

The possible advantage of two or more wavenumbers over a single one is indicated by the following considerations. Suppose we have several wavenumbers $\alpha_1, \alpha_2, \cdots$ and corresponding velocity fields u_i, w_i, with $\Sigma\langle u_i w_i\rangle = 1$. A slight modification of the argument leading to (18) gives

$$\begin{aligned}&\langle u_{i_z}{}^2 + \alpha_i{}^2 u_i{}^2\rangle\langle \alpha_i{}^{-2}w_{i_{zz}}{}^2 + 2w_{i_z}{}^2 + \alpha_i{}^2 w_i{}^2\rangle\\ &\qquad\geq 3\cdot 2^{-2/3}\langle u_{i_z}{}^2\rangle^{1/3}\langle w_{i_{zz}}{}^2\rangle^{1/3}\langle\, |\, u_i w_i\,|\,\rangle^{1/3}\end{aligned} \tag{20}$$

Now to make F large the denominator $\langle(1-\overline{uw})^2\rangle$ must be made small, so $\overline{uw}=\Sigma\overline{u_i w_i}$ must be nearly 1 except close to the boundaries (where it must approach zero); the sharp rise of $\overline{uw}$ at the boundaries is limited by the need to keep $\langle|\nabla \mathbf{u}|^2\rangle^2$ sufficiently less than R that the numerator is not too small. With only one wavenumber, the relationship between the steepness of the rise of $\overline{uw}$ and $\langle|\nabla \mathbf{u}|^2\rangle$ is rather directly obtained using Lemmas 1 and 3, together with Equation (18). With two wavenumbers the situation is a little more complicated. If we concentrate on the question of getting $\overline{uw}$ up to 1 as rapidly as possible, it appears that a second wavenumber might well be used to advantage: $\overline{u_2 w_2}$ can rise more rapidly than $\overline{u_1 w_1}$ only by making $\langle u_{2_z}{}^2\rangle\langle w_{2_{zz}}{}^2\rangle$ larger, but the deleterious effect of this on $\langle|\nabla \mathbf{u}|^2\rangle$ [cf Equation (20)] is mitigated if $\langle|\overline{u_2 w_2}|\rangle$ is small. Since the normalizing constraint $\Sigma\langle\overline{u_i w_i}\rangle=1$ can be satisfied almost entirely by the α_1 fields, for instance by choosing them to be nearly the optimal *single* wavenumber functions, we presumably *can* make $\langle|\overline{u_2 w_2}|\rangle$ small. However, it cannot be made arbitrarily small, for though it is necessary to make $\overline{u_2 w_2}$ rise rapidly to almost 1 in order to make $\langle(1-\overline{uw})^2\rangle$ smaller than it would be with the α_1 fields alone, this is not sufficient; $\overline{u_2 w_2}$ must *stay* fairly large also as one goes away from the boundary, out to the distance from the wall at which $\overline{u_1 w_1}$ can take over the job of holding $\overline{wu}$ almost at 1. Setting $\zeta_i=[(4/3)\langle u_{i_z}{}^2\rangle\cdot\langle w_{i_{zz}}{}^2\rangle]^{-1/4}$ and estimating the various terms roughly, we thus see that it is possible to make $\langle(1-\overline{uw})^2\rangle\cong(8/15)\zeta_2$ (with $\zeta_2\ll\zeta_1$) provided $\langle|\overline{u_1 w_1}|\rangle\cong 1$ (or more) and $\langle|\overline{u_2 w_2}|\rangle=2\zeta_1/3$ (or more), and using these with Equation (20) we may expect a bound for two wavenumbers of the form

$$F-1\leq\left[1-R^{-1}3\cdot 2^{1/3}\left(\tfrac{3}{4}\right)^{1/3}\left(\zeta_1{}^{-4/3}+\zeta_2{}^{-4/3}\left(\tfrac{2}{3}\zeta_1\right)^{1/3}\right)\right]\cdot\frac{15}{8}\zeta_2{}^{-1}$$

Maximizing the right-hand side of this with respect to ζ_1 and ζ_2 leads to

$$F-1\leq\frac{10}{13}\cdot 2^{5/2}\left(\frac{3R}{62}\right)^{15/16}$$

While this has not been obtained rigorously, a true bound with this exponent and a somewhat larger coefficient can be obtained (for the two-wavenumber case) by handling the above rough estimates more carefully; similar considerations give bounds with the exponents $1-4^{-N}$ found by Busse's asymptotic analysis with N wavenumbers, but it seems to be increasingly difficult, as N increases, to get rigorous values for the coefficients that are anywhere near as close to Busse's values as we obtained above for $N=1$. It seems likely that some different approach is needed to obtain rigorously a numerically good estimate of the upper bound with the continuity constraint but without reference to wavenumbers. It is not too difficult to reduce the bound (15) by a factor of 2 by using the continuity constraint, but this is still well above Busse's value.

The structure of the maximizing fields found by Busse's asymptotic analysis is very interesting, and while it cannot be fully discussed here, a couple of aspects should be pointed out. As suggested by the qualitative considerations just given, for fixed N and large R a set of boundary layers is found; $\ddot{u}\dot{w}$ differs significantly from 1 only in the thinnest of these, but $\overline{u_N w_N}$ is important both here and in the next boundary layer, where it co-operates with $\overline{u_{N-1}w_{N-1}}$ to produce $u\ddot{w}\cong 1$. Similarly, $\overline{u_{N-1}w_{N-1}}$ is important in both the second- and third-thinnest boundary layers, in the latter of which $\overline{u_N w_N}$ has disappeared but $\overline{u_{N-2}w_{N-2}}$ appears, and so on until the interior is reached. For any fixed N, as $R\rightarrow\infty$ each boundary layer becomes infinitely thin compared to its successor, but if for a fixed (though large) R one chooses that N which (according to the asymptotic theory) is optimal one finds that $N\rightarrow\infty$ with R, and in such a way that in fact the thickness ratio of successive boundary layers does not go to zero with R, but instead approaches 1/4. This of course casts a certain doubt on the use of the asymptotic results in this way, but Busse makes a fairly convincing case for the assertion that nevertheless the results are essentially correct. Although the momentum transport of the maximizing solution is significantly above that observed, nevertheless several features of this solution have a striking similarity with the corresponding things in the real flow—a most notable one is that the mean profile in the interior for the maximizing solution has a slope that is not zero but one-quarter that of the laminar plane Couette flow. While this is at variance with popular ideas about what turbulent flow "ought" to be like, Reichardt's experiments seem to agree remarkably well with it.

POISEUILLE FLOW

We turn now to a brief consideration of two-dimensional Poiseuille flow and axisymmetric pipe flow. If such flows are regarded as being driven by a fixed pressure gradient or body force, the enhanced friction produced by turbulence leads to a *decreased* mean flow or equivalently to a decreased rate of viscous dissipation, and the analogue of the variational problem for Couette flow is here to *minimize* the dissipation among the solutions of the equations and boundary conditions (or among such larger classes of functions as may be more convenient). The apparent difference comes only from transferring the driving inhomogeneity from the boundary conditions to the equations, and to state the problem in terms of maximization of the dissipation it is only necessary to fix, say, the mean flux through the channel or pipe instead of the pressure gradient. We may ask: among all flows with steady mean properties through a pipe (or channel) that carries a given total flux, is there an upper limit to the rate of viscous dissipation, and if so can we estimate it? As with Couette flow, the stability estimate given by the energy method shows that at sufficiently low Reynolds number (based on the flux through the pipe) the laminar Poiseuille flow is the only possibility, and gives the maximum (and minimum) flow resistance. Again it is unlikely that a universal principle of maximum dissipation for given flux (or minimum dissipation for given pressure gradient) can be tenable as a rule for selecting flows, when more than one solution exists (at high enough Reynolds number), though it is possible that this or some equally simple rule might have an asymptotic validity. Still, it is of interest to have an upper bound on the flow resistance and we shall now describe the formulation of the problem of obtaining such a bound by enlarging the class of flows to functions satisfying the boundary conditions, the energy integral relation, and (possibly) the continuity equation. For definiteness we consider the case of pipe flow (channel flow is almost the same; while slightly simpler, it is perhaps of somewhat less direct physical interest).

We consider an infinite pipe of radius a along the x axis through which the fluid is flowing under a mean overall pressure gradient (or body force) $\Delta P/\Delta x$. If we use a as length scale, ν/a as velocity scale, a^2/ν as time scale, and $\rho\nu^2/a^2$ as pressure scale, the dimensionless momentum equation becomes

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p = \mathbf{i}A + \nabla^2\mathbf{v} \tag{21}$$

where

$$A = \frac{1}{\rho}\,\frac{\Delta p}{\Delta x}\,\frac{a^3}{\nu^2}$$

Using cylindrical coordinates r, θ, x and denoting averages over a cylinder $r=$constant by the overbar, and averages over the pipe by $\langle\ \rangle$, we assume that $\bar{\mathbf{v}}=u(r)\mathbf{i}$, and write $\mathbf{v}=\bar{\mathbf{v}}+\mathbf{u}$. The mean flow equation is then found to be

$$r\,\frac{dU}{dr} = r\overline{uw} - \frac{1}{2}\,Ar^2 \tag{22}$$

where u and w are the axial and radial components of $\mathbf{u}$. The energy integral is

$$A\langle U\rangle = \langle|\nabla\mathbf{v}|^2\rangle \equiv \langle|\nabla\mathbf{u}|^2\rangle + \langle U'(r)^2\rangle$$

or, in view of (22) and $\langle r^2\rangle = 2\int_0^1 r^2 r\,dr = 1/2$,

$$A\langle U\rangle = \langle|\nabla\mathbf{u}|^2\rangle + \langle\overline{uw}^2\rangle - A\langle r\overline{uw}\rangle + \frac{A^2}{8} \tag{23}$$

From (22) we find by averaging over the pipe that

$$\langle ruw\rangle - \frac{A}{4} = \left\langle r\,\frac{dU}{dr}\right\rangle = 2\int_0^1 r^2\,\frac{dU}{dr}\,dr = -4\int_0^1 rU\,dr = -2\langle U\rangle$$

or

$$\frac{A}{8} - \langle U\rangle = \tfrac{1}{2}\langle ruw\rangle \tag{24}$$

The conventional Reynolds number for pipe flow is $R=2a[\langle U\rangle\nu/a]/\nu=2\langle U\rangle$, and in place of A we may introduce the "friction factor"

$$f=\frac{4a}{\rho}\,\frac{\Delta P}{\Delta x}\Big/\left(\frac{\langle U\rangle\nu}{a}\right)^2=\frac{16A}{R^2}$$

Our problem may now be stated as follows: for given $\langle U\rangle$ (or R), find an upper bound on A (or f) among all $\mathbf{u}$ consistent with (23), (24), and the boundary condition $\mathbf{u}=0$ on $r=1$ (and possibly also restricted by $\nabla\cdot\mathbf{u}=0$). To put this problem into a homogeneous form we note that from (23) and (24) $4\langle U\rangle\langle ruw\rangle-\langle|\nabla\mathbf{u}|^2\rangle-\langle\overline{uw}^2\rangle=-2\langle ruw\rangle^2$, and so since $\langle\overline{uw}^2\rangle-2\langle ruw\rangle^2=\langle(\overline{uw}-2r\langle ruw\rangle)^2\rangle$,

$$A/8\langle U\rangle-1=2\langle ruw\rangle[\langle ruw\rangle-\langle|\nabla\mathbf{u}|^2\rangle/4\langle U\rangle]/\langle(\overline{uw}-2r\langle ruw\rangle)^2\rangle$$

Thus

$$\frac{A}{8\langle U\rangle}-1=\frac{2\langle ruw\rangle[\langle ruw\rangle-\langle|\nabla\mathbf{u}|^2\rangle/4\langle U\rangle]}{\langle(\overline{uw}-2r\langle ruw\rangle)^2\rangle} \tag{26}$$

If A is maximized for fixed $\langle U\rangle$ by maximizing the right-hand side of this for all $\mathbf{u}$ satisfying the boundary conditions, then from the homogeneous form of (26) we see that $\mathbf{u}$ can always be normalized so as to satisfy (24); likewise we may if we wish temporarily normalize $\mathbf{u}$ so that, say, $\langle ruw\rangle=1/2$, thereby putting the problem in the form:

Maximize f for fixed R and $\langle ruw\rangle=1/2$, where

$$\frac{fR}{64}-1=\frac{1}{2}\,\frac{1-R^{-1}\langle|\nabla\mathbf{u}|^2\rangle}{\langle(r-\overline{uw})^2\rangle} \tag{27}$$

The close similarity with the form (13) for Couette flow is apparent. Incidentally, for any admissible field $\mathbf{u}$ in the original problem, Equation (25) shows that the right-hand side of (27) is nonnegative, so, as expected, f cannot be *less* than the laminar value $64/R$;
interested in obtaining a bound from *above*. The required analogue of Lemma 2 for the present case is that $\langle\psi'^2\rangle\langle(r-\psi^2)^2\rangle\geq 16/9$ (if $\psi(1)=0$ and $\langle r\psi^2\rangle=1/2$), and this leads to the bound

$$f\leq\frac{64}{R}+\frac{9}{4} \tag{28}$$

Results equivalent to this were given by Busse (1968). The friction factor for real turbulent flow in smooth pipes continually decreases as R increases, apparently toward zero, so (28) is not a particularly good upper bound. Busse has also attacked this problem and the two-dimensional one (Busse 1970) with the additional constraint $\nabla\cdot\mathbf{u}=0$, using the same type of multiple boundary-layer technique as in the Couette case, again assuming that the maximum is attained for functions independent of x. With a single wavenumber (in the azimuthal direction) he obtains a bound for the friction factor that decreases like $R^{-1/4}$, the same exponent that appears in the Blasius formula, though the coefficient is about four times larger. When many wavenumbers are allowed, however, the asymptotic form is $f \leq$ constant, the same form as (28) though the constant is about eight times smaller. Note that for flow in *rough* pipes the friction factor does seem to approach a constant, depending on the roughness (but ordinarily well below the above value), as $R \rightarrow \infty$.

Thermal Convection

The sort of investigation we have been discussing was actually first undertaken with thermal convection in mind, rather than shear flow, though as mentioned above the relationship between the two is closer than one might think offhand. My own interest in these problems was stimulated by certain aspects of Malkus' early work on turbulent convection (see Howard 1963). Malkus had suggested maximum heat transport, here equivalent to maximum viscous dissipation, as a sort of selection principle within the framework of the mathematical model he was trying to develop, and evidently had it in mind as a possibility in a more general context. In trying to give a mathematically clearer formulation of this I was led to emphasize the upper-bound aspect and leave aside the question of "selection principles," as well as other ideas suggested by Malkus; this does not of course necessarily get one nearer to the essentials of the physics. Indeed, while I feel that the upper-bound studies that have arisen from Malkus' suggestions about heat transport have an interesting and useful contribution to make to the study of turbulence, it seems to me that some of the other ideas he presented at that time may well turn out to be of greater significance for the development of a physically satisfactory picture of turbulent flow. In particular, his emphasis on the stability properties of the mean flow as significant in fully developed turbulence, and not merely relevant to transition, seems important; the fundamental role of bursts in maintaining the turbulence has of course been recognized only relatively recently but it does seem increasingly clear that the turbulent motions are rather well described for a large fraction of the time by the Orr–Sommerfeld equation. Exploitation of this in conjunction with a better understanding of bursts looks like one of the most promising approaches to a better physical picture.

Upper-bound studies in convection have been directed almost entirely at the "Bénard problem": a horizontal layer of fluid, mathematically modeled with the Boussinesq equations, is driven by a fixed temperature difference (hotter on the bottom) applied across the boundaries. The "mean flow" equation gives here an expression for the mean temperature gradient in terms of the "turbulent" (deviations from the mean) quantities, which in dimensionless form is

$$\frac{dT}{dz} = 1 + \langle w\theta \rangle - \overline{w\theta} \tag{29}$$

Here the flow domain is taken to be $0 \leq z \leq 1$, the temperature scale is that of the applied temperature difference ΔT, and the velocity $\mathbf{u}$ (of which w is the vertical component) is scaled with κ/d, the ratio of the thermal diffusivity to the dimensional layer thickness; the total temperature field is the (horizontal) mean $T(z)$ plus θ. In this problem there are two "energy integrals," obtained from the momentum and temperature equations, which can be put in the form

$$R\langle w\theta\rangle = \langle |\nabla\mathbf{u}|^2\rangle \tag{30}$$

$$\langle w\theta\rangle = \langle(\overline{w\theta} - \langle w\theta\rangle)^2\rangle + \langle |\nabla\theta|^2\rangle \tag{31}$$

R is here the Rayleigh number $\alpha g\Delta T d^3/(\kappa\nu)$, α being the expansion coefficient and g the gravitational acceleration. The mean heat flux is measured, relative to that due to conduction alone, by the Nusselt number N, which is the value of the dimensionless temperature gradient at the boundaries: $N = 1 + \langle w\theta\rangle$ [see Equation (29)]. In view of (30) a maximum of N for fixed R corresponds to maximum viscous dissipation. Although we have here the *two* integral relations, there are also now both velocity and temperature fields and it is possible to put the problem of maximizing N into a homogeneous form in which no separate integral constraints have to be considered. Indeed if $\mathbf{u}$ and θ satisfy (30) and (31) and if $N-1$ is defined as $\langle w\theta\rangle$, we have (assuming $\langle w\theta\rangle > 0$)

$$N - 1 = \langle w\theta\rangle = \langle w\theta\rangle \frac{\langle w\theta\rangle - \langle |\nabla\theta|^2\rangle}{\langle(\overline{w\theta} - \langle w\theta\rangle)^2\rangle}$$

$$= \frac{\langle w\theta\rangle^2 - R^{-1}\langle |\nabla\mathbf{u}|^2\rangle\langle |\nabla\theta|^2\rangle}{\langle(\overline{w\theta} - \langle w\theta\rangle)^2\rangle}$$

The functional on the right here is homogeneous in both $\mathbf{u}$ and θ, and it is not difficult to see that any trial functions $\mathbf{u}$ and θ that on being inserted in this functional give a positive value (and have $\langle w\theta\rangle > 0$; in any case $\langle w\theta\rangle \neq 0$ for the functional to be well defined, and by reversing the sign of θ if necessary one may then assume $\langle w\theta\rangle > 0$) can be renormalized so that (30) and (31) are satisfied. Thus we may ignore the integral relations and simply ask for the maximum of this homogeneous functional; or to simplify its form slightly we may temporarily normalize so that $\langle w\theta\rangle = 1$ and ask for the maximum of

$$\mathfrak{F}\{\mathbf{u}, \theta\} = \frac{1 - R^{-1}\langle |\nabla\mathbf{u}|^2\rangle\langle |\nabla\theta|^2\rangle}{\langle(1 - \overline{w\theta})^2\rangle} \tag{32}$$

with $\langle w\theta\rangle = 1$, the boundary conditions, and if desired $\nabla\cdot\mathbf{u} = 0$ as constraints. The close similarity with the Couette-flow problem is here apparent. Of course if R is small enough, there are no fields with $\langle w\theta\rangle = 1$ that make $\mathfrak{F} > 0$, and in fact the largest R for which this is true (with $\nabla\cdot\mathbf{u} = 0$) is the energy-method estimate for the minimum critical Rayleigh number. (In the present case the energy method gives the exact value $R_c \cong 1708$.)

If the constraint $\nabla \cdot \mathbf{u} = 0$ is neglected, it is apparent from (32) that the optimum has $\mathbf{u} = w(z)\mathbf{k}$ and $\theta = \theta(z)$; one can also show that in this case the optimum may be assumed to have $w(z) = \theta(z)$, and an application of Lemma 2 (slightly modified for the interval (0, 1) instead of $(-1, 1)$—4/9 becomes 16/9) leads to the bound

$$N - 1 \leq \left(\frac{3R}{64}\right)^{1/2} \tag{33}$$

This bound is in fact the asymptotic form of the exact solution of the maximum problem (without $\nabla \cdot \mathbf{u} = 0$) obtained by Howard (1963).

With the continuity equation included as a constraint but assuming that there is only a single horizontal wavenumber, we apply Lemma 4 [modified for the interval (0, 1)] to give the result

$$N - 1 \leq \frac{5 \cdot 2^{1/2}}{22}\left(\frac{3R}{11}\right)^{3/8} \tag{34}$$

The coefficient here is about 1.56 times that obtained by a boundary-layer calculation for this problem (Howard 1963).

As mentioned above, however, Busse found that at least for large enough R the assumption that the optimum is obtained with a single horizontal wavenumber is incorrect. It was in his attack on the problem with several wavenumbers that he developed the multiple-boundary-layer techniques that were subsequently applied to Couette and Poiseuille flow. For the convection problem the asymptotic bound with n wavenumbers has the exponent $(1-4^{-n})/2$, and an overall bound of $(R/1035)^{1/2}$, which is about 1/7 of the value (33).

Where comparison with experiment is possible, Busse's bounds are somewhat less than a factor of 10 above the heat transport that actually occurs, and (given a little good will) it is possible to find considerable similarities between mean properties of the optimal solutions and those of real convection. Nevertheless, it would of course be desirable to find bounds still closer to the observed heat transports; in particular, the experimental data are fairly well described for large R by $N = 0.1\ R^{1/3}$, and since there are fairly good heuristic reasons for expecting that N should vary as $R^{1/3}$ as $R \rightarrow \infty$ one would hope to be able to obtain a bound with this exponent. This evidently cannot be done by using only the two "energy integrals" and the continuity equation as constraints: if a bound $KR^{1/3}$ exists, it can only be obtained by using more consequences of the full equations to further restrict a class of competitors.

This does not seem to be very easy to do, but a quite interesting step in this direction has been taken recently by Chan (1971). The dimensionless

Boussinesq equations contain two dimensionless parameters—the Rayleigh number and also the Prandtl number $\sigma \equiv \nu/\kappa$. If in these equations we let $\sigma \rightarrow \infty$, the momentum equation becomes linear and not explicitly dependent on time (one may think of $\sigma \rightarrow \infty$ as being $\nu \rightarrow \infty$, so that the hydrodynamics becomes the Stokes equation of slow viscous flow, but the heat transport remains nonlinear). The relative simplicity of the momentum equation in this case suggests the possibility of using the "energy integral" obtained from the heat equation as before, but retaining the full momentum equation (as well as the continuity equation) as constraints for the variational problem, which will give an upper bound. This problem was attacked by Chan, using the multiple-boundary-layer techniques of Busse. His results give asymptotic bounds with n wavenumbers that vary as $[R^{1/3}(\log R)^{2/3}]^{(1-10^{-n})}$, and an overall bound, when n is chosen optimally for a given R, of $0.152\ R^{1/3}$. This bound is very comfortingly close to the observations, both with regard to the coefficient and the exponent, and Chan also finds very satisfactory agreement of both the mean temperature profile and the rms temperature deviations $(\overline{\theta^2})^{1/2}$ of his optimal solution with the corresponding mean quantities, as observed experimentally. Of course experiments are not done with fluids of infinite Prandtl number, so the precise status of this type of "bound" is still uncertain. It would also be most desirable to have a directly obtained and rigorous bound for this problem of the form $N-1 \leq KR^{1/3}$ [analogous to the bound (33) for the pure "energy-integral" problem] even with a poor value for K, to reduce any lingering doubts about the validity of the rather complicated boundary-layer solution. Such a rigorous estimate has not yet been found. On the whole, however, Chan's results seem to suggest rather strongly that real turbulent convection, at least for high Prandtl number, does come pretty close to maximizing the heat transport among the possible motions, and that fairly good estimates of heat transport can be obtained from relatively tractable variational problems.

Other Problems

All the cases discussed above are characterized by a fairly high degree of spatial homogeneity, and this no doubt plays a role in making the variational problems more or less tractable, but the general ideas we have been considering are certainly applicable to many other kinds of flow, at least some of which should be accessible. Recent and imminent papers deal with upper-bound results for cylindrical Couette flow (Nickerson 1969), thermohaline convection (Lindberg 1971), and convection in a porous medium (Busse & Joseph 1972). There would seem to be possibilities of obtaining useful results for parallel shear flow or thermal convection in the presence of magnetic fields, and perhaps for some types of electroconvection. Nothing much seems to have been attempted along these lines for compressible fluids. Can one obtain an upper bound for the drag on a sphere that is moved at constant speed through an infinite viscous incompressible fluid? Many interesting questions like this can be raised, but probably one cannot say that the cases studied so far have yet provided anything like a general method for answering them. In addition to the need for other specific examples from which to try to induce a more general approach, the following three aspects would seem to be worthy of further study: (a) Can Busse's multiple-boundary-layer methods be put on a mathematically firmer foundation? (b) Can more comprehensive direct methods, fragmentary examples of which are given by the lemmas in this article, be developed? (c) Can variational problems of the kind discussed here, which apparently have a rather complex boundary-layer structure, bc understood well enough in general terms to permit the development of effective numerical methods for their solution?

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