GEOFYSISKE PUBLIKASJONER

GEOPHYSICA NORVEGICA

VOL. XXII NO. 3

ON THE TRANSFER OF ENERGY IN STATIONARY MOUNTAIN WAVES

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FREMLAGT I VIDENSKAPS-AKADEMIETS MØTE DEN 9DE DESEMBER 1960

Summary. The flow of wave energy in stationary, two-dimensional gravity waves of small amplitude in a basic current where the velocity and stability varies with height, is studied. The vertical flux of wave energy is found to vary with height in proportion to the wind speed. In layers where the wave motion of a particular wave length is of the internal type, the motion may be subdivided into two parts, one wave carrying wave energy upward, and the other carrying wave energy downward. In the case of mountain waves, the wave with upward energy flow may be interpreted as the incident wave, set up by the mountain, whereas the wave carrying energy downward is caused by reflection of the incident wave in higher layers in the atmosphere. Such reflection is generally found to take place when wind or stability varies with height. The reflection coefficients in two- and threelayer atmospheres are calculated.

The results are applied to a distribution of wind and stability typical of situations in which mountain waves occur. It was found that, depending on the wave length, 65 – 100 per cent of the wave energy was reflected from the layers of strong wind in the upper troposphere. In middle latitudes in winter, wave energy may be transmitted to the lower ionosphere.

A study is also made of the energy transfer for long quasi-static mountain waves.

1. Introduction. When air flows over a mountain ridge, a stationary wave disturbance is set up in the air current. These waves, when analysed with respect to their wave length, or period, fall into three categories: short waves of period much smaller than the half pendulum day are gravity waves; longer waves of period comparable with the half pendulum day are gravity-inertia waves; and longer waves with longer periods are quasi-geostrophic, planetary waves of the Rossby-wave type. Mountains of small extent in the direction of the flow will excite only gravity waves, while the large mountain ranges of continental scale will excite all types of waves; however, if these mountain ranges are assumed to be sufficiently smooth, only long planetary waves will be excited.

The gravity-type mountain waves have been subject to several theoretical studies in recent years, mostly on the basis of the linearized perturbation equations. The wave motion depends strongly on the distribution with height of wind velocity and static stability. The simplest case is that of constant wind and constant static stability (isothermal atmosphere with pressure and density exponentially decreasing with height). This case, which leads to perturbation equations with constant coefficients, was solved by LYRA (1943) and QUENEY (1947). It turns out that the steady-state equations alone do not suffice to give a unique solution; some additional principle has to be utilized in order to select the «correct» solution. LYRA and QUENEY showed that a unique solution can be obtained by introducing a small RAYLEIGH-friction, and that the solution thus obtained is monotoneous on the wind-ward side of the mountain, and oscillatory (but damped) on the lee side, in fair agreement with observed flow patterns. However, the method could hardly be said to be fully convincing.

Following a suggestion by HØILAND (1951), PALM (1953) and WURTELE (1953) assumed the motion to start from rest, and found the solution to the initial-value problem; this solution turned out to approach uniquely the steady LYRA-QUENEY solution for large values of time. This method of solving the uniqueness problem is physically satisfactory, but also very labourious. A. ELIASSEN and PALM (1954) pointed out that the gravity waves can convey wave energy in vertical direction, and that the LYRA-QUENEY solution can be arrived at by selecting those steady-state solutions which transport wave energy upwards, and rejecting those who transport wave energy downwards, since the energy source (the mountain) is at the ground. This radiation condition has also been used by ZIEREP (1957) for solving the uniqueness problem. WILKES (1949) has earlier made use of the same principle in the theory of atmospheric tides[1]; he likewise assumes that the source of wave energy (produced by tidal forces and diurnal heating and cooling) is predominantly located at low levels. WILKES explains furthermore the resonance which gives rise to the large amplitude of the 12-hour oscillation as a consequence of trapping of energy in low layers, caused by reflection by higher layers of sufficiently small static stability. He also points out the analogy between the propagation of tidal waves and the propagation of radio waves in the atmosphere.

The tides in the atmosphere, and mountain waves in a constant basic current are very similar problems; they are both steady flows in one frame of reference, and perturbations of the state of rest in another frame. In both problems the flux of wave energy is non-divergent throughout most of the fluid, outside the relatively limited regions where such energy is being produced, and the useful concepts of reflection and transmission of wave energy may be used without difficulty.

The matter becomes much more complicated when the basic current is non-uniform as in the case of mountain waves in a basic current which varies with height. Exchange of energy between the wave motion and the basic current is then possible, and sources or sinks of wave energy, depending upon the wave motion itself, may occur at all

[1] The authors are indebted to J. S. SAWYER for bringing this to their attention.

levels. We shall refer to these sources as *secondary*, in contrast to the *primary sources* (the mountains) which cause the wave motion in the first place. With the secondary sources present, the concepts of transmission and reflection seem a priori to have no clear interpretation. Thus for instance, a part of the wave motion which corresponds to energy transfer downward needs not be due to reflection, but might perhaps be the result of secondary energy sources located at high levels.

Yet some of the results which have been obtained in the study of lee waves in a basic current which varies with height are very suggestive of an interpretation in terms of reflection and transmission of wave energy. Thus the results by SCORER (1949) that an undamped wave train on the lee side of the mountain will appear if the static stability decreases, or the wind speed increases with height, might be interpreted as a resonance wave due to trapping of energy in the lowest layer, caused by total reflection in layers above. The phenomenon might thus be very similar to the resonance suggested by WILKES for tidal waves in the atmosphere. Likewise, PALM and FOLDVIK (1960) have computed the undamped resonance waves produced by a mountain for a realistic variation with height of wind and static stability. They found that the wave motion in the lower troposphere consists primarily of one or a few resonance waves of relatively short wave length which were almost completely determined by the distribution of wind and temperature up to about 8 km, and practically independent of wind and temperature distribution in the stratosphere. The wave motion in the stratosphere, on the other hand, was found to consist to a large extent of much longer waves, which depend critically upon the distribution of wind and temperature up to very high levels. It thus looks as if the very short waves are almost totally reflected by the upper troposphere, whilst somewhat longer waves can penetrate more easily into the stratosphere.

The present paper deals with the transfer of wave energy when the basic current changes with height. It is found that the radiation condition is under certain circumstances still of basic importance, that the reflection coefficient can still be defined, but that the concept of transmission needs a re-interpretation. The results are believed to be of interest for the theory of mountain waves, since many features of the motion can be explained without much calculation. The results also throw some light upon the question of the possible propagation of gravity waves up to very high levels. In a recent paper, HINES (1959) has suggested that certain irregularities observed in the ionosphere at about 80—100 km are gravity waves set up by energy sources at low levels.

It is noteworthy that long quasi-geostrophic waves can convey energy in vertical direction in much the same way as gravity waves. The energy transfer brought about by such long waves in a non-uniform basic current is discussed in Chapter II.

CHAPTER I

WAVE ENERGY TRANSFER IN STATIONARY GRAVITY WAVES WHEN THE BASIC CURRENT VARIES WITH HEIGHT.

2. Perturbation equations and energy equation. The air is assumed piezotropic (adiabatic), so that the density ϱ of each particle is a function of its pressure p. Using the notation of V. BJERKNES, the compressibility may be characterized by the coefficient of piezotropy

$$\gamma = D\varrho/Dp \tag{2.1}$$

where D denotes the individual differential. The velocity of sound is then $\gamma^{-\frac{1}{2}}$.

The basic flow is taken to be a straight current parallel with the horizontal x-axis; the velocity $U(z)$ is assumed to be a function of height z. The pressure $p_o(z)$ and density $\varrho_0(z)$ fulfil the hydrostatic relation

$$\frac{dp_0}{dz} = -g\varrho_0 = -\frac{p_0}{H} \tag{2.2}$$

where g is the acceleration of gravity, and H the scale height.

The density stratification may be characterized by V. BJERKNES' coefficient $\Gamma = d\varrho_0/dp_0$; the density gradient is then

$$\frac{d\varrho_0}{dz} = \Gamma \frac{dp_0}{dz} = -\Gamma g\varrho_0 \tag{2.3}$$

The static stability of the basic flow may now be characterized by the buoyancy frequency (VÄISÄLA–BRUNT frequency) ν_0, which is given by

$$\nu_0{}^2 = (\Gamma - \gamma)\, g^2 \tag{2.4}$$

The wave motion will be assumed to take place in the xz-plane and to depend only upon the coordinates x and z; it will be characterized by the vertical displacement ζ, the horizontal and vertical perturbation velocities u and w, the pressure p and the density perturbation ϱ. (Primes will be omitted, since there is no danger of confusion with the total pressure and the total density).

Retaining only first order terms, one finds from (1), (2), (3) and (4),

$$\varrho = \frac{\nu_0{}^2}{g}\,\varrho_0\zeta + \gamma p \tag{2.5}$$

When this equation is used to eliminate ϱ, the equations of motion for stationary flow become

$$\varrho_0 U u_x + \varrho_0 U_z w + p_x = 0 \tag{2.6}$$

$$\varrho_0 U w_x + \varrho_0 \nu_0^2 \zeta + p_z + \gamma g p = 0 \tag{2.7}$$

and the continuity equation

$$u_x + w_z - \gamma g w + \frac{\gamma U}{\varrho_0} p_x = 0 \tag{2.8}$$

Here we have

$$w = U \zeta_x \tag{2.9}$$

The wave energy equation is obtained by multiplying (6) by u, (7) by w, (8) by p, and adding. One finds

$$\frac{\partial}{\partial x}(EU + pu) + \frac{\partial}{\partial z}(pw) = -\varrho_0 U_z uw \tag{2.10}$$

where

$$E = \tfrac{1}{2}\varrho_0 \left[u^2 + w^2 + \nu_0^2 \zeta^2 + \gamma \varrho_0^{-2} p^2\right] \tag{2.11}$$

is the total *wave energy* per unit volume. The wave energy is made up by a kinetic part $\frac{1}{2}\varrho_0(u^2 + w^2)$ and an available potential and internal part $\frac{1}{2}\varrho_0(\nu_0^2\zeta^2 + \gamma\varrho_0^{-2}p^2)$.

Equation (2.10) has the following interpretation. The left-hand side is the divergence of the wave-energy flux, i.e. the horizontal wave-energy flux through a unit vertical area is $(EU + pu)$ and the vertical flux through a unit horizontal area is pw. The right-hand side represents the "secondary" source of wave energy referred to in the introduction; it represents a conversion of kinetic energy of the basic current (or rather the mean current, averaged with respect to x) into wave energy. Depending upon the kinematics of the wave motion, this secondary energy source may be positive or negative.

In the case when no resonance waves occur, the wave energy E, (and hence, all perturbation quantities) tend toward zero as $x \to \pm\infty$. Therefore we obtain, when the wave energy equation (2.10) is integrated with respect to x from $-\infty$ to $+\infty$:

$$\frac{d}{dz}\int_{-\infty}^{+\infty} pw\,dx = -U_z\varrho_0\int_{-\infty}^{+\infty} uw\,dx \tag{2.12}$$

On the other hand we obtain, when (2.6) is multiplied by $(\varrho_0 U u + p)$ and integrated from $x = -\infty$ to $x = +\infty$:

$$\int_{-\infty}^{+\infty} pw\,dx = -U\varrho_0\int_{-\infty}^{+\infty} uw\,dx \tag{2.13}$$

Comparison of (2.12) and (2.13) shows that

$$\varrho_0 \int_{-\infty}^{+\infty} uw\, dx = \text{constant when } U \neq 0, \tag{2.14}$$

i.e. the vertical flux of horizontal momentum does not change with height, except possibly at levels where $U = 0$. Hence it follows that within a layer where $U \neq 0$, the total vertical flux of wave energy $\int_{-\infty}^{+\infty} pw\, dx$ must vary with height in proportion to U. Moreover, the right-hand member of (2.12), which represents "secondary" sources of wave energy, integrated with respect to x, is seen to vary with height in proportion to U_z. The possible distributions with height of these secondary energy sources are thus strongly restricted. The vertical distributions of the secondary energy sources and the vertical flux of wave energy are completely determined within a layer where $U \neq 0$, when the constant momentum flux is known.

Moreover, in a layer where U is everywhere positive, it is seen from (2.13) that the vertical fluxes of wave energy and of momentum are of opposite sign. If wave energy is transferred upwards, momentum is transferred downwards; the secondary energy sources are then positive where U increases with height, and negative where U decreases with height.

These results are no longer valid if resonance waves occur, since the wave motion then does not tend to zero as $x \to +\infty$. In this case the wave energy produced by the mountain is not only carried upwards, but also downstream.

3. Fourier representation of the solution. We assume that the mountain profile may be represented as a Fourier integral with respect to x. If no resonance waves occur, the dependent variables (perturbation velocities, pressure, density) may likewise be expressed as Fourier integrals; and when resonance waves do occur, they can be expressed as Fourier-Stieltjes integrals. In any case, we may consider a single Fourier component with wave number k. Such a component will also satisfy eqs. (2.6, 2.7, 2.8, 2.9 and 2.10). We may now derive equations which are very similar to (2.12) and (2.13) except that the integrals are taken over one wave length instead of the entire x-axis. Denoting the average value over one wave length by a bar, we obtain from (2.10) by averaging

$$\frac{d}{dz}\overline{pw} = -U_z \varrho_0 \overline{uw} \tag{3.1}$$

and from (2.6) by multiplying with $(\varrho_0 Uu + p)$ and averaging

$$\overline{pw} = -U\varrho_0 \overline{uw} \tag{3.2}$$

Hence

$$\varrho_0 \overline{uw} = \text{constant when } U \neq 0 \tag{3.3}$$

The vertical wave energy flux brought about by a single Fourier component must therefore vary with height in proportion to U.

It is convenient to express $\overline{pw}$ for one Fourier component in terms of w alone. For this purpose we multiply the continuity equation (2.8) by w_x and average:

$$\overline{w_x w_z} = -\overline{u_x w_x} - \frac{\gamma U}{\varrho_0}\overline{p_x w_x} = -k^2\overline{uw} - \frac{\gamma U}{\varrho_0}k^2\,\overline{pw} \tag{3.4}$$

Elimination of $\overline{uw}$ between (3.2) and (3.4) gives the desired relation

$$(1-\gamma U^2)\,\overline{pw} = \frac{U\varrho_0}{k^2}\overline{w_x w_z} \tag{3.5}$$

For mountain waves it may be assumed that the wind speed U is small compared with the velocity of sound, i.e. $\gamma U^2 \ll 1$, so that (3.5) may be simplified to

$$\overline{pw} = \frac{U\varrho_0}{k^2}\overline{w_x w_z} \tag{3.6}$$

Comparison with (3.2) shows that when $U \neq 0$, $\varrho_0\overline{w_x w_z}$ must be constant with height. This expression is seen to depend upon the tilt of the wave with height. If the phase is shifted upstream with height, wave energy flows upward. If furthermore U increases with height, then kinetic energy of the mean motion is converted into wave energy. If the system were confined between horizontal rigid planes, a wave of this kind would necessarily grow. In the present case, however, the wave remains stationary because the wave energy produced is transferred upwards to higher layers.

The differential equation satisfied by w for a Fourier component of the solution is found from eqs. (2.6 – 2.9) by eliminating u, p and ζ. With good approximation this equation may be written (see. e.g SCORER (1949) or PALM and FOLDVIK (1960))

$$\hat{w}_{zz} + (l^2 - k^2)\,\hat{w} = 0 \tag{3.7}$$

where $\hat{w}$ is defined by

$$w = \text{Real part}\left\{\hat{w}(z)\, e^{\frac{1}{2}\Gamma gz}\, e^{ikx}\right\} \tag{3.8}$$

and

$$l^2 = \frac{\nu_0^{\,2}}{U^2} - \frac{U_{zz}}{U} \tag{3.9}$$

In the approximation used, Γ is considered constant, which is equivalent to assuming an exponential density distribution with height. It should be noted that this approximation does not imply that the static stability is nearly constant with height; the density distribution is nearly exponential, even if the static stability varies considerably.

In therms of $\hat{w}$, the mean vertical wave energy flux (3.6) may be expressed as

$$\overline{pw} = \tfrac{1}{2}\,\varrho_0(0)\; Uk^{-1}\ \text{Imaginary part}\left\{\hat{w}^*\,\hat{w}_z\right\} \tag{3.10}$$

where the asterisk denotes the complex conjugate value.

It is noteworthy that the differential equation (3.7) for stationary gravity waves is formally similar to the equation which governs the propagation of radio waves in the ionosphere, as was pointed out by WILKES (1949). In the absence of a magnetic field, and with sufficiently low frequency of collisions between electrons and air molecules and atoms, the propagation of waves of frequency ν travelling vertically is governed by the equation

$$\mathbf{E}_{zz} + \frac{\nu^2 - f_N{}^2}{c^2}\,\mathbf{E} = 0 \tag{3.11}$$

where $\mathbf{E}$ is the electric field intensity, c the velocity of light in vacuum and f_N is the plasma frequency, which varies with height in proportion with the square root of the electron density. Comparison between (3.11) and (3.7) shows that a ionospheric layer of high electron density corresponds to a layer of small l^2 in the case of gravity waves. The energy flow connected with the radio waves is given by Poynting's vector, which can be expressed in terms of $\mathbf{E}$ as

$$-\,(2\nu)^{-1}\,\text{Imaginary part}\,\{\mathbf{E}^{*}\cdot\mathbf{E}_z\}. \tag{3.12}$$

This expression is in close formal agreement with the formula (3.10) for the wave energy flux connected with gravity waves, apart from a difference in sign owing to the fact that radio waves convey energy in the direction of the phase velocity, in contrast to gravity waves.

Moreover, radio waves are also similar to gravity waves regarding the conditions to be satisfied at interfaces between media of different properties. As a result, radio waves and gravity waves show similar behaviour with respect to transmission and reflection. Examples of such analogous behavious will be found below.

4. Properties of the solutions in a layer of constant l^2.

As proposed by SCORER (1949), it is of interest to investigate the case of constant l^2, which leads to very simple solutions.

For $k^2 > l^2$, the stationary waves will be of the external type. The general solution may then be written as

$$\hat{w} = A\,e^{\mu z} + B\,e^{-\mu z} \tag{4.1}$$

with

$$\mu^2 = k^2 - l^2, \qquad \mu > 0 \tag{4.2}$$

and A, B denoting complex constants. For this solution, (3.10) gives

$$\overline{pw} = \varrho_0(0)\,U\frac{\mu}{k}\,\text{Imaginary part}\,\{A\,B^{*}\} \tag{4.3}$$

Thus the vertical flux of wave energy may have any value; it will vanish only when A and B have the same arguments, or when either A or B is zero.

Next we consider the case of internal waves, $k^2 < l^2$. In this case, the general solution may be written in the form

$$\hat{w} = A\, e^{i\lambda z} + B\, e^{-i\lambda z} \tag{4.4}$$

with

$$\lambda^2 = l^2 - k^2, \qquad \lambda > 0 \tag{4.5}$$

Application of (3.10) gives in this case

$$\overline{pw} = \tfrac{1}{2}\, \varrho_0(0)\; U \frac{\lambda}{k} \left(|A|^2 - |B|^2 \right) \tag{4.6}$$

Thus we see that the solution $\exp(i\lambda z)$, which represents a wave with the lines of constant phase tilted windward (assuming $U > 0$) transports energy upwards, whereas a solution of the form $\exp(-i\lambda z)$ representing waves with phase-lines tilted downstream, transports wave energy downwards (A. ELIASSEN and PALM, 1954).

For a superposition of both types of solutions, the resulting vertical wave-energy flux is additive. It is noteworthy that this is a unique property of the particular fundamental set of solutions $\exp(i\lambda z)$, $\exp(-i\lambda z)$ employed in (4.4); with any other choice of fundamental set, the energy fluxes would not be additive.

The additivity of the energy fluxes enables us to consider the solution $A\exp(i\lambda z)$ as an incident wave set up by surface corrugation or by low-level heat sources, and $B\exp(-i\lambda z)$ as a wave caused by reflection from higher layers of the atmosphere, provided these layers do not contain any forcing effects such as heat sources. We may define the reflection coefficient r of these higher layers as the ratio of the downwards to the upwards energy flux, i.e.

$$r = \left| \frac{B}{A} \right|^2 \tag{4.7}$$

Thus, r is a constant within a layer of constant l^2, in spite of the fact that energy is extracted from, or given to the mean current within the layer. This is because the upward and the downward fluxes both vary in proportion to U.

The net upward flux of wave energy is thus $(1 - r)$ times the flux associated with the incident wave.

5. Transmission and reflection in layered model atmospheres. When l^2 varies continuously with height, we cannot in general find analytical solutions of (3.7). The problem may then be solved in an approximate manner by replacing the continuous distribution of l^2 by a fitting discontinuous stepcurve, i.e. by dividing the atmosphere in a number of layers with a constant value of l^2 in each layer. The solution must then in each layer be of the form discussed in section 4; at the interfaces, they may be linked together by requiring

$$w \text{ and } w_z \text{ continuous.} \tag{5.1}$$

These interface conditions may be looked upon as being purely mathematical, serving the purpose that the solution of (3.7) for a step-wise distribution of l^2 shall approximate the solution of the same equation for a continuous distribution of l^2. It is therefore not necessary to consider the discontinuous distribution of l^2 as belonging to a real physical system and derive the interface conditions from the hydrodynamical boundary conditions of such a system.

We shall consider the propagation of wave energy for such a step-wise distribution of l^2. The uppermost layer will be assumed to be of infinite extent. Let l_∞^2 denote the value of l^2 in this layer; the motion within this layer will be of the external or internal type according as $k^2 \gtrless l_\infty^2$.

Consider first the case $k^2 > l_\infty^2$, so that the solution in the uppermost layer is of the form (4.1). In order that the wave energy shall be finite at $z = \infty$, we must require $A = 0$. Thus the solution in the uppermost layer of the form

$$\hat{w} = B\, e^{-\mu z} \tag{5.2}$$

It follows that $\overline{pw}$ is zero in the uppermost layer, and since $\overline{pw}$ varies in proportion to U (section 3), it must be zero also in the layers below. Suppose that the motion in one of these layers is of internal type. Since there is no vertical flux of wave energy, the solution in such a layer must be of the form (4.4) with $|A| = |B|$. Hence $r = 1$, and it follows that *the reflection is always total when the motion is of external type in the uppermost layer.*

Consider next the case $k^2 < l_\infty^2$. The solution in the uppermost layer is then of the type (4.4). If the wave motion is set up by mountains or by low-level heat and cold sources, we here select the solution which represents upward flux of wave energy, i.e. $B = 0$, since there is no reflection from layers above. Therefore, the solution in the uppermost layer is now of the form.

$$\hat{w} = A\, e^{i\lambda z} \tag{5.3}$$

In this case, wave energy is continually lost by the lower atmosphere. The reflection coefficient must necessarily be less than one in all lower layers (although it may approach unity under suitable conditions).

We shall now calculate the reflection coefficient for a two- and a three-layer atmosphere.

a. Two-layer model. Let subscript 1 refer to the lower layer, and subscript 2 to the upper layer. We assume that $k^2 < l_1^2$, so that the wave in the lower layer is of the internal type. The solution in the lower layer may then be written

$$\hat{w}_1 = A_1\, e^{i\lambda_1 z} + B_1\, e^{-i\lambda_1 z} \tag{5.4}$$

where A_1 is the amplitude factor of the incident wave, and B_1 the amplitude factor of the reflected wave.

In the upper layer, the wave motion will be of the internal or external type, depending on $k^2 \lessgtr l_2^2$. Consider first the case of internal wave motion $k^2 < l_2^2$. According to what is said above, the solution in the upper layer is then

$$\hat{w}_2 = A_2 \, e^{i\lambda_2 z} \tag{5.5}$$

The reflection coefficient is found by applying the boundary conditions at the interface. Choosing the origin of the reference system at the interface, these conditions give

$$\left.\begin{aligned} A_1 + B_1 &= A_2 \\ \lambda_1 (A_1 - B_1) &= \lambda_2 A_2 \end{aligned}\right\} \tag{5.6}$$

By elimination of A_2

$$A_1 (\lambda_1 - \lambda_2) - B_1 (\lambda_1 + \lambda_2) = 0 \tag{5.7}$$

Hence

$$r = \frac{|B_1|^2}{|A_1|^2} = \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)^2} \tag{5.8}$$

We see from this formula that if $\lambda_1 = \lambda_2$, i.e. l^2 is constant throughout the atmosphere, no reflection occurs. On the other hand, if $\lambda_1 = 0$ or $\lambda_2 = 0$, the reflection is total. In all other cases a partial reflection takes place.

The case $\lambda_2 = 0$ forms the transition to external waves in the upper layer. For external waves in the upper layer, the solution is

$$\hat{w}_2 = B_2 \, e^{-\mu_2 z} \tag{5.9}$$

The reflection coefficient is readily found to be

$$r = \left| \frac{\lambda_1 - i\mu_2}{\lambda_1 + i\mu_2} \right|^2 = 1 \tag{5.10}$$

which corresponds to total reflection, in agreement with the deductions above.

It will be noted that (5.8) is identic with the formula for the reflection coefficient in optics, if λ is interpreted as the refraction index. The case of total reflection (5.10) may be compared with radio waves reflected from a layer with plasma frequency higher than the wave frequency.

In case of internal waves in the upper layer the energy is reflected from the interface itself, whereas the case of external waves in the upper layer, the energy is reflected from the entire upper layer. Thus in order that strict total reflection shall take place, the upper layer must be infinite vertical extent, as in the present model. To apply the notion of reflection to wave-motions in the atmosphere, it is important to examine the reflection from a layer of finite depth. We are thus led to study the behaviour of the wave motion in a three-layer model.

b. Three-layer model. We shall now consider a three-layer model with l^2 constant in each layer. Let subscript 1 refer to the lower layer, subscript 2 to the middle layer and subscript 3 to the upper layer. It has been demonstrated above that if the waves are external in the upper layer, total reflection takes place. This case therefore needs no further consideration. Furthermore, as in the previous section, we assume that the wave in the lower layer is of the internal type. Hence, in this section $k^2 < l_1^2$ and $k^2 < l_3^2$.

Choosing in the upper layer the solution which represents energy flux upwards, the solution of (3.7) in the three layers may now be written

$$\hat{w}_1 = A_1 e^{i\lambda_1 z} + B_1 e^{-\lambda_1 z} \tag{5.11}$$

$$\left.\begin{aligned} \hat{w}_2 &= A_2 e^{i\lambda_2 z} + B_2 e^{-i\lambda_2 z} \qquad && \text{if } k^2 < l_2^2 \\ \hat{w}_2 &= A_2 e^{\mu_2 z} + B_2 e^{-\mu_2 z} \qquad && \text{if } k^2 > l_2^2 \end{aligned}\right\} \tag{5.12}$$

$$\hat{w}_3 = A_3 e^{i\lambda_3 z} \tag{5.13}$$

where A_1 is the amplitude of the incident wave and B_1 the amplitude of the reflected wave. Applying the boundary conditions at the interfaces, A_2, B_2 and A_3 may be eliminated and we end up with a relation between A_1 and B_1, from which the reflection coefficient is obtained:

$$r = \frac{\lambda_2^2 (\lambda_1 - \lambda_3)^2 - (\lambda_2^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2)\sin^2 \lambda_2 h}{\lambda_2^2 (\lambda_1 + \lambda_3)^2 - (\lambda_2^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2)\sin^2 \lambda_2 h} \qquad \text{if } k^2 < l_2^2 \tag{5.14}$$

$$r = \frac{\mu_2^2 (\lambda_1 - \lambda_3)^2 + (\mu_2^2 + \lambda_1^2)(\lambda_3^2 + \mu_2^2)\,\mathrm{Sin}_2\, \mu_2 h}{\mu_2^2 (\lambda_1 + \lambda_3)^2 + (\mu_2^2 + \lambda_1^2)(\lambda_3^2 + \mu_2^2)\,\mathrm{Sin}^2 \mu_2 h} \qquad \text{if } k^2 > l_2^2 \tag{5.15}$$

It will be seen that (5.14) is identic with the formula for the reflection coefficient in optics if λ is interpreted as the refraction index. The formula (5.15) corresponds to the reflection coefficient for radio waves in a three-layer model where the plasma frequency in the middle layer is higher than the wave frequency.

In order to obtain a formula for the reflection coefficient suitable for graphical representation, we introduce

$$\varkappa = \frac{k^2 - l_2^2}{l_1^2 - l_2^2}, \quad a = \frac{l_3^2 - l_2^2}{l_1^2 - l_2^2}, \quad b = h\sqrt{l_1^2 - l_2^2} \tag{5.16}$$

assuming $l_1^2 > l_2^2$.

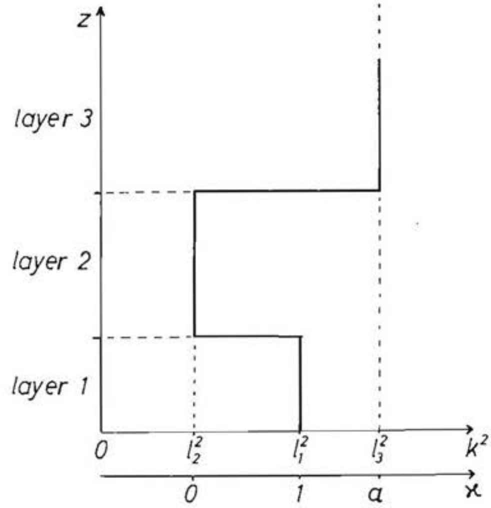

Fig. 1. The three-layer model.

The interpretation of $\varkappa$ and a will be clear by inspection of Fig. 1. The reflection coefficient now takes the form

$$r = \frac{\dfrac{\sin^2 b\sqrt{-\varkappa}}{-\varkappa} + \dfrac{(\sqrt{1-\varkappa} - \sqrt{a-\varkappa})^2}{a}}{\dfrac{\sin^2 b\sqrt{-\varkappa}}{-\varkappa} + \dfrac{(\sqrt{1-\varkappa} + \sqrt{a-\varkappa})^2}{a}} \quad \text{if } \varkappa < 0 \tag{5.17}$$

$$r = \frac{\dfrac{\operatorname{Sin}^2 b\sqrt{\varkappa}}{\varkappa} + \dfrac{(\sqrt{1-\varkappa} - \sqrt{a-\varkappa})^2}{a}}{\dfrac{\operatorname{Sin}^2 b\sqrt{\varkappa}}{\varkappa} + \dfrac{(\sqrt{1-\varkappa} + \sqrt{a-\varkappa})^2}{a}} \quad \text{if } \varkappa > 0 \tag{5.18}$$

Figs. 2, 3 and 4 show the reflection coefficient r as a function of $\varkappa$ (i.e. of wave number), for selected values of a and b. It should be kept in mind that $\varkappa > 0$ and $\varkappa < 0$ correspond to external and internal waves, respectively. The curves reveal that the reflection power of the middle layer is mainly determined by the parameter b. In Fig. 2, $b = 1$, and the amount of energy reflected is relatively small, except for $\varkappa$-values close to those corresponding to external waves in the lowest or upper layer. For increasing values of b the reflection power for positive $\varkappa$ increases, as revealed by Figs. 3 and 4. For negative values of $\varkappa$, r is an oscillating function of $\varkappa$, the oscillations becoming more rapid as b increases.

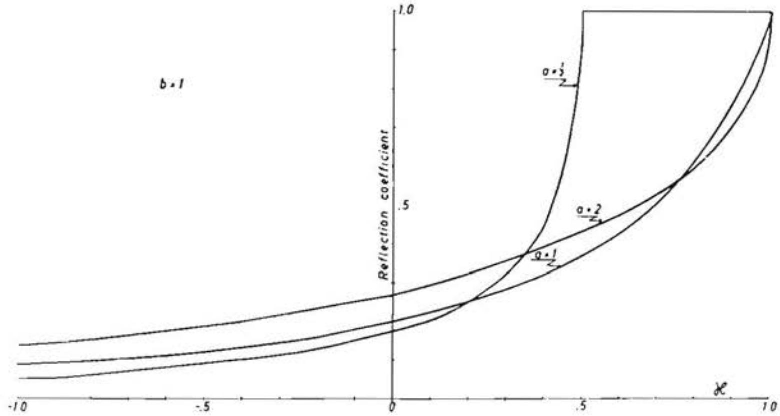

Fig. 2. The reflection coefficient in the three-layer model as function of $\varkappa$ ($a = \frac{1}{2}$, 1, 2, $b = 1$).

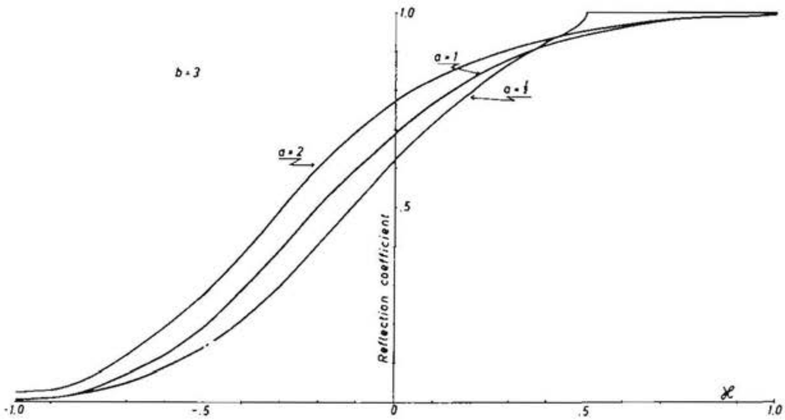

Fig. 3. The reflection coefficient in the three-layer model as a function of $\varkappa$ ($a = \frac{1}{2}$, 1, 2, $b = 3$).

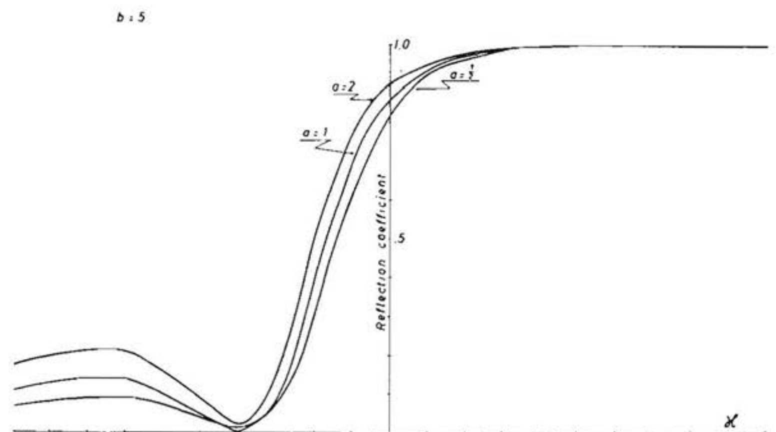

Fig. 4. The reflection coefficient in the three-layer model as function of $\varkappa$ ($a = \frac{1}{2}$, 1, 2, $b = 5$).

6. An illustrative example. We shall now make an attempt to apply the concepts of transmission and reflection to mountain waves, and shall take as an example a case studied by the Sierra Wave Project (HOLMBOE and KLIEFORTH, 1957). The actual distribution of l^2 up to 11 km, as shown by the full curve in Fig. 5, is based on observed winds and temperatures on a day when pronounced mountain waves occurred on the lee side of Sierra Nevada. Above 11 km a probable distribution of l^2 has been computed on the basis of mean curves of wind and stability for the season, and is shown by dashed curve in Fig. 5. The wave system for this distribution of l^2 has been examined by PALM and FOLDVIK (1960). It will be seen that l^2 decreases with height in the troposphere as a result of the increase of westerly wind with height. Above the tropopause, l^2 increases due to increased static stability and decreases because of increasing wind speed up to 50 km.

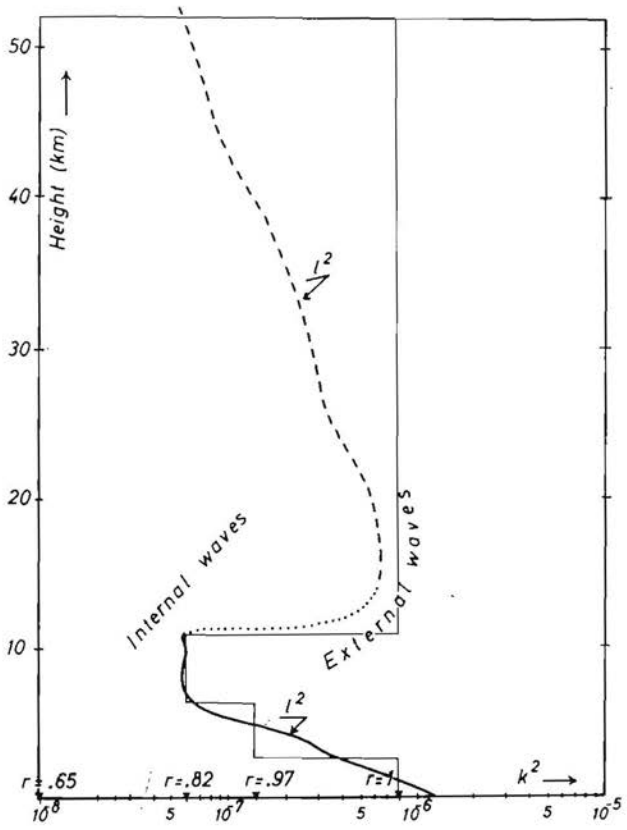

Fig. 5. l^2 as function of height. Solid line is computed from observed winds and temperatures in the Sierra Nevada regions, dashed line from the mean values of wind and temperature, and dotted line is obtained by interpolation. The l^2-curve separates the regions of internal and external waves. Thin line shows the values of l^2 for a four-layer model.

In order to examine the reflection of wave energy flux due to the low values of l^2 in the upper troposphere, we have calculated the reflection coefficient for a hypothetical discontinuous distribution of l^2 shown by the thin solid line in Fig. 5. It should be noted that this distribution does not approximate the values of l^2 in the higher stratosphere, and consequently does not give reflection from these levels. One may assume, however, that the step-curve will yield realistic values for the reflection from the troposphere.

The reflection coefficient was computed from this four-layer model for four different wave numbers. The result is shown in the following Table 1, which also gives the wave lengths corresponding to the four selected wave numbers.

Table 1.

$k^2(m^{-2})$	1×10^{-8}	6×10^{-8}	$1{,}4 \times 10^{-7}$	$0{,}8 \times 10^{-6}$
L km	63	26	17	7
r	0,65	0,82	0,97	1

The values of r are also shown in Fig. 5.

Waves shorter than about 26 km ($k^2 > 6 \times 10^{-8} m^{-2}$) are of the external type in the upper troposphere. These waves are seen to be very effectively reflected by the upper troposphere; for waves shorter than about 17 km, the reflection is almost total. These short waves are therefore confined to the lower and middle troposphere, and very little energy is transmitted into the stratosphere. When such total reflection occurs, the reflected wave will interfere with the incident wave and give horizontal nodal planes where $w = 0$. Resonance waves will occur for those wave lengths, for which the phase of the reflected wave is such that ground becomes a nodal plane; the wave lengths of the resonance waves may be calculated on this basis.

It follows that the short waves which are trapped in the lower troposphere must be completely determined by the distribution of l^2 in the troposphere and thus be independent of the conditions which prevail at higher elevations. This was also found by the case by PALM and FOLDVIK (1960).

Waves longer than 26 km ($k^2 > 6 \times 10^{-8} m^{-2}$) are seen to be of the internal type throughout the troposphere and up to very high elevations. For these waves, the reflection coefficient for the troposphere was found to vary between 0,65 and 0,82, depending upon their wave length. A considerable part of their energy will therefore be transmitted into the stratosphere. The motion in the stratosphere will thus consist primarily of such longer waves. These waves will depend upon the distribution of wind and stability up to very high altitudes.

7. Remarks concerning the use of the radiation condition and the possible propagation of waves to very high altitudes. It was assumed in section 5 that l^2 was constant in the uppermost layer, so that no reflection takes place there. The radiation condition then requires that the solution for internal waves is of the form (5.2). Such a mathematical model implies that the wave energy is lost to an infinite outer space. This is of course not possible in the real atmosphere; in fact, it is easy to show on the basis of the model that the ratio between perturbation pressure and total pressure increases upward as $\varrho_0^{-\frac{1}{2}}$. The ratio will thus increase by a factor of 100 from the ground up to about 70 km, and by a factor of 1000 from the ground up to about 100 km. It is well known that the linear theory must for this reason break down at sufficiently high levels. Many mountain waves have a considerable amplitude already in the troposphere so that we must expect the breakdown to occur in the upper mesosphere and lower ionosphere. At these levels, non-linear terms must become important, and the wave energy will be transferred to other wave lengths. It is conceivable that the wave energy can be converted into turbulent energy and ultimately into heat.

It thus seems plausible to assume that these non-linear effects should cause an absorption of wave energy, and not a reflection. If this is so, we are justified in applying the radiation condition at levels just below the absorbing layers, and the motion below will take place just as though the wave energy were radiated into an infinite space.

The amount of energy absorbed by the upper atmosphere in this manner depends upon the reflection caused by the atmosphere below (i.e. upon the distribution of l^2 from the surface to the level where the energy is absorbed) and on the wind profile. It has been shown above that the vertical flux of wave energy varies with height in proportion to U. A necessary condition for application of the radiation condition seems to be that U stays positive at all levels from the surface to the layers where the energy is absorbed, up to 80—90 km, say. According to MURGATROYD (1957), this happens normally only in winter when the westerlies prevail from the surface and up to 90—100 km. At other times of the year, the westerlies change into easterlies at much lower levels. To the authors' knowledge, the problem of the steady-state mountain waves has not been solved for the latter case. The level where $U = 0$ represents a singularity in the differential equation ($l^2 = \infty$) which in a statically stable atmosphere cannot even be removed by addition of friction; it is likely that the problem requires non-linear treatment. However, below the singular level, the upward flux of wave energy must vary with height in proportion to U and must therefore tend to zero when the singular level is approached. On this basis, transfer of wave energy from the lower westerlies to the upper easterlies seems very unlikely.

We must therefore conclude that transfer of energy by stationary mountain waves into the high atmosphere is probably possible only in mid-winter. On the other hand, non-steady gravity waves, which may be generated when the wind speed over mountaneous areas varies with time, may reach the ionosphere at all times of the year.

8. Definition of reflection coefficient when l^2 varies continuously with height. In a layer of constant l^2, the reflection coefficient (4.7) may be expressed as

$$\mathrm{r} = \left| \frac{i\lambda - s}{i\lambda + s} \right|^2 \tag{8.1}$$

where

$$s = \frac{\hat{w}_z}{\hat{w}} \tag{8.2}$$

This formula may be used to define the reflection coefficient also in layers where l^2 is not constant, provided $k^2 < l^2$. To justify this, we note that the vertical energy flux (eq. 3.10) depends only on $\hat{w}$ and $\hat{w}_z$; furthermore we note that A and B in formula (4.4) can always be chosen such that at any given level $\hat{w}$ and $\hat{w}_z$ obtained from this formula will be equal to $\hat{w}$ and $\hat{w}_z$ for the exact solution.

Although r is defined only at levels where $k^2 < l^2$, the function s is defined at all levels. The change of s with height is governed by the Ricatti equation:

$$s_z = (k^2 - l^2) - s^2 = -\lambda^2 - s^2 = \mu^2 - s^2 \tag{8.3}$$

This is verified by substitution of (8.2) and application of (3.7).

Since $\hat{w}$ and $\hat{w}_z$ are continuous even where l^2 changes abruptly, it follows that s is continuous everywhere except at nodal lines where $\hat{w} = 0$.

If it can be assumed that the solution (5.2) or (5.3) holds at a sufficiently high level, we obtain at this level

$$\begin{aligned} s &= -\mu \qquad \text{when } k^2 > l^2, \\ s &= i\lambda \qquad \text{when } k^2 < l^2. \end{aligned} \tag{8.4}$$

Starting with these values as boundary conditions, s may be determined at lower levels by numerical integration of (8.3), and r is then obtained from (8.1). For a resonance wave, s becomes infinite at the surface.

CHAPTER II

LONG QUASI-STATIC STATIONARY WAVES.

9. Perturbation and energy equations for stationary quasi-static waves in a non-uniform basic current.

a. Basic current. For simplicity, the motion will be referred to a beta-plane with horizontal Cartesian coordinates x east, y north. Pressure p is used as vertical coordinate. The basic current is a straight westerly flow with velocity $U(y,p)$. Geostrophic and hydrostatic equilibrium is expressed by

$$fU = -\Phi_y \tag{9.1}$$

$$\varrho_o^{-1} = - \Phi_p \tag{9.2}$$

Here f is the Coriolis parameter, $\varrho_0(y, p)$ the equilibrium density and $\Phi(y,p)$ the equilibrium geopotential. Derivatives with respect to x, y and t are understood to be taken at constant p.

Elimination of Φ between these equations gives the thermal wind equation

$$f\, U_p = (\varrho_0{}^{-1})_y \tag{9.3}$$

b. Perturbation equations. Let u, v denote the perturbation velocities, ω the individual rate of change of pressure, and φ the perturbation geopotential. The perturbation equations for stationary, adiabatic and quasi-static motion without friction then are:

$$Uu_x + (U_y - f)\, v + U_p\omega + \varphi_x = 0 \tag{9.4}$$

$$fu + Uv_x + \varphi_y = 0 \tag{9.5}$$

$$- fU_pv + \sigma\omega + U\varphi_{px} = 0 \tag{9.6}$$

$$u_x + v_y + \omega_p = 0 \tag{9.7}$$

The coefficient σ appearing in the thermodynamic energy equation (9.6) is a measure of static stability, defined by

$$\sigma = - \frac{\theta_p}{\varrho_0\theta} = \left(\frac{\nu_0}{g\varrho_0}\right)^2 = \left(\frac{RT\nu_0}{gp}\right)^2 \tag{9.8}$$

were θ is equilibrium potential temperature, T temperature, R gas constant, and

$$\nu_0 = \sqrt{\frac{g}{\theta}\frac{\partial\theta}{\partial z}} \tag{9.9}$$

is the buoyancy frequency. The basic current is assumed to be statically stable ($\sigma > 0$).

10. The flow of wave energy. Multiplying (9.4) by u, (9.5) by v, (9.6) by $\sigma^{-1}\varphi_p$, (9.7) by φ and adding, we find

$$(EU + \varphi u)_x + (\varphi v)_y + (\varphi\omega)_p = - U_yuv - U_pu\omega + \frac{fU_p}{\sigma}\, v\, \varphi_p \tag{10.1}$$

Here

$$\mathrm{E} = \frac{1}{2}\,(u^2 + v^2) + \frac{1}{2\sigma}\varphi_p^2 \tag{10.2}$$

is the wave energy per unit mass (kinetic plus available potential energy). The left-hand side of (10.1) is the wave-energy flux divergence, and the terms on the right are source terms; the two first of these represent conversion of kinetic energy of the basic current into wave energy, and the last source term represents conversion of available potential energy of the basic current into wave energy.

Since x corresponds to longitude, the motion will be considered periodic in x. Denoting the average over one period (zonal average) by a bar, we obtain from (10.1) by averaging, since $\overline{\frac{\partial}{\partial x}} = 0$,

$$(\overline{\varphi v})_y + (\overline{\varphi \omega})_p = - U_y \overline{uv} - U_p \overline{u\omega} + \frac{fU_p}{\sigma} \overline{v\varphi_p} \tag{10.3}$$

We shall now derive a relation between the meridional fluxes of energy and momentum. Elimination of ω between (9.4) and (9.6) gives

$$[Uu + \varphi - \sigma^{-1} UU_p \varphi_p]_x = [f - U_y - \sigma^{-1} f U_p^2] v \tag{10.4}$$

Multiplication of this equation by $[Uu + \varphi - \sigma^{-1} UU_p \varphi_p]$ and averaging gives

$$\overline{\varphi v} = U [\sigma^{-1} U_p \overline{v\varphi_p} - \overline{uv}] \tag{10.5}$$

Next we multiply (9.4) by $(Uu + \varphi)$ and average:

$$(f - U_y)(U\overline{uv} + \overline{\varphi v}) = U_p (U\overline{u\omega} + \overline{\varphi\omega}) \tag{10.6}$$

Elimination of $\overline{\varphi v}$ between (10.5) and (10.6) gives

$$\overline{\varphi\omega} = U [\sigma^{-1}(f - U_y) \overline{v\varphi_p} - \overline{u\omega}] \tag{10.7}$$

The formulae (10.5) and (10.6) relate the horizontal and vertical wave-energy fluxes to the meridional eddyflux of sensible heat and to the meridional and vertical flux of momentum. The first term in the brackets of (10.5) gives a meridional wave-energy flux in the same direction as the flux of sensible heat when $|U|$ increases with height, and in the opposite direction if $|U|$ decreases with height.

The first term in the brackets of (10.7) gives an upward flux of wave energy if the heat flux is positive (northward), corresponding to a westward tilt of the waves with height, and a downward energy transport if the heat flux is negative (assuming U, $(f - U_y)$ and σ positive). The last terms of the brackets of (10.5) and (10.7) represent horizontal and vertical energy fluxes in directions opposite to the corresponding momentum fluxes.

When the expressions (10.5) and (10.7) for the energy fluxes are entered into the energy equation (10.3), we obtain

$$\frac{\partial}{\partial y}[\sigma^{-1} U_p \overline{v\varphi_p} - \overline{uv}] + \frac{\partial}{\partial p}[\sigma^{-1}(f - U_y) \overline{v\varphi_p} - \overline{u\omega}] = 0 \tag{10.8}$$

We may therefore write

$$\sigma^{-1} U_p \overline{v\varphi_p} - \overline{uv} = \psi_p, \qquad \sigma^{-1}(f - U_y) \overline{v\varphi_p} - \overline{u\omega} = -\psi_y \tag{10.9}$$

Thus we have arrived at the result that the components of the meridional energy flux (10.5) and (10.9) can be written in the form

$$\overline{\varphi v} = U\psi_p, \qquad \overline{\varphi\omega} = -U\psi_y \tag{10.10}$$

The curves $\psi =$ constant are streamlines for the flow of wave energy in the meridional plane. The flux of wave energy in a channel between two adjacent streamlines is not constant, but varies along the channel in proportion to U as indicated by arrows in Fig. 6. Thus if energy flows towards increasing values of U, the wave energy flux is divergent and wave energy is being produced from the energy of the basic flow through the source terms on the right of (10.3). The opposite energy conversion takes place in regions where wave energy flows towards smaller values of U. The distribution of the energy source terms on the right of (10.3) is thus strongly restricted and related to the

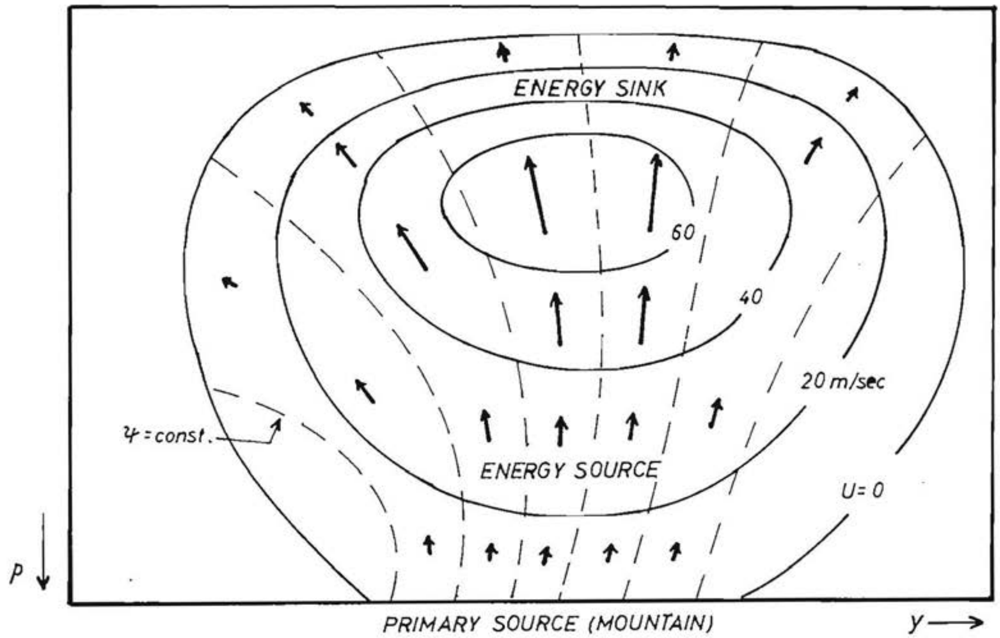

Fig. 6. Meridional flow of wave energy. Solid lines: isotachs of zonal wind. Dashed lines: Wave energy streamlines. Arrows indicate wave energy flux.

distribution of zonal wind in the meridional plane. We notice in particular that wave energy flows along each streamline in the same direction throughout the whole length of the streamline within the region of positive U in the meridional plane.

The differential equations for stationary wave motions have a singularity in points where $U = 0$. The wave-energy flux tends towards zero as one approaches a singular line $U = 0$ in the meridional plane, so that it seems that wave energy cannot be transported across such a line.

So far we have only made use of the quasi-static approximation, and the results obtained apply to quasi-static gravity-inertia waves as well as planetary waves.

If we restrict our consideration to waves long enough so that their period is considerably longer than the half pendulum day, we may apply the quasi-geostrophic approximation. This may be done by replacing the convective acceleration terms in (9.4) and (9.5), as well as the meridional temperature advection term in (9.6) by their geostrophic values. Proceeding as before, we now obtain instead of (10.5) and (10.7) the following expressions for the flux of wave energy:

$$\overline{\varphi v} = U\psi_p = Uf^{-2}\,\overline{\varphi_x\varphi_y} = -\,U\overline{uv} \tag{10.11}$$

$$\overline{\varphi\omega} = -\,U\psi_y = U\sigma^{-1}\,\overline{\varphi_x\varphi_p} = -\,U\frac{fR}{\sigma p}\,\overline{Tv} \tag{10.12}$$

It follows from (10.12) that long quasi-geostrophic stationary waves will transport wave energy upwards if the flux of sensible heat is directed northward, i.e. if the waves tilt westward with height (assuming U and σ positive). In this respect, quasi-geostrophic waves are similar to short gravity waves, although the mechanism connecting wave tilt and energy flux is quite different for the two types of waves.

11. The long stationary waves in the atmosphere. Normal weather maps show a predominantly zonal circulation with a superimposed system of stationary long wave disturbances. These are generally believed to be set up by the large mountain barriers (CHARNEY and A. ELIASSEN 1949) and by geographically fixed heat sources and sinks, mainly at low levels (SMAGORINSKY 1953). From these primary sources of wave energy, located mainly within the zone of westerlies, wave energy must flow upward and possibly also to the sides (north and south) in the meridional plane, within the region of westerly flow.

It is important to note, however, that the stationary motion displayed by the normal maps satisfy time averaged equations, which contain a number of eddy terms representing the effect of transient motions upon the stationary waves. This influence cannot be assessed quantitatively at the present time, since the necessary statistics have not yet been computed (as far as the authors know). Considering that the basic flow is normally unstable, it does not seem unlikely that part of the wave energy formed by the primary source will be absorbed by transient growing disturbances. There may thus be important sinks (and perhaps also sources) of the energy of the stationary waves, other than those appearing on the right (10.3). With this reservation, we may nevertheless attempt to apply the theoretical considerations of the preceding section to the long stationary waves in the atmosphere.

E. ELIASEN (1958) has performed a Fourier analysis of the stationary disturbances up to 500 mb. He found a pronounced westward tilt of the waves with wave numbers 1, 2 and 3 in January, corresponding to an upward flux of wave energy according to (10.12). His calculations for July gave an eastward tilt, however, corresponding to a downward flux of wave energy, in apparent contradiction to the hypothesis that the primary source is located at or near the ground. The contradiction is resolved if it can be assumed that the geographically fixed heat sources and sinks are important in the layer from the ground and as high as 500 mb., because (10.12) is no longer true in a layer where heating is significant. In such a layer additional terms will appear on the right of (10.12), as will be seen by repeating the derivation of this equation with a stationary heat source term added on the right of (9.6). The vertical flux of wave energy may therefore still be upward at all seasons, as required by the theory.

The horizontal flux of wave energy should, according to (10.11), be directed opposite to the "standing" flux of momentum $\overline{uv}$. According to E. ELIASEN (1958), the horizontal momentum flux is predominantly northward up to 50°N for wave numbers 1, 2 and 3, with the largest values in winter. This agrees with values given by STARR and WHITE (1952) for 31°N. Their figures are indicative of a significant southward flux of wave energy in the troposphere from the middle latitudes into the subtropics.

Another part of the wave energy may flow into the stratosphere, and may penetrate to great heights in the winter season, when the westerly flow extends to the lower ionosphere.

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