Hydrodynamic Stability of Parallel Flow of Inviscid Fluid

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I. Introduction

Stability of parallel flow of inviscid fluid was first studied in the last third of the nineteenth century, notably by Helmholtz [1], Kelvin [2] and Rayleigh [3]. They considered the inertial instability of a homogeneous incompressible fluid, and its modification—Kelvin-Helmholtz instability—when there is variation of density of the fluid transverse to the basic flow. Subsequent authors have continued this work and gone on to consider other modifications of inertial instability, such as those due to compressibility of the fluid, to rotation of the system, and to magnetohydrodynamic effects. There is a wide class of such problems, which have been considered piecemeal by research workers ranging from sanitary engineers to astrophysicists. General and particular results by hundreds of authors have appeared in dozens of journals, and there has been much duplication of work on the same mathematical problems in different physical contexts. Our approach will be the fluid dynamical one of studying the phenomenon of instability rather than its practical applications or natural occurrence. In this way we shall emphasize the unity of the various problems discussed.

We begin in Section II with the fundamental theory of inertial instability of plane parallel flow of inviscid fluid. Euler's equations of motion are linearized with respect to small perturbations of the basic parallel flow. We first discuss the method of normal modes, whereby it is assumed that each perturbation can be resolved into dynamically-independent wave components. A linear eigenvalue problem is then posed to determine the typical component. Squire's theorem shows that the most rapidly growing component in an unstable flow is two-dimensional. Thus, in seeking a criterion for instability, one may assume that the typical wave component is two-dimensional, and thereby simplify the eigenvalue problem. The eigenvalue problem is singular, and the singularity admits solutions with discontinuous derivatives and a continuous spectrum of eigenvalues in addition to well-behaved solutions with a discrete spectrum. All these solutions are necessary to form a complete set to represent an arbitrary initial disturbance. The eigenvalue problem for an inviscid fluid is related to that for a slightly viscous fluid, though the two are formally independent, this relation being discussed briefly. Many general properties of the eigenvalue problem are given, the most notable being Rayleigh's necessary condition for instability that the basic velocity profile has a point of inflection. We describe these properties both mathematically and physically before giving details of stability characteristics for particular velocity profiles.

In Section II.3 we discuss the stability problem from the point of view of the initial-value problem posed by the linearized equations of motion. Rayleigh's inflection point theorem is reconsidered, and obtained in a quite general form. Somewhat more detailed results for particular velocity profiles can be obtained by taking the Laplace transform with respect to time of the equations, and in this way the relation between the initial-value problem and the equivalent normal mode solution is brought out explicitly.

In Section III we briefly pose the analogous eigenvalue problems when various external force fields act on the inertial instability. We also give the solutions for two important basic flows, those of static equilibrium and of a vortex sheet. Unbounded disturbances of static equilibrium are neutrally-stable waves, and those of a vortex sheet are instabilities which in some cases may be stabilized by the force field. We add a survey of results and literature for each force field. The fields we consider are compressibility in a fluid of variable temperature, buoyancy due to variations of density, Coriolis force due to rotation of the system in which the parallel flow is placed, variations of this Coriolis force transverse to the flow, and magnetohydrodynamic forces.

The similarity of all these problems is brought out in Section IV. Mechanisms of instability are discussed, and analyzed dimensionally to give some general stability characteristics. Some of these dimensional arguments are elaborated by physical ones.

Lack of space and time prevent our treating in detail the case of each force field, so we have picked the single case of buoyancy due to variations of density for detailed study in Section V. This case is as typical as any, and has the advantages of practical importance and of advanced theoretical development. We discuss this case in Section V much as we did inertial instability in Section II. Finally, in Section VI, we give some results on non-planar parallel flows.

II. Inertial Instability

1. *Eigenvalue Problem for Inertial Modes*

The first work on instability of parallel flow seems to be a physical remark of Helmholtz [**1**] in 1868, though he and others had studied neutrally-stable waves previously. In 1871 Kelvin [**2**] gave a complete analysis of

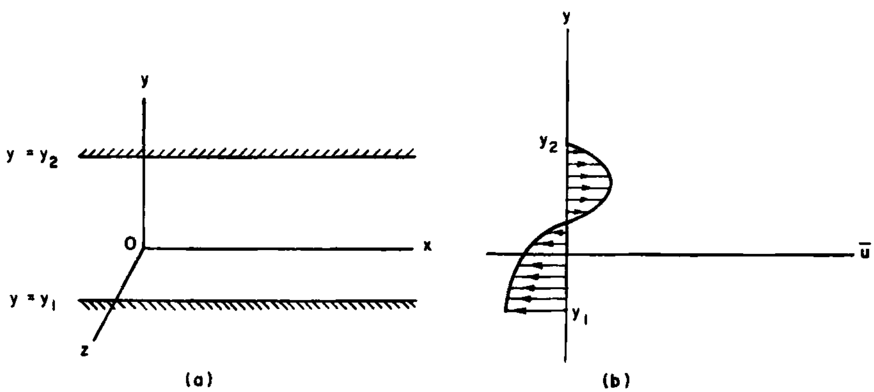

Fig. 1. (a) Channel of flow. (b) Velocity profile of basic flow.

the instability of a vortex sheet of inviscid incompressible fluid, allowing for surface tension and a discontinuity of density at the sheet. Later Rayleigh [cf. **3**] wrote a series of fundamental papers on hydrodynamic stability, and by the beginning of this century the theory was well formed.

A wide range of problems has been solved since, the theory has been extended to viscous fluids, and applications of hydrodynamic stability are numerous. This abundance of work makes a chronological account impractical, so we develop the subject logically, referring to authors where appropriate. Strictly speaking, a logical account should begin with the formulation of the problem for an arbitrary, not necessarily parallel, basic flow. However, since the vast majority of work in stability theory has been on the parallel flow case and only a few general results are known otherwise, we shall defer our remarks on non-parallel flow to Section II.3, and begin here in the traditional manner.

We consider the stability of a basic plane parallel flow of inviscid fluid with given velocity

$$\bar{\mathbf{u}}_* = (\bar{u}_*(y_*),0,0) \qquad (y_{1*} \leqslant y_* \leqslant y_{2*}).$$

This is illustrated in Figure 1. We take the flow bounded by the two planes $y_* = y_{1*}, y_{2*}$ parallel to the flow. Each of these planes is either rigid or free, so that either the normal velocity of the fluid is zero or the pressure constant there. However, we allow one or both of the planes to be at infinity. In this section we suppose the fluid is incompressible and homogeneous.

It is convenient to choose some velocity scale V of the basic flow $\bar{u}_*(y_*)$ and some length scale L of its transverse variations in order to introduce dimensionless variables. Thus we define in the usual way the dimensionless time, position vector, velocity, basic velocity and pressure as

$$t \equiv t_* V/L, \qquad \mathbf{r} \equiv \mathbf{r}_*/L, \qquad \mathbf{u} \equiv \mathbf{u}_*/V, \qquad \bar{\mathbf{u}} \equiv (\bar{u},0,0) \equiv \bar{\mathbf{u}}_*/V,$$
$$p \equiv p_*/\rho V^2$$

respectively, where ρ is the density of the fluid. Now the Euler equations of motion can be written

$$\partial\mathbf{u}/\partial t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p.$$

Also the equations of continuity and incompressibility give

$$\nabla\cdot\mathbf{u} = 0.$$

The basic flow satisfies these equations and the boundary conditions with uniform pressure $\bar{p}$. To study the instability of this flow one puts

$$\mathbf{u}(\mathbf{r},t) = \bar{\mathbf{u}}(y) + \mathbf{u}'(\mathbf{r},t), \qquad p(\mathbf{r},t) = \bar{p} + p'(\mathbf{r},t)$$

and neglects quadratic terms in the perturbations, denoted by primes. Linearizing the equations of motion in this way for small perturbations, we find

$$\begin{aligned}
\partial u'/\partial t + \bar{u}\partial u'/\partial x + v' d\bar{u}/dy &= -\partial p'/\partial x, \\
\partial v'/\partial t + \bar{u}\partial v'/\partial x &= -\partial p'/\partial y, \\
\partial w'/\partial t + \bar{u}\partial w'/\partial x &= -\partial p'/\partial z, \\
\partial u'/\partial x + \partial v'/\partial y + \partial w'/\partial z &= 0.
\end{aligned}$$

The stability problem involves the growth of an arbitrary infinitesimal perturbation, but it has been generally assumed that such perturbations can be resolved into independent wave-like components. Each component is supposed to satisfy the linearized equations of motion and boundary conditions separately. So we consider a typical wave component with

$$\mathbf{u}' = \hat{\mathbf{u}}(y) \exp [i(\alpha x + \gamma z - \alpha ct)], \qquad p' = \hat{p}(y) \exp [i(\alpha x + \gamma z - \alpha ct)]$$

for some functions $\hat{\mathbf{u}}$, $\hat{p}$, where α, γ are real wave-numbers and $c = c_r + ic_i$ is a complex velocity. It is to be understood that real parts are to be taken to get physical quantities, this being permissible for the linear problem. Thus each component travels as a wave in the direction of $(\alpha,0,\gamma)$ with phase speed $\alpha c_r/(\alpha^2 + \gamma^2)^{1/2}$ and grows or dies away in time like $\exp (\alpha c_i t)$. Therefore the wave is unstable when $\alpha c_i > 0$ and stable when $\alpha c_i \leqslant 0$. It is said to be neutrally stable when $\alpha c_i = 0$.

This assumption that any disturbance can be represented by wave components, according to the method of normal modes, serves to separate the variables and reduce the linearized equations of motion from partial to ordinary differential equations. They now become

$$i\alpha(\bar{u} - c)\hat{u} + \hat{v}d\bar{u}/dy = - i\alpha\hat{p}, \tag{2.1}$$

$$i\alpha(\bar{u} - c)\hat{v} = - d\hat{p}/dy, \tag{2.2}$$

$$i\alpha(\bar{u} - c)\hat{w} = - i\gamma\hat{p}, \tag{2.3}$$

$$i\alpha\hat{u} + d\hat{v}/dy + i\gamma\hat{w} = 0. \tag{2.4}$$

The boundary conditions are that $\hat{v}$ or $\hat{p}$ vanishes on each boundary according as it is rigid or free. We shall usually consider rigid walls, so that

$$\hat{v} = 0 \qquad (y = y_1, y_2). \tag{2.5}$$

These four equations and the two-point boundary conditions in general pose an eigenvalue problem to determine an eigenvalue or values c for given $\alpha,\gamma,\bar{u}(y)$. If $\alpha c_i \leqslant 0$ for every wave-number vector $(\alpha,0,\gamma)$, then the basic flow $\bar{u}(y)$ is stable to any wave disturbance, and is said to be stable.

Some care must be taken in this method of normal modes because of the occurrence of "improper" modes associated with concentrated layers of vorticity and of the corresponding continuous part of the c-spectrum, as well as the occurrence of ordinary stable or unstable waves with the discrete c-spectrum. The singularity of the equations where $\bar{u} = c$ leads to a continuous spectrum of eigenvalues c, whose eigenfunctions can be found in terms of generalized functions. These real eigenvalues are in addition to the discrete spectrum of eigenvalues, which may be real or complex. The eigenfunctions for all these eigenvalues are needed to form a complete set to represent an arbitrary initial disturbance with bounded vorticity. The initial disturbance may be represented as an integral of components, some of which separately have infinite vorticities. The existence of the continuous spectrum was known to Rayleigh [3, pp. 391–400] but its importance and connection with the initial-value problem has been widely appreciated only recently. We shall discuss this matter more fully in Section II.3, and assume the method of normal modes meanwhile.

Squire [4] has proved that for each unstable three-dimensional wave component there is a more unstable two-dimensional one (i.e. one with $\gamma = 0$). His proof runs essentially as follows for inviscid fluid. First define

$$\tilde{\alpha}^2 = \alpha^2 + \gamma^2, \qquad \tilde{\alpha}\tilde{u} = \alpha\hat{u} + \gamma\hat{w}, \qquad \tilde{p}/\tilde{\alpha} = \hat{p}/\alpha, \qquad \tilde{v} = \hat{v}, \qquad \tilde{c} = c.$$

Then equation (2.1) plus the product of γ/α and (2.3), and equations (2.2), (2.4) give

$$i\tilde{\alpha}(\bar{u} - \tilde{c})\tilde{u} + \tilde{v}\,d\bar{u}/dy = -\,i\tilde{\alpha}\tilde{p}, \tag{2.6}$$

$$i\tilde{\alpha}(\bar{u} - \tilde{c})\tilde{v} = -\,d\tilde{p}/dy, \tag{2.7}$$

$$i\tilde{\alpha}\tilde{u} + d\tilde{v}/dy = 0; \tag{2.8}$$

and boundary conditions (2.5) give

$$\tilde{v} = 0 \qquad (y = y_1, y_2). \tag{2.9}$$

It can be seen that the eigenvalue problem (2.6)–(2.9) has the same form as (2.1)–(2.5) when $\gamma = 0$, $w = 0$. Thus $\tilde{c}(\tilde{\alpha})$ has the same functional form as $c(\alpha)$ when $\gamma = 0$. Thus to each three-dimensional wave, growing in time like $\exp(\alpha c_i t)$ for given α,γ, there corresponds a two-dimensional wave, growing like $\exp(\tilde{\alpha} c_i t)$. Now $\tilde{\alpha}^2 > \alpha^2$ if $\gamma \neq 0$. Therefore to each unstable three-dimensional wave there is a faster-growing two-dimensional one.

Lin [5, pp. 3–4] has described this result qualitatively. A three-dimensional wave travels in the direction $(\alpha,0,\gamma)$, making angle $\theta = \tan^{-1}(\gamma/\alpha)$ with the x-axis. If the coordinate frame is rotated about the y-axis so that the new $\tilde{x}$-axis is in the direction of the wave, then the basic flow has components

$$\tilde{\bar{\mathbf{u}}} = (\bar{u}(y)\cos\theta, 0, -\,\bar{u}(y)\sin\theta).$$

The wave now propagates in the $\tilde{x}$-direction and is independent of $\tilde{z}$. Further, the equations governing $\tilde{u}$, $\tilde{v}$, $\tilde{p}$ are independent of $\tilde{\bar{w}}$, $\tilde{w}$, so that we have essentially a two-dimensional wave-disturbance of a basic flow $(\bar{u}\cos\theta, 0, 0)$. Thus the velocity of the basic flow is effectively reduced by the factor $\cos\theta$, and the growth rate of the three-dimensional is less than that of a two-dimensional disturbance by the same factor.

Henceforth, in seeking a sufficient criterion for instability, we shall confine our attention to two-dimensional disturbances. Although the fastest growing small disturbances are two-dimensional, it should not be forgotten that three-dimensional disturbances may be of practical significance. In particular, three-dimensional effects are known to be important in determining the nonlinear growth of a disturbance.

With $\gamma = 0 = \hat{w}$, equation (2.4) may be integrated by use of the stream function of the disturbance,

$$\psi'(x,y,t) = \varphi(y)e^{i\alpha(x-ct)}$$

such that

$$u' = \partial\psi'/dy, \qquad v' = -\partial\psi'/\partial x,$$

i.e.

$$\hat{u} = D\varphi, \qquad \hat{v} = -i\alpha\varphi$$

where $D = d/dy$. We may also write $\bar{u} = w(y)$ without ambiguity henceforth. Then equation (2.1) gives

$$\hat{p} = \varphi Dw - (w - c)D\varphi. \tag{2.10}$$

Now equation (2.2) gives

$$(w - c)(D^2 - \alpha^2)\varphi - (D^2w)\varphi = 0, \tag{2.11}$$

which is in fact the perturbation of the vorticity equation of inviscid fluid in two-dimensional motion,

$$\frac{\partial(\Delta\psi)}{\partial t} + \frac{\partial(\Delta\psi,\psi)}{\partial(x,y)} = 0.$$

Equation (2.11) is called the *Rayleigh stability equation.* Its generalization for viscous fluid is the Orr-Sommerfeld equation,

$$(D^2 - \alpha^2)^2\varphi = i\alpha R\{(w - c)(D^2 - \alpha^2)\varphi - (D^2w)\varphi\},$$

where $R = VL/\nu$ is the Reynolds number of the basic flow of fluid of kinematic viscosity ν.

Boundary conditions (2.9) give

$$\alpha\varphi = 0 \qquad (y = y_1, y_2). \tag{2.12}$$

The eigenvalue problem comprises the singular second-order linear differential equation (2.11) and the two-point boundary conditions (2.12). The equation has two independent solutions φ_1, φ_2 which are analytic functions of y, α^2, c over domains in which the equation is non-singular throughout the field of flow, i.e. over domains in which c lies outside the range of $w(y)$ for $y_1 \leqslant y \leqslant y_2$. Thus

$$\varphi = A_1\varphi_1(y;\alpha^2,c) + A_2\varphi_2(y;\alpha^2,c)$$

for some complex constants A_1, A_2. Substitution into boundary conditions (2.12) and elimination of A_1, A_2 gives the eigenvalue relation

$$F(\alpha^2,c) = \begin{vmatrix} \varphi_1(y_1;\alpha^2,c) & \varphi_2(y_1;\alpha^2,c) \\ \varphi_1(y_2;\alpha^2,c) & \varphi_2(y_2;\alpha^2,c) \end{vmatrix} = 0.$$

In general this can be inverted to give a many-valued function $c = c(\alpha^2)$, continuous in any domain excluding the range of w.

If w or Dw is discontinuous, at y_0 say, then the pressure must be continuous at the material fluid interface with mean position $y = y_0$. Therefore, to first degree in the small perturbation,

$$[(w - c)D\varphi - (Dw)\varphi] = 0 \tag{2.13}$$

at $y = y_0$, where square brackets here and henceforth denote the "jump" of their contents at a possible discontinuity. Also the normal velocity of the fluid must be continuous at the material interface. Let this interface have equation

$$y = y_0 + \eta(x,t), \quad \text{where} \quad \eta = F_0 e^{i\alpha(x-ct)}.$$

Then

$$v' = D\eta/Dt = \partial\eta/\partial t + w\partial\eta/\partial x = i\alpha(w - c)\eta,$$

to first degree in the perturbation. But $v = i\alpha\varphi$, so it follows that

$$[\varphi/(w - c)] = 0 \tag{2.14}$$

at $y = y_0$.

Conditions (2.13), (2.14) can alternatively be proved mathematically. We may take $y_0 = 0$ without loss of generality. Then stability equation (2.11) has integral

$$[(w - c)D\varphi - (Dw)\varphi]_{-\varepsilon}^{\varepsilon} = \alpha^2 \int_{-\varepsilon}^{\varepsilon} (w - c)\varphi dy.$$

On taking the limit $\varepsilon \to 0+$, condition (2.13) follows. Division by $(w - c)^2$ and further integration gives

$$[\varphi/(w - c)]_{-\varepsilon}^{\varepsilon} = \alpha^2 \int_{-\varepsilon}^{\varepsilon} dy(w - c)^{-2} \int_{-\varepsilon}^{y} dy_1(w_1 - c)\varphi_1,$$

and thence condition (2.14) as $\varepsilon \to 0+$. These ideas may be extended to generalized functions $w(y)$, functions which do not represent real flows but do approximate some properties of real flows and give sensible stability characteristics easily. For example, when $w^2(y) = \delta(y - y_0)$, the Dirac delta function, we get from the above

$$c^2\,[DF] = \alpha^2(F)_{y=y_0}, \qquad [F] = 0 \tag{2.15}$$

at $y = y_0$, where $F = \varphi/(w - c)$, a function which has no worse behavior than a discontinuous derivative at $y = y_0$.

Explicit solutions of the eigenvalue problem are difficult to find in practice for smoothly-varying functions $w(y)$. However, when $w(y)$ is a piecewise linear function, one can find exponential solutions of the stability equation piecewise, and join them up by use of conditions (2.13), (2.14). This method, first used by Kelvin, has led to explicit eigenvalue relations for many problems. This use of profiles which do not vary smoothly can be justified [**6**; **7**, p. 221; **8**] sometimes as an approximation to similar smoothly-varying profiles when the wave-number is small.

Note the symmetry of the stability equation and boundary conditions in α and $(-\alpha)$. So we shall henceforth take $\alpha \geqslant 0$ without loss of generality. Then the criterion for instability of the basic flow is that there be a solution with $c_i > 0$ for some $\alpha > 0$.

Note further that for each eigenfunction φ, with eigenvalue c, for given α there is another complex conjugate eigenfunction φ^* with eigenvalue $c^* = c_r - ic_i$ for the same α. Thus to each damped stable wave there corresponds an amplified unstable wave, and *vice versa*. This expresses the time symmetry of the problem, comprising periodic motion of inviscid fluid with steady boundaries. It follows that the condition for stability is that c is real, and for instability that c is complex. So we shall write $c_i > 0$ when there is instability and ignore the associated conjugate eigenvalue, though it should be remembered that on the inviscid theory the result $c_i < 0$ equally implies instability.

We emphasize that we are entirely concerned with hydrodynamic stability of inviscid fluid. Since the Orr-Sommerfeld equation is not invariant under complex conjugation like the Rayleigh stability equation, stability for viscous fluid does not necessarily correspond to real c. Indeed, the relation to the solutions of the Orr-Sommerfeld equation is complicated. For this we refer to Lin's book [**9**, Chap. 8], where it is shown that at least for analytic $w(y)$ a solution of the Rayleigh equation for $c_i > 0$ is a limit of some solution of the Orr-Sommerfeld equation, though its complex conjugate may not be, throughout the domain of flow. Further, when solution of the Rayleigh equation gives a stable basic flow of inviscid fluid, solution of the Orr-Sommerfeld equation may give *instability* of the same flow of viscous fluid, in accord with Heisenberg's criterion. These and other subtle mathematical questions raised by the inclusion of viscosity are important in some cases, but are not considered here, so our results must be taken with this in mind. Nevertheless, a full understanding of the inviscid theory is a desirable preliminary to any study of the viscous theory.

The fundamentals of this section are due to Rayleigh [**3**]. For more recent accounts, the book of Lin [**9**, Chaps. 4, 8] and chapter of Stuart [**10**] are recommended. These two works, and Reid's survey [**11**], also place the inviscid theory in its viscous context.

2. *General Stability Characteristics of Plane Parallel Flow*

Rayleigh [12] proved in 1880 that a *necessary* condition for instability is that the velocity profile $w(y)$ should have a point of inflection. This can be proved by multiplying the Rayleigh stability equation (2.11) by φ^* and integrating from y_1 to y_2. Thus, after use of the boundary conditions (2.12)

$$\int_{y_1}^{y_2} |D\varphi|^2 + \alpha^2|\varphi|^2 dy = \int_{y_1}^{y_2} \frac{(D^2w)|\varphi|^2}{w - c} dy. \tag{2.16}$$

The imaginary part of this equation gives

$$c_i \int_{y_1}^{y_2} \frac{(D^2w)|\varphi|^2}{|w - c|^2} dy = 0. \tag{2.17}$$

If $c_i > 0$ it now follows that D^2w must change sign at one or more points in the field of flow. On assuming that D^2w is continuous, there must be at least one inflection point on (y_1, y_2) and indeed an inflection point at which the velocity profile crosses its tangent, i.e. a relative maximum or minimum in the *basic vorticity* $\bar{\omega} = - Dw$. With the (weaker) assumption that $\bar{\omega}$ is only piecewise continuously differentiable (which has really been tacitly made anyway in writing down the stability equation) we can still say that $\bar{\omega}$ must have a relative maximum or minimum.

In 1950 Fjørtoft [13, p. 25] proved the stronger necessary condition for instability that $(D^2w)(w - w_s) < 0$ somewhere in the field of flow, where y_s is a point at which D^2w vanishes, and where $w_s = w(y_s)$. A proof comes from the real part of equation (2.16),

$$\int_{y_1}^{y_2} \frac{(D^2w)(w - c_r)|\varphi|^2}{|w - c|^2} dy = - \int_{y_1}^{y_2} |D\varphi|^2 + \alpha^2|\varphi|^2 dy.$$

Adding

$$(c_r - w_s) \int_{y_1}^{y_2} (D^2w)|\varphi|^2/|w - c|^2 dy = 0$$

to the left-hand side, we get

$$\int_{y_1}^{y_2} \frac{(D^2w)(w - w_s)|\varphi|^2}{|w - c|^2} dy = - \int_{y_1}^{y_2} |D\varphi|^2 + \alpha^2|\varphi|^2 dy < 0.$$

The result follows. In particular, if $w(y)$ is a monotone function and D^2w vanishes only once in the field of flow, a necessary condition for instability is that $(D^2w)(w - w_s) \leqslant 0$ throughout the flow, with equality only where $y = y_s$; this result is depicted in Figure 2.

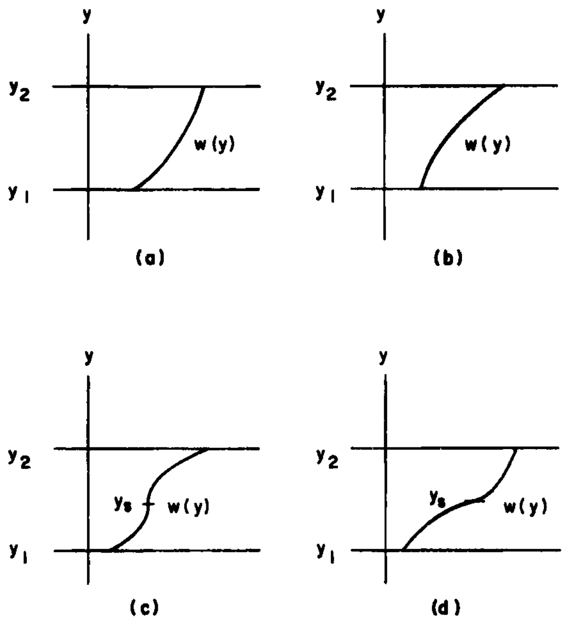

FIG 2. (a) Stable: $D^2w < 0$. (b) Stable: $D^2w > 0$. (c) Stable: $D^2w = 0$ at y_s, but Dw increases where $w < w_s$. (d) Possibly unstable: $D^2w = 0$ at y_s and $(w - w_s)D^2w < 0$ elsewhere.

Fjørtoft's extension of Rayleigh's theorem can be shown to be equivalent to the statement: If $\bar{\omega}$ is piecewise continuously differentiable, a necessary condition for instability is that $|\bar{\omega}|$ should somewhere have a ***maximum*** inside the flow domain.

Neither condition for instability is sufficient in general. We shall present a counter-example (c) to sufficiency with $w = \sin y$ in Section II.4. However, Tollmien [**14**] proved sufficiency in **1935** for symmetric velocity profiles in a channel and for boundary layers. The basis of Tollmien's argument is first to show that there exists a neutrally stable eigensolution with $c = w_s$, and then to construct unstable solutions for which $c \to w_s$ as $c_i \to 0$ through positive values.

Friedrichs [15] has given an elegant alternative proof of the existence of the neutrally stable eigensolution,

$$\varphi = \varphi_s, \qquad \alpha = \alpha_s > 0, \qquad c = w_s,$$

say. For the proof we suppose that $K(y) = -D^2w/(w - w_s)$ is integrable over the field of flow, and put $c = w_s$, $\lambda = -\alpha^2$ in the Rayleigh stability equation to get

$$D^2\varphi + K(y)\varphi + \lambda\varphi = 0,$$

a real non-singular equation which makes up a Sturm-Liouville problem with boundary condition (2.12). The associated variational principle gives the least eigenvalue

$$\lambda_s = \min\left\{\left(\int_{y_1}^{y_2} (Df)^2 - Kf^2 dy\right)\Big/\left(\int_{y_1}^{y_2} f^2 dy\right)\right\}, \tag{2.18}$$

the minimum being for functions f that vanish at the walls and have square-integrable derivatives. Therefore a neutral eigensolution with positive $\alpha = \alpha_s = (-\lambda_s)^{1/2}$ exists if and only if $\lambda_s < 0$. There may be a finite number of other eigensolutions for larger eigenvalues λ, provided these are negative. Also there may be other series of eigensolutions when $c = w_s$ for other values of w_s at other points of inflection, and sometimes eigenfunctions with real c not equal to the value of w at an inflection point, though these have slightly singular behavior.

The existence of the neutral eigensolution with $\lambda_s < 0$ follows easily when $K(y) > \pi^2/(y_2 - y_1)^2$ everywhere on account of the well-known inequality,

$$(y_2 - y_1)^2 \int_{y_1}^{y_2} (Df)^2 dy \geqslant \pi^2 \int_{y_1}^{y_2} f^2 dy.$$

Again, when $K(y) > 0$ over the field of flow and w vanishes at the walls but not between, trial of $f = w$ shows that $\lambda_s < 0$.

Tollmien also demonstrated heuristically the existence of unstable waves, whose limit as $c_i \to 0$ is the neutral s-wave above. This has been considered alternatively by Lin [7, pp. 223–224; 9, pp. 122–123] as follows. The stability equation for φ_s gives

$$D^2\varphi_s + \{\lambda_s - (D^2w)/(w - w_s)\}\varphi_s = 0. \tag{2.19}$$

Multiplying this by φ and subtracting φ_s times the equation for φ, we get

$$D(\varphi_s D\varphi - \varphi D\varphi_s) - (\lambda - \lambda_s)\varphi\varphi_s - (D^2w)\{(w - c)^{-1} - (w - w_s)^{-1}\}\varphi\varphi_s = 0.$$

Integrating from y_1 to y_2, we now find

$$(\lambda - \lambda_s)\int_{y_1}^{y_2} \varphi\varphi_s dy = (c - w_s)\int_{y_1}^{y_2} \frac{(D^2w)\varphi\varphi_s}{(w-c)(w-w_s)}\,dy.$$

To find the unstable solutions near the neutral one, we take the limits $\alpha \to \alpha_s$, $c \to w_s$, $\varphi \to \varphi_s$. Then

$$\left(\frac{d\alpha^2}{dc}\right)_{\alpha=\alpha_s}\int_{y_1}^{y_2} \varphi_s^2 dy = \lim_{c\to w_s}\int_{y_1}^{y_2} \frac{K(y)\varphi_s^2}{w-c}\,dy$$

$$= \lim_{c\to w_s}\left\{\int_{y_1}^{y_2} \frac{(w-c_r)K\varphi_s^2}{(w-c_r)^2+c_i^2}\,dy + i\int_{y_1}^{y_2} \frac{c_iK\varphi_s^2 dy}{(w-c_r)^2+c_i^2}\right\}$$

$$= -\mathscr{P}\int_{y_1}^{y_2} \frac{(D^2w)\varphi_s^2}{(w-w_s)^2}\,dy - i\pi\{\operatorname{sgn}(Dw)_{y=y_s}\}\{(D^3w)\varphi_s^2/(Dw)^2\}_{y=y_s}$$

as $c_i \to 0$ through positive values, provided $(Dw)_{y=y_s} \neq 0$, where $\mathscr{P}$ denotes the principal value of the integral. Therefore

$$\left(\frac{dc}{d\alpha^2}\right)_{\alpha=\alpha_s} = -\left\{\int_{y_1}^{y_2} \varphi_s^2 dy\right\}\Big/\left\{\mathscr{P}\int_{y_1}^{y_2} \frac{(D^2w)\varphi_s^2}{(w-w_s)^2}\,dy\right.$$
$$\left. + i\pi \operatorname{sgn}(Dw)_{y=y_s}(\varphi_s^2D^3w/(Dw)^2)_{y=y_s}\right\}. \tag{2.20}$$

For a known neutral eigensolution this gives eigenvalues $c(\alpha)$ near w_s in the complex c-plane and thence the associated eigenfunctions in the limit $c_i \to 0+$. If $K(y) > 0$ throughout the field of flow, then the imaginary part of the denominator is positive, and it follows that $(dc_i/d\alpha^2)_{\alpha=\alpha_s} < 0$, with instability for α just less than α_s.

Further, it can be proved as follows that there is instability only when $\alpha < \alpha_s$. Suppose $K(y) > 0$ throughout the field of flow. Then, when $c_i \neq 0$, the real part of equation (2.16) plus $(w_s - c_r)/c_i$ times equation (2.17) gives

$$\int_{y_1}^{y_2} |D\varphi|^2 + \alpha^2|\varphi|^2 dy = \int_{y_1}^{y_2} \frac{(w-c_r)^2-(w_s-c_r)^2}{(w-c_r)^2+c_i^2}K|\varphi|^2 dy < \int_{y_1}^{y_2} K|\varphi|^2 dy.$$

Therefore

$$\alpha^2 < \max \left\{ \int_{y_1}^{y_2} - |D\varphi|^2 + K|\varphi|^2 dy \Big/ \int_{y_1}^{y_2} |\varphi|^2 dy \right\} = \alpha_s^2.$$

It follows that there is stability ($c_i = 0$) when $\alpha \geqslant \alpha_s$.

This argument can be extended to prove the following result, applicable to flows for which the function K exists and is non-negative [**16**]. Let $\lambda_1, \lambda_2, \ldots$ be the eigenvalues of the Sturm-Liouville problem $f'' + Kf + \lambda f = 0$, $f(y_1) = f(y_2) = 0$, arranged in increasing order. Then there can be no more than $n - 1$ linearly independent unstable eigenfunctions of the stability problem if $\alpha^2 \geqslant - \lambda_n$. Thus if α^2 is larger than the absolute value of the lowest (negative) eigenvalue λ_1, the flow is *stable*; if α^2 lies between $- \lambda_2$ and $- \lambda_1$ there can be only one unstable mode, if it lies between $- \lambda_3$ and $- \lambda_2$, at most two, and so on. Of course eventually λ_n becomes positive and thereafter the relevance to the stability problem ceases. In particular if $\lambda_1 \geqslant 0$, which may happen even for $K \geqslant 0$ if the boundaries are sufficiently close, then the flow must be stable even though it has an inflection point with $w''(w - w_s) < 0$. (For a different and interesting approach to a related result in the case of a monotone $w(y)$, see [17].)

We shall next consider the energy of a disturbance. If one multiplies the Rayleigh stability equation by φ^*, integrates from y_1 to y_2, and uses the boundary conditions, one finds

$$\int_{y_1}^{y_2} (w - c)(|D\varphi|^2 + \alpha^2|\varphi|^2) + (D^2 w)|\varphi|^2 - \varphi^*(Dw)(D\varphi) dy = 0.$$

The imaginary part of this gives

$$c_i \int_{y_1}^{y_2} |D\varphi|^2 + \alpha^2|\varphi|^2 dy = \frac{1}{2} i \int (Dw)(\varphi D\varphi^* - \varphi^* D\varphi) dy. \tag{2.21}$$

This is in fact the x-average of the energy equation of the disturbance,

$$\frac{\partial}{\partial t} \int_{y_1}^{y_2} \int \frac{1}{2} (u'^2 + v'^2) dx dy = - \int_{y_1}^{y_2} \int u'v' Dw dx dy.$$

Foote and Lin [18] noted that the average of the Reynolds shear stress over a wavelength

$$\tau = \overline{- u'v'} = - (\alpha/2\pi) \int_0^{2\pi/\alpha} u'v' dx = \frac{1}{4} i\alpha(\varphi D\varphi^* - \varphi^* D\varphi) e^{2\alpha c_i t}.$$

The stability equation now gives

$$(2.22) \quad D\tau = \tfrac{1}{4}\, i\alpha(\varphi D^2\varphi^* - \varphi^* D^2\varphi)e^{2\alpha c_i t} = \tfrac{1}{2}\,\alpha c_i (D^2 w)|w - c|^{-2}|\varphi|^2 e^{2\alpha c_i t}.$$

Since $\tau = 0$ at the boundaries, the integral of $D\tau$ over (y_1, y_2) is zero: this is in fact just the integral (2.17) used to prove Rayleigh's theorem. But consider now a neutrally stable mode which is adjacent to unstable modes, that is, suppose we have an unstable mode for some α, and as $\alpha \to \alpha_s$, say, the corresponding c approaches a real value: $c_i(\alpha) \to 0$. Then for this neutral mode, (2.22) shows that $D\tau$ must be zero everywhere, except possibly at $y = y_c(w(y_c) = c)$ where the limit of the right-hand side of (2.22) does not exist. By consideration of the nature of the singularity in $D\tau$ which appears as $c_i \to 0+$ it can be shown that at the "critical layer" $y = y_c$ (or layers) τ has (in the limit $c_i \to 0+$) a jump $[\tau]$, of magnitude

$$[\tau] = \tfrac{1}{2}\alpha\pi\{(D^2 w)|\varphi|^2/|Dw|\}_{y=y_c}.$$

Because of the boundary conditions satisfied by τ, the algebraic sum of all such jumps must be zero. If the profile is monotone, there can be only one jump, which must thus be zero, and this implies that $D^2 w$ vanishes at y_c since it can be shown that the alternative $\varphi(y_c) = 0$ is impossible. Thus for monotone profiles the neutral value of c must be the value of w at the inflection point. This is true also for some non-monotone profiles, for example the symmetrical jet $w = \operatorname{sech}^2 y$, but is not always the case. For many non-monotone profiles, notably most non-symmetrical jets, there is no possibility that $D^2 w = 0$ at *all* places where $w = c$. A neutral c adjacent to unstable modes doubtless exists in such cases, but it is not the value of w at any inflection point, and the corresponding eigenfunction must exhibit a certain weak singular behavior so that τ can have two compensating jumps. This must in fact be regarded as the typical case for non-monotone profiles; it is fortunate that many profiles of interest are either monotone or sufficiently symmetrical so that the neutral c can be identified at once as the value of w at the inflection point. In general *both* the neutral c (adjacent to unstable modes) and the corresponding α have to be determined by numerical solution of the equation taking proper account of the singularities at the critical layers. Some further discussion of the neutral eigenfunctions is given in Section II.3.

In 1915 Taylor [**19**, pp. 23–26] gave a physical interpretation of Rayleigh's necessary condition for instability. Taylor noted how the frictionless slipping of the fluid at the boundaries prevented the transfer of x-momentum necessary to maintain an unstable disturbance when $D^2 w$ is always of one sign. Essentially the momentum is transferred by the Reynolds stress, which must vanish near the walls and whose gradient can only vanish at a point where $D^2 w$ vanishes. (Lighthill [**20**] has applied these physical ideas to the instability of wind whereby ocean waves are generated.) Taylor went on to note that viscosity allowed momentum to be diffused from the boundaries, and suggested that a given basic flow might thus be stable for inviscid but unstable for viscous fluid. This suggestion has since been verified, for plane Poiseuille flow as an example.

Lin [7, pp. 226–227] also has interpreted physically the mechanism of inertial instability by consideration of the migration of vorticity. He regarded the flow due to a neutrally-stable disturbance in Kelvin's "cat's-eye" di-

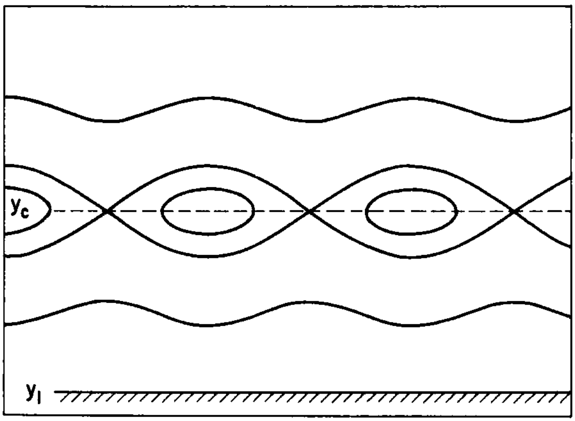

FIG. 3. Kelvin's cat's-eye diagram. The streamlines viewed by an observer moving with the neutral wave.

agram [21], the pattern of streamlines viewed by an observer moving with the phase velocity c of the wave (Figure 3). This observer sees a stationary flow, with $\bar{\psi} = \int_{y_c}^{y} w dy$ and $\psi' = \varphi(y)e^{i\alpha x}$. (It should be remembered that the physical quantity ψ' is understood to be the real part of its complex representation.) Let us assume that the critical layer $y = y_c$ lies within the field of flow, and that φ does not vanish in that layer. Then there will be some closed streamlines, and the streamline $\psi = \bar{\psi} + \psi' = 0$ will intersect at points on $y = y_c$ periodically separated by $2\pi/\alpha$. Now the flow is inviscid and two-dimensional. Therefore the total vorticity $\omega = \bar{\omega} + \omega' = -Dw - \partial u'/\partial y + \partial v'/\partial x$ is uniform on each streamline, and in particular on the intersecting streamline. But $\partial\omega/\partial y = 0$ at the points of intersection. Therefore $\partial\omega/\partial y = O(|\boldsymbol{\omega}'|)$ at the critical layer, i.e. $(D^2w)_{y=y_c} = 0$ to zeroth degree in the perturbed quantities, or $\partial\omega'/\partial y$ is singular. It follows that it is possible to find a non-singular neutral disturbance in inviscid fluid only if $D^2w = 0$ where $w = c$. In reality a singular disturbance with large $\alpha^2|\mathbf{u}'|^{-1}\partial\omega'/\partial y$ would be damped by viscosity. Lin [7; 9, pp. 56–58] has also gone on to discuss the two-dimensional motion of vortices during instability, and a more complete discussion of this physical mechanism has been given by Gill [22].

Rayleigh limited the possible range of eigenvalues in the complex c-plane, proving that $w_{\min} < c_r < w_{\max}$ when $c_i \neq 0$. Howard [23] generalized this result with his semicircle theorem. For its proof, suppose $F = \varphi/(w - c)$ is non-singular, and rewrite the Rayleigh stability equation as

$$D\{(w - c)^2 DF\} - \alpha^2 (w - c)^2 F = 0. \tag{2.23}$$

Multiply this equation by F^* and integrate from y_1 to y_2, using the conditions that F vanishes on the boundaries. Then

$$\int_{y_1}^{y_2} (w - c)^2 \{|DF|^2 + \alpha^2 |F|^2\} dy = 0. \tag{2.24}$$

This equation implies that c cannot be real when F is non-singular and therefore that c cannot lie beyond the range of w. Next suppose $c_i \neq 0$ and take the real and imaginary parts of (2.24). This gives

$$\int_{y_1}^{y_2} \{(w - c_r)^2 - c_i^2\} Q dy = 0, \qquad 2c_i \int_{y_1}^{y_2} (w - c_r) Q dy = 0,$$

where $Q \equiv |DF|^2 + \alpha^2 |F|^2 > 0$. Therefore

$$\int_{y_1}^{y_2} wQdy = \int_{y_1}^{y_2} c_r Q dy, \qquad \int_{y_1}^{y_2} w^2 Q dy = \int_{y_1}^{y_2} (c_i^2 + c_r^2) Q dy.$$

But

$$0 \geqslant \int_{y_1}^{y_2} (w - w_{\min})(w - w_{\max}) Q dy$$

$$= \int_{y_1}^{y_2} \{(c_i^2 + c_r^2) - (w_{\min} + w_{\max}) c_r + w_{\min} w_{\max}\} Q dy,$$

the maximum and minimum being taken over the field of flow $y_1 \leqslant y \leqslant y_2$. Therefore

$$c_i^2 + c_r^2 - (w_{\min} + w_{\max}) c_r + w_{\min} w_{\max} \leqslant 0,$$

i.e. for unstable waves c lies in the semicircle

$$\{c_r - \tfrac{1}{2}(w_{\min} + w_{\max})\}^2 + c_i^2 \leqslant \{\tfrac{1}{2}(w_{\max} - w_{\min})\}^2 \qquad (c_i > 0). \tag{2.25}$$

This shows that any eigenvalue c, real or complex, must lie in or on the circle with center $\frac{1}{2}(w_{\max} + w_{\min})$ and radius $\frac{1}{2}(w_{\max} - w_{\min})$.

Again, with $G = \varphi/(w - c)^{1/2}$, the stability equation can be written

$$D\{(w - c)DG\} - \{\tfrac{1}{2}D^2w + \alpha^2(w - c) + \tfrac{1}{4}(Dw)^2/(w - c)\}G = 0. \tag{2.26}$$

This has an integral

$$\int_{y_1}^{y_2} (w - c)\{|DG|^2 + \alpha^2|G|^2\} + \tfrac{1}{2}(D^2w)|G|^2 + \tfrac{1}{4}(Dw)^2(w - c^*)|G/(w - c)|^2\}dy = 0,$$

whose imaginary part gives

$$\alpha^2\int_{y_1}^{y_2} |G|^2dy = \int_{y_1}^{y_2}\left[\frac{1}{4}(Dw)^2|G/(w - c)|^2 - |DG|^2\right]dy \leqslant \frac{1}{4}c_i^{-2}\max (Dw)^2\int_{y_1}^{y_2} |G|^2dy,$$

because $|w - c|^{-2} = \{(w - c_r)^2 + c_i^2\}^{-1} \leqslant c_i^{-2}$. It follows that

$$\alpha c_i \leqslant \tfrac{1}{2}\max |Dw|.$$

This result is due to Høiland [**24**, p. 11], this proof to Howard [**23**]. A more general analogue will be given in Section II.3.

The stability problem has certain symmetries when the basic flow is symmetric, i.e. when it is possible to choose coordinate axes so that $y_1 = -y_2$ and $w(y)$ is an even function. In that event, if $\varphi(y)$ is an eigenfunction with eigenvalue c for any given α, it follows that the even part

$$\varphi_e = \tfrac{1}{2}\{\varphi(y) + \varphi(-y)\}$$

and the odd part

$$\varphi_0 = \tfrac{1}{2}\{\varphi(y) - \varphi(-y)\}$$

of φ are also eigenfunctions for the same c, α. This can be seen at once from the symmetric stability equation and boundary conditions. It can be shown further that either φ_0 or φ_e is identically zero. To show this, we multiply the stability equation (2.11) for φ_e by φ_0, and subtract φ_e times the same equation for φ_0. This gives

$$\varphi_eD^2\varphi_0 - \varphi_0D^2\varphi_e = 0$$

where $w \neq c$. Therefore

$$\begin{aligned}\varphi_eD\varphi_0 - \varphi_0D\varphi_e &= \text{constant} = \text{value at wall}\\ &= 0.\end{aligned}$$

Therefore φ_e, φ_0 are linearly dependent in general, which is only possible if one of them is identically zero. Thus we have proved that an eigenfunction is either odd or even, except possibly when c is real.

In fact both even and odd eigenfunctions are found for the same symmetric basic flow, but they have different eigenvalues c for each α. An even eigenfunction is associated with a disturbance named *sinuous* by Rayleigh, the pattern of streamlines being antisymmetric about the line $y = 0$. Similarly, an odd eigenfunction is associated with a *varicose* disturbance, the streamlines being symmetric. This oddness or evenness of φ allows one to assume that φ is even (or odd) and reduce the effective field of flow to the half range, $0 \leqslant y \leqslant y_2$, applying the symmetry condition $D\varphi = 0$ (or $\varphi = 0$) at $y = 0$ and the original boundary condition $\varphi = 0$ at $y = y_2$. This is a convenient method to find eigenvalues for the sinuous and varicose modes of instability. It can be seen from the variational principle (2.18) with even $K(y)$ that the least eigenvalue λ corresponds to an even function f, and that therefore the first sinuous submode is more unstable than any varicose mode of a given basic flow.

Next we suppose that the profile is antisymmetric, with $y_1 = -y_2$ and odd $w(y)$. Then for each eigenfunction $\varphi(y)$ with eigenvalue c there is an eigenfunction $\varphi^*(-y)$ with value $-c^* = -c_r + ic_i$ for the same α. When the eigensolution is unique, this Hermitian symmetry implies that $c_r = 0$ and $\varphi^*(-y) = \varphi(y)$. Otherwise, there may be a pair of eigensolutions with phase velocities $\pm c_r(\alpha)$ and the same $c_i(\alpha)$, one function the Hermitian conjugate of the other. Howard [25] gave a physical argument for a situation when the latter must occur. In Section II.4.1 we give an example of a discontinuous shear layer for which it occurs. We also know that it may occur for the s-eigensolution at least when the profile has a point of inflection, other than that at $y = 0$, where $w_s \neq 0$. At any rate for the neutral eigensolution with $y_s = 0$, $K(y)$ is an even function, and the variational principle (2.18) gives the greatest wave-number α_s for an even eigenfunction φ_s; Lin's argument to deduce the perturbation formula (2.20) gives $(dc_r/d\alpha^2)_{\alpha=\alpha_s-0} = 0$. So one might conjecture that for this mode associated with the point of inflection $y_s = 0$ there is exchange of stabilities such that $c_r = 0$ when $c_i \neq 0$, i.e. when $0 \leqslant \alpha < \alpha_s$. This can in fact be proved for monotone antisymmetric profiles with $K(y) > 0$; cf. [16].

Let us now revert to general basic flows, not necessarily with any symmetry. Equation (2.24) was derived on the assumption that $F = \varphi/(w - c)$ had a square-integrable derivative over the interval $[y_1, y_2]$. It shows that, when c is real and F not identically zero, c lies in the range of w and either (i) $\alpha = 0$, $F = \text{constant} = A$, say, or (ii) DF is not square-integrable. In the latter case, φ itself might be singular where $w = c$ or it might have a lower order zero than $(w - c)$. Our previous work now shows that as $c_i \to 0+$ either $\alpha \to 0+$ or $\alpha \to \alpha_s -$.

If $\alpha = 0$ and $F = A$, we get the ***trivial eigensolution*** with $\psi' = \varphi = A(w - c)$. This is really a form of the basic flow, for the total x-component of the velocity of the perturbed flow is

$$u = w(y) + \partial\psi'/\partial y = w(y) + ADw(y)$$
$$= w(y + A) + O(A^2)$$

and the y-component is $v = -\partial\psi'/\partial x = 0$. Thus the trivial solution is really the basic solution displaced laterally by the small distance A. In fact, for any solution it is readily seen that the vertical displacement at (x,t) of the material surface with mean level y is

$$\eta(x,t) = F(y)e^{i\alpha(x-ct)}.$$

The trivial solution appears as the first term in a power series expansion of φ for small α. Heisenberg [6] found formally two solutions of the Rayleigh stability equation:

$$\varphi_j(y;\alpha^2,c) = (w - c)\{q_{j0}(y,c) + \alpha^2 q_{j1}(y,c) + \ldots + \alpha^2 q_{jn}(y,c) + \ldots\} \quad (j = 1,2)$$

where

$$q_{10}(y,c) = 1, \qquad q_{20}(y,c) = \int (w - c)^{-2}\,dy,$$

$$q_{jn+1}(y,c) = \int (w - c)^{-2}dy \int (w - c)^2 q_{jn}(y)dy \qquad (n = 0,1,2,\ldots).$$

In these formulae the lower limit of integration is arbitrary, but may conveniently be taken as y_1. The zeroth approximation for small α^2 gives the eigenvalue relation

$$\int_{y_1}^{y_2} (w - c)^{-2}dy = 0. \tag{2.27}$$

However this result depends only on heuristic analysis and is equivocal [7, pp. 220–221]. In fact Heisenberg used the series chiefly for the viscous solution at high Reynolds numbers.

Heisenberg was not concerned with the case of one infinite boundary, for which it can be seen that his series are not uniformly convergent. On taking the limit as $\alpha \to 0$ for fixed $w(y)$, the stability equation in the form (2.23) gives

$$(w - c)^2 DF = \text{constant} = \text{value at boundary} = 0,$$

and therefore F is constant between critical layers $y = y_c$. Therefore $F = 0$ from infinity down to the largest value of y_c. On the other hand, the stability equation and boundary condition (2.12) at infinity give

$$F \sim \text{constant} \times e^{-\alpha y} \quad \text{as} \quad y \to \infty$$

for any positive α, however small, provided that $w \to$ constant smoothly. Thus the order of the limits $\alpha \to 0$, $y \to \infty$ cannot be changed without changing the limit of the eigenfunction F, which, like $e^{-\alpha y}$, does not admit a power series expansion in α^2 uniformly for y large.

In 1962 Drazin and Howard [8] considered long-wave disturbances of unbounded and semi-bounded flows. The basis of their work for unbounded flows is as follows. The stability equation (2.23) has two solutions $F_\pm(y;\alpha,c)$ defined by their asymptotic properties

$$F_\pm \sim e^{\mp\alpha y}, \qquad DF_\pm \sim \mp\alpha e^{\mp\alpha y} \quad \text{as} \quad y \to \pm\infty.$$

These solutions are defined by the stability equation for given $\alpha, c, w(y)$, and are in general independent. However, for an eigenfunction which vanishes at $y = \pm\infty$,

$$F \equiv K_+F_+ \equiv K_-F_- \qquad (-\infty < y < \infty)$$

for some complex constants $K_\pm$, the solutions $F_\pm$ being linearly dependent when c is an eigenvalue corresponding to given α. Therefore the Wronskian

$$F_+DF_- - F_-DF_+ = 0 \tag{2.28}$$

at each and every point y, and at $y = 0$ in particular. This is the exact eigenvalue relation. At this stage one may assume α is small and seek to expand $F_\pm$ as power series in α. To avoid the non-uniformity of convergence at $y = \pm\infty$ we put

$$F_\pm = e^{\mp\alpha y} \sum_{n=0}^{\infty} (\pm\alpha)^n \chi_{\pm n}(y,c) \qquad (0 \leqslant \pm y < \infty),$$

the two series being used in semi-infinite intervals which just overlap at $y = 0$. The coefficients $\chi_{\pm n}$ can be found formally from the stability equation as repeated integrals of $w(y)$, c etc. Now the eigenvalue relation (2.28) can be expanded in powers of α, the coefficients involving c, $w(y)$ in explicit integrals. This method can be shown to give one mode for which

$$c \to \tfrac{1}{2}\{w(\infty) + w(-\infty)\} + \tfrac{1}{2}i|w(\infty) - w(-\infty)| \quad \text{as} \quad \alpha \to 0. \tag{2.29}$$

Thus there is instability when $w(-\infty) \neq w(\infty)$, or the flow is of *shear-layer* or *half-jet type*. On the large scale of the long wave (with small α)

a general smoothly-varying shear layer behaves like the vortex sheet with basic velocity

$$w = \begin{cases} w(\infty) & (y > 0) \\ w(-\infty) & (y < 0). \end{cases}$$

Indeed the limit (2.29) of c gives the exact result for this vortex sheet, as is given in Section II.4.d. In fact there is also instability (but of smaller growth) when $w(-\infty) = w(\infty)$, i.e. when the flow is of *jet type*, the next approximation then giving

$$c - w(\infty) \sim i\left(\frac{1}{2}\alpha \int_{-\infty}^{\infty} \{w - w(\infty)\}^2 dy\right)^{1/2} \quad \text{as} \quad \alpha \to 0. \tag{2.30}$$

Other modes were also considered [8], it being found that

$$c - w(y_m) = O(\alpha^{r/(2r-1)})$$

as $\alpha \to 0$, where

$$D^p w = 0 \qquad (1 \leqslant p \leqslant r - 1), \qquad D^r w \neq 0$$

at $y = y_m$.

In discussing the eigenvalue relation for a general profile, bounded or unbounded, we mentioned that $c(\alpha)$ may be a many-valued function. The variational principle (2.18) suggested that there might be many values of α for each $c = w_s$ and many values of $c = w_s$. For symmetric profiles we mentioned sinuous and varicose disturbances, when c is at least double valued. We shall meet many-valued c in several examples of the Section II.4, finding that each branch of $c(\alpha)$ is well behaved and corresponds to a distinct mode of instability. By continuity in α one might expect that for each neutral eigensolution with $c = w_s$, and therefore for approximately each point of inflection, there is one mode of instability. Drazin and Howard [8] considered unbounded flows and associated heuristically the neutral eigensolutions having $c = w_s$ for each zero y_s of D^2w with the small-α eigensolutions having $c = w(y_m)$ for each zero y_m of Dw. However the general problem defies oversimplification, and the modes have not been satisfactorily classified.

3. *The Initial-Value Problem, and the Stability of Non-parallel Flow*

Hydrodynamic stability theory is by far most highly developed for the case of a parallel basic flow, but there are a few more general results and we insert here a brief description of some of these. We shall also in this subsection regard the problem as an initial-value problem, though elsewhere in this article we generally follow the more usual normal mode approach.

In the parallel flow case, the flow domain is taken as the strip $-\infty < x < \infty$, $y_1 \leqslant y \leqslant y_2$, and the boundary conditions are usually taken to correspond to rigid walls at $y = y_1$ and y_2. In the general case, it seems appropriate to take the flow domain to be some region R in the plane or in space, and to allow the possibility of flow *across* the boundary B of R. In this case, more attention has to be given to the boundary conditions. We shall assume that R is fixed and that we have given in R a basic flow $\mathbf{U}_0$ which is steady, incompressible, and inviscid. We shall also not consider any body forces. The stability problem is then formulated as follows: given some initial conditions which differ from $\mathbf{U}_0$ by a small amount, in terms of some appropriate measure (e.g., the L_2 norm of the difference), we find the time dependent flow $\mathbf{U}$ determined by these initial conditions and some suitable boundary conditions which are satisfied by $\mathbf{U}_0$. The flow is *stable* if $\mathbf{U}$ continues to differ from $\mathbf{U}_0$ by a small amount in terms of the selected measure. By "suitable boundary conditions" we mean such as assure that the initial-value problem for the flow equations will have a solution, and a unique one. The mathematical questions of existence and uniqueness of the initial-value problem for the inviscid flow equations with various boundary conditions do not appear to have received as much study as they deserve, but it is not appropriate here to embark on such a discussion. We shall give only some brief heuristic remarks. The most familiar case is that the boundary B consists entirely of a rigid wall, so that $\mathbf{U} \cdot \mathbf{n} = 0$ on it. This condition probably is, in itself, a "suitable boundary condition" in the above sense. More generally, if the flow crosses B, we should expect to prescribe $\mathbf{U} \cdot \mathbf{n}$ as a boundary condition on B, subject only to the requirement that its integral over B should vanish (otherwise the continuity equation alone would have no solution). However, this condition alone is in general *not* sufficient to insure uniqueness of the solution to the initial-value problem—consider for instance the plane flows in the annulus $1 \leqslant r \leqslant 2$ (polar coordinates) whose radial velocity component is $1/r$ and whose azimuthal component is $(1/r)f(r^2 - 2t)$ where f is a function which is zero for values of its argument $\geqslant 1$, but is otherwise arbitrary (it may be as smooth as desired). It is easily checked that these velocity fields do satisfy the flow equations, they all have the same normal component on the boundaries of the annulus, and are identical at $t = 0$.

In the example just given (two-dimensional and axisymmetric flow in an annulus), uniqueness can be insured by prescribing, in addition to the normal velocity component on the complete boundary, the *tangential* component on the part ($r = 1$) of the boundary through which the fluid *enters* the flow region, for all $t > 0$. It is clear that this additional information is just sufficient to determine the function f. Note also that this example shows that one may *not* in general prescribe the tangential component where the fluid leaves the region ($r = 2$).

We shall now show that these boundary conditions, namely the prescription of the normal velocity component over the complete boundary (in a manner consistent with the continuity equation) and of the tangential velocity components over that part of the boundary where the flow is ***inward***, together with one additional assumption, are sufficient to insure the uniqueness of the solution to the initial-value problem for general three-dimensional incompressible inviscid flow.

Let $\mathbf{U}_0$ be such a flow, in a region R with boundary B, and let $\mathbf{U} = \mathbf{U}_0 + \mathbf{u}$ be another. We assume that $\mathbf{U}$ and $\mathbf{U}_0$ satisfy the same boundary conditions, so that $\mathbf{u} \cdot \mathbf{n} = 0$ on B and $\mathbf{u} = 0$ on that part B_1 of B on which $\mathbf{U}_0 \cdot \mathbf{n} < 0$. Write B_2 for the rest of B, on which $\mathbf{U}_0 \cdot \mathbf{n} \geqslant 0$. Consider now the deformation tensor D of $\mathbf{U}_0$, with components $D_{ij} = U_{0i,j} + U_{0j,i}$. Since D_{ij} has zero trace, at least one of its eigenvalues is $\leqslant 0$; let $C(t)$ be the supremum over R of the absolute value of the most negative eigenvalue of D—we call $C(t)$ the ***maximum shear*** of $\mathbf{U}_0$, and we assume that $C(t)$ is finite, initially and thereafter. Writing p for the difference of the pressure fields of the flows $\mathbf{U}$ and $\mathbf{U}_0$, divided by density, one obtains the following equation for the "perturbation" $\mathbf{u}$ by subtracting the momentum equation for $\mathbf{U}_0$ from that for $\mathbf{U}$:

$$\mathbf{u}_t + \mathbf{U} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{U}_0 + \nabla p = 0. \tag{2.31}$$

Note that the "perturbation" $\mathbf{u}$ is not necessarily small. Multiplying (2.31) by $\mathbf{u}$ and using $\nabla \cdot \mathbf{U} = 0$ we get

$$\frac{\partial}{\partial t}\left(\frac{1}{2}|\mathbf{u}|^2\right) + \mathbf{u} \cdot (\nabla \mathbf{U}_0) \cdot \mathbf{u} + \nabla \cdot \left[\mathbf{U}_0 \frac{1}{2}|\mathbf{u}|^2 + \mathbf{u}\left(\frac{1}{2}|\mathbf{u}|^2 + p\right)\right] = 0. \tag{2.32}$$

Integrating this over R, applying the divergence theorem to the last term, and using the boundary conditions, we get

$$\frac{\partial}{\partial t}\int_R \frac{1}{2}|\mathbf{u}|^2 dV = \int_R -\frac{1}{2}\mathbf{u} \cdot D \cdot \mathbf{u}\, dV - \int_{B_2} \mathbf{U}_0 \cdot \mathbf{n} \frac{1}{2}|\mathbf{u}|^2 dS. \tag{2.33}$$

Write $E(t) \equiv \int_R \frac{1}{2}|\mathbf{u}|^2 dV$; we call this the "energy of the perturbation," and shall use it as a measure of how large the perturbation is. Since $\mathbf{U}_0 \cdot \mathbf{n} \geqslant 0$ on B_2 and $-\mathbf{u} \cdot D \cdot \mathbf{u} \leqslant C(t)|\mathbf{u}|^2$, (2.33) gives

$$\frac{dE}{dt} \leqslant C(t)E, \tag{2.34}$$

and thus

$$E(t) \leqslant E(0) \exp \int_0^t C(t')dt'. \tag{2.35}$$

Inequality (2.35) shows that if the perturbation is initially zero, i.e. if $\mathbf{U}$ satisfies the same initial conditions as $\mathbf{U}_0$, then it remains zero, and so the solution to the initial value problem is unique.

(2.35) can be applied in another way. Suppose $\mathbf{U}_0$ is a *steady* flow, whose stability we are studying. In this case $C(t)$ is constant and we see that the growth of the energy of any perturbation is limited by an exponential with growth rate C, and this is true not only for the initial growth in the range of the linear stability theory, but also for any subsequent non-linear development of the initial perturbation. In the particular case of parallel plane flow with velocity profile $w(y)$, the quantity C is easily seen to be $\max |Dw|$; for a linear perturbation with exponential growth rate αc_i, the energy E (per wave-length, say) has exponential growth rate $2\alpha c_i$, and the result given by (2.35) thus reduces to Høiland's estimate $\alpha c_i \leqslant \frac{1}{2} \max |Dw|$ given in Section II.2.

Though we do not have the existence theorem that ideally should accompany it, this uniqueness theorem suggests rather strongly that the boundary conditions of prescribed $\mathbf{U} \cdot \mathbf{n}$ over all of B and prescribed $\mathbf{U}$ over the "incoming" part of B are "suitable boundary conditions" in the sense of the formulation of the stability problem given above. However, this is not to say that these are the *only* suitable boundary conditions; in particular, it is probable that instead of giving the tangential velocity components on the incoming part of B one might equally well prescribe instead the tangential components of the vorticity vector there. This becomes particularly clear in the case of plane flow. Formulating the problem in terms of the stream function $\Psi(\mathbf{U} = \nabla\Psi \times \mathbf{k})$, and eliminating the pressure by going over to the vorticity equation we have the pair of equations:

$$\Delta\Psi + \Omega = 0 \tag{2.36}$$

$$\Omega_t + \mathbf{U} \cdot \nabla\Omega = 0. \tag{2.37}$$

Now one might imagine the following step-by-step process (similar to one used in numerical weather forecasting) for computing the solution to the initial-value problem: Using the initial values of the velocity field for $\mathbf{U}$, integrate the first-order equation (2.37) to find Ω at a slightly later time. It is clear from the structure of (2.37), which says that the vorticity field moves with the fluid particles, that what is needed to do this is the initial values of Ω in R (which follow from those of $\mathbf{U}$), *plus* the values of Ω carried by the new fluid particles which enter the region. Having found the vorticity at the slightly later time, we then calculate the new flow field by solving the Poisson equation (2.36); to do this, we need a boundary condition, and the most natural one is to prescribe Ψ on B, which is equivalent to giving $\mathbf{U} \cdot \mathbf{n}$ on B. Thus this hypothetical computation scheme suggests quite definitely that suitable boundary conditions, with which one might expect to be able to prove existence and uniqueness of the solution of the initial-value problem for equations (2.36) and (2.37), are the prescription of Ψ (or $\mathbf{U} \cdot \mathbf{n}$) on B, and of Ω on B_1. In fact, with these boundary conditions and the assumptions that $|\mathbf{U}_0|$ and Ω_0 remain finite, one can prove uniqueness by a method similar to that used above for the case of the tangential velocity components being given on B_1.

We now consider the stability problem for plane flow, using this vorticity boundary condition, to establish a result which may be regarded as giving a generalization to non-parallel plane flow of Rayleigh's inflection point theorem. Let the basic steady flow be $\mathbf{U}_0 = \nabla\Psi_0 \times \mathbf{k}$. Its vorticity $\Omega_0 = -\Delta\Psi_0$ is constant along streamlines, and we shall write $\Omega_0 = f(\Psi_0)$, though in some cases such a representation is not literally possible unless f is regarded as multiple valued—different streamlines might carry the same value of Ψ_0 but different values of Ω_0. We are going to prove that if $f'(\Psi_0) < 0$ throughout the field of flow, then the flow is *stable* to two-dimensional disturbances. As with Rayleigh's theorem, this will be a sufficient, but not necessary, condition for stability. The perturbation momentum equation is:

$$\mathbf{u}_t + \Omega\mathbf{k} \times \mathbf{u} + \omega\mathbf{k} \times \mathbf{U}_0 + \nabla h = 0, \tag{2.38}$$

where h is the perturbation $H - H_0$ of the total head $H = \frac{1}{2}|\mathbf{U}^2| + P$. From (2.38) we deduce:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}|\mathbf{u}|^2\right) + \omega\mathbf{u} \cdot (\mathbf{k} \times \mathbf{U}_0) + \nabla \cdot (\mathbf{u}h) = 0. \tag{2.39}$$

The perturbation vorticity equation is

$$\omega_t + \mathbf{U} \cdot \nabla\omega + \mathbf{u} \cdot \nabla\Omega_0 = 0, \tag{2.40}$$

and from this we get:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\omega^2\right) + \omega\mathbf{u} \cdot \nabla\Omega_0 + \nabla \cdot \left(\frac{1}{2}\omega^2\mathbf{U}\right) = 0. \tag{2.41}$$

Now

$$\omega\mathbf{u} \cdot \nabla\Omega_0 = f'(\Psi_0)\omega\mathbf{u} \cdot \nabla\Psi_0 = f'\omega(\mathbf{u} \times \mathbf{k}) \cdot (\nabla\Psi_0 \times \mathbf{k}) = f'\omega\mathbf{u} \cdot (\mathbf{k} \times \mathbf{U}_0).$$

If we now assume $f' < 0$ we can rewrite (2.41), using the fact that f' is independent of t and constant along the basic streamlines, as:

$$\frac{\partial}{\partial t}\left(\frac{\omega^2}{2f'}\right) + \omega\mathbf{u} \cdot (\mathbf{k} \times \mathbf{U}_0) + \nabla \cdot \left(\mathbf{U}\frac{\omega^2}{2f'}\right) - \frac{\omega^2}{2}\mathbf{u} \cdot \nabla\left(\frac{1}{f'}\right) = 0. \tag{2.42}$$

Subtracting (2.42) from (2.39) and integrating over the flow region R, using the divergence theorem and the boundary conditions $\mathbf{u} \cdot \mathbf{n} = 0$ on B and $\omega = 0$ on B_1 we get:

$$\frac{\partial}{\partial t}\int_R \frac{1}{2}\left[|\mathbf{u}|^2 - \frac{\omega^2}{f'}\right] dA + \int_R \frac{1}{2}\,\omega^2 \mathbf{u}\cdot\nabla\left(\frac{1}{f'}\right) dA = \int_{B_2} \mathbf{U}_0\cdot\mathbf{n}\,\frac{\omega^2}{2f'}\,ds. \tag{2.43}$$

So far we have not assumed that the perturbation is small. We now note that the second integral on the left in (2.43) is of *third* order in the perturbation, while the remaining terms are of second order. Thus within the framework of the linear stability theory, this second integral is to be dropped relative to the other terms. Since $\mathbf{U}_0\cdot\mathbf{n} \geqslant 0$ on B_2 and $f' < 0$ we thus obtain, for the linear stability theory:

$$\frac{\partial}{\partial t}\int_R \frac{1}{2}\,[|\mathbf{u}|^2 - \omega^2/f']dA \leqslant 0. \tag{2.44}$$

Since $f' < 0$, this integral is positive definite, and it follows that the energy of the perturbation, though it may possibly increase somewhat over its initial value at the expense of the term $\int_R -\omega^2/2f'dA$, must remain bounded; thus the flow is stable. It is interesting to note that the restriction to the linear stability theory is not necessary for the (rather special) class of basic flows which have f' constant and negative.

The relation of this result to Rayleigh's theorem is easy to see. For a parallel flow with velocity profile $w(y)$ we have

$$f' = \frac{d\Omega_0}{d\Psi_0} = \Omega_0'(y)/\Psi_0'(y) = -\,w''/w.$$

If there are no inflection points, w'' is of one sign, say $w'' > 0$. By adding a suitable uniform translation if necessary which obviously does not affect the stability properties of a parallel flow, we can assume that $w > 0$ throughout, and so $f' < 0$ and the flow is stable. (If $w'' < 0$ we can add a suitable uniform translation so that $w < 0$; but if w'' changes sign this is not possible.) If there is just one inflection point we can assume that $w = 0$ there, and our result thus implies stability if $w''/w > 0$; we thus also obtain Fjørtoft's extension of Rayleigh's theorem.

The above argument is not applicable if $f' \equiv 0$, i.e. for constant vorticity (in particular irrotational) flows. However such flows, like their parallel prototype the plane Couette flow, are always stable, at least with the boundary conditions we have assumed here. For when $\nabla\Omega_0 = 0$, the vorticity equation (2.40) shows that the *perturbation* vorticity ω is constant following particles. Since no new perturbation vorticity is brought in by entering fluid particles, ω cannot grow, and since the perturbation stream function is determined from the Poisson equation $\Delta\psi + \omega = 0$ with $\psi = 0$ on B, ψ cannot grow either.

We conclude this subsection with a brief account of some investigations in which a direct attack on the initial-value problem for plane parallel flow is made.

First we take the solution for plane Couette flow due essentially to Orr [**26**, pp. 26–27; cf. **15**, p. 209] in 1907. Here we put $w(\equiv \bar{u}) = y\,(-1 \leqslant y \leqslant 1)$ in the perturbation of the vorticity equation for two-dimensional flow to get

$$\left(\frac{\partial}{\partial t} + y\frac{\partial}{\partial x}\right)\left(\frac{\partial^2\psi'}{\partial x^2} + \frac{\partial^2\psi'}{\partial y^2}\right) = 0. \tag{2.45}$$

Therefore

$$\frac{\partial^2\psi'}{\partial x^2} + \frac{\partial^2\psi'}{\partial y^2} = F(x - yt, y)$$

for an arbitrary function F differentiable with respect to x. Now any given well-behaved initial velocity distribution satisfying the equation of continuity and the boundary conditions can be expressed in terms of the Fourier integral in x and series in y,

$$\psi'(x,y,0) = \int_{-\infty}^{\infty} d\alpha \cos \alpha x \sum_{n=1}^{\infty} b_n(\alpha) \sin \tfrac{1}{2}n\pi(y+1).$$

This given initial distribution determines $F(x,y)$, and thence $F(x - yt, y)$. The resultant time-dependent equation for ψ' above and the boundary conditions can be shown to have the solution

$$\begin{aligned}\psi'(x,y,t) = \int_{-\infty}^{\infty} d\alpha \sum_{n=1}^{\infty} &\tfrac{1}{2}b_n(\alpha^2 + \tfrac{1}{4}n^2\pi^2) \operatorname{cosech} 2\alpha \\ &\cdot [(\sinh 2\alpha \sin\{\alpha x + (\tfrac{1}{2}n\pi - \alpha t)(y+1)\} \\ &- \sinh \alpha(1-y) \sin \alpha x - \sinh \alpha(y+1) \\ &\cdot \sin\{\alpha x + 2(\tfrac{1}{2}n\pi - \alpha t)\})/\{\alpha^2 + (\tfrac{1}{2}n\pi - \alpha t)^2\} \\ &- (\sinh 2\alpha \sin\{\alpha x - (\tfrac{1}{2}n\pi + \alpha t)(y+1)\} - \sinh\alpha(1-y)\sin\alpha x \\ &- \sinh\alpha(y+1)\cdot\sin\{\alpha x - 2(\tfrac{1}{2}n\pi + \alpha t)\})/\{\alpha^2 + (\tfrac{1}{2}n\pi + \alpha t)^2\}].\end{aligned}$$

Evaluation of this solution for large t shows that $\psi' = O(t^{-1})$ and therefore that plane Couette flow is stable.

A more systematic approach to the initial-value problem, more suitable for application to basic flows other than plane Couette, comes from use of the Laplace transform with respect to time. This approach has been developed for stability problems by Miles [27], Carrier and Chang [28], Case [29, 30] and Dikii [31, 32]. It has been reconciled with the method of normal modes, both for inviscid and slightly viscous fluid, by Case [33] and Lin [34].

First let us take Case's [29] solution of plane Couette flow, for comparison with the other solution above. Let the Fourier transform with respect to x and the Laplace transform with respect to t give

$$\bar{\Psi}(y,\alpha,t) \equiv \int_{-\infty}^{\infty} e^{-i\alpha x}\psi'(x,y,t)dx,$$

$$\Psi(y;\alpha,p) \equiv \int_{0}^{\infty} e^{-pt}\bar{\Psi}(y,\alpha,t)dt.$$

Equation (2.45) has the Fourier-Laplace transform

$$\left(\frac{\partial^2}{\partial y^2} - \alpha^2\right)\Psi(y;\alpha,p) = \frac{1}{p + i\alpha y}\left(\frac{\partial^2}{\partial y^2} - \alpha^2\right)\bar{\Psi}(y,\alpha,0). \tag{2.46}$$

Next invert the Laplace transform, using the results that

$$\bar{\Psi}(y,\alpha,t) = \frac{1}{2\pi i}\int_{C} \Psi(y;\alpha,p)e^{pt}dp,$$

$$\frac{1}{2\pi i}\int_{C} \frac{e^{pt}}{p + i\alpha y}\,dp = e^{-i\alpha yt};$$

where C is a Bromwich contour parallel to the imaginary axis and on the right of all singularities of the integrand. Then

$$\left(\frac{\partial^2}{\partial y^2} - \alpha^2\right)\bar{\Psi}(y,\alpha,t) = e^{-i\alpha yt}\left(\frac{\partial^2}{\partial y^2} - \alpha^2\right)\bar{\Psi}(y,\alpha,0). \tag{2.47}$$

Given the conditions that $\psi' = 0$ on the boundaries for all x,t and therefore that

$$\Psi, \bar{\Psi} = 0 \qquad (y = \pm 1),$$

we show in the usual way that the solution is

$$\bar{\Psi}(y,\alpha,t) = \int_{-1}^{1} G(y,y_0)e^{-i\alpha y_0 t}\left(\frac{\partial^2}{\partial {y_0}^2} - \alpha^2\right)\bar{\Psi}(y_0,\alpha,0)dy_0,$$

where the Green's function

$$G(y,y_0) = \begin{cases} -\dfrac{\sinh\alpha(1+y)\sinh\alpha(1-y_0)}{\alpha\sinh 2\alpha} & (-1 \leqslant y \leqslant y_0) \\[2ex] -\dfrac{\sinh\alpha(1+y_0)\sinh\alpha(1-y)}{\alpha\sinh 2\alpha} & (y_0 \leqslant y \leqslant 1). \end{cases}$$

In principle we know $\tilde{\Psi}(y,\alpha,0)$ as the Fourier transform of $\psi'(x,y,0)$, the given distribution of ψ' at $t = 0$. So we have $\tilde{\Psi}(y,\alpha,t)$, and $\psi'(x,y,t)$ on inverting the Laplace transform, for all $t \geqslant 0$. To study the behavior of ψ' for large t, Case [**29**, p. **145**] evaluated the inverted transforms asymptotically and showed that $\psi' = O(1/t)$ for fairly general $\psi'(x,y,0)$. This result agrees with Orr's solution above and with the solution in Section II.4.a below, obtained with normal modes.

To extend this approach for a general basic velocity profile $w(y)$, we follow the work of Case [**29**] and Dikii [**32**]. Here the Fourier-Laplace transform of the linearized vorticity equation is

$$\left(\frac{\partial^2}{\partial y^2} - \alpha^2 - \frac{i\alpha D^2 w}{p + i\alpha w}\right)\Psi(y;\alpha,p) = \frac{1}{p + i\alpha w}\left(\frac{\partial^2}{\partial y^2} - \alpha^2\right)\tilde{\Psi}(y,\alpha,0)$$

$$\equiv H(y,\alpha)/(p + i\alpha w), \tag{2.48}$$

say, where we may suppose H known in terms of the Fourier transform of the initial distribution $\psi'(x,y,0)$. The transforms of the boundary conditions are

$$\Psi(y;\alpha,p) = 0 \qquad (y = y_1,y_2). \tag{2.49}$$

The solution of (2.48), (2.49)

$$\Psi(y;\alpha,p) = \int_{y_1}^{y_2} G(y,y_0)H(y_0,\alpha)/(p + i\alpha w_0)dy_0; \tag{2.50}$$

where

$$G(y,y_0) = \begin{cases} -W^{-1}(\Psi_1,\Psi_2)\Psi_1(y;\alpha,p)\Psi_2(y_0;\alpha,p) & (y_1 \leqslant y \leqslant yy_0) \\ -W^{-1}(\Psi_1,\Psi_2)\Psi_1(y_0;\alpha,p)\Psi_2(y;\alpha,p) & (y_0 \leqslant y \leqslant y_2); \end{cases}$$

the (constant) Wronskian

$$W(\Psi_1,\Psi_2) \equiv \Psi_1 d\Psi_2/dy - \Psi_2 d\Psi_1/dy;$$

and $\Psi_1(y;\alpha,p)$, $\Psi_2(y;\alpha,p)$ are any two given solutions of the homogeneous form of equation (2.48),

$$\left(\frac{\partial^2}{\partial y^2} - \alpha^2 - \frac{i\alpha^2 D^2 w}{p + i\alpha w}\right)\Psi = 0, \tag{2.51}$$

such that

$$\Psi_1(y_1;\alpha,p) = 0, \qquad \Psi_2(y_2;\alpha,p) = 0.$$

As in the case of plane Couette flow, we can now invert the transforms, etc., to find $\psi'(x,y,t)$ as an explicit multiple integral of $\psi'(x,y,0)$.

The crucial integral is the inversion of the Laplace transform,

$$\tilde{\Psi}(y,\alpha,t) = \frac{1}{2\pi i}\int_C \Psi(y;\alpha,p)e^{pt}dp. \tag{2.52}$$

We seek to evaluate this integral for large t to see whether the flow is stable. When t is large, $|e^{pt}|$ is large along C where $\operatorname{Re} p > 0$. However, as we shall see, cancellation of the large components of the integrand means that the integral may not be large. The cancellation makes it advantageous to move the contour close to the imaginary axis and thereby reduce the magnitude of $|e^{pt}|$. Following the usual method of deforming the contour C, we must study the singularities of Ψ, which is given by equation (2.50). Since Ψ_1,Ψ_2 satisfy equation (2.51), their only singularities can be at the singularities of this equation, namely where $w(y) = ip/\alpha = c$. These singularities lie only on the imaginary axis of the p-plane. Thus it is convenient to take a new contour on the imaginary axis with indentations to the right of each singularity or just to the right of the imaginary axis, say from $\varepsilon - i\infty$ to $\varepsilon + i\infty$ for small ε. To find $\tilde{\Psi}$ in terms of the integral along the new contour we must know the other singularities of the integrand in the half-plane $\operatorname{Re} p > 0$; these coming from zeros of the denominator of the Green's function $G(y,y_0)$, i.e. from the solutions of

$$W(\Psi_1,\Psi_2) = 0 \qquad (\operatorname{Re} p > 0).$$

This can be seen to be just the eigenvalue relation for the discrete spectrum of normal modes with $c_i > 0$. [Of course, equation (2.51) is essentially the Rayleigh stability equation with $p = -i\alpha c$.] Supposing these eigenvalues are already known, we may evaluate the residues at these singularities of the integrand in equation (2.52) and find

$$\tilde{\Psi}(y;\alpha,t) = \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \Psi(y,\alpha,p)e^{pt}dt + \sum_{\text{discrete spectrum}}\begin{pmatrix}\text{exponentials} & \\ \text{growing like} & e^{\alpha c_i t}\end{pmatrix}.$$

The fastest growing component will be dominant for large t. It will come from the discrete spectrum if there is one, being the term growing like $e^{\alpha c_i t}$ for the largest value of αc_i. Case [**29**, p. 148] and Dikii [**32**, p. 1180] have indicated that the integral above over the continuous spectrum decays like $1/t$, so the discrete spectrum alone is associated with instability. Thus, in seeking a criterion for instability, we may use the method of normal modes and ignore the continuous spectrum.

4. *Stability Characteristics of Various Basic Flows*

As will be seen, most of the basic flows with known stability characteristics are piecewise linear. However, the characteristics of many smoothly-

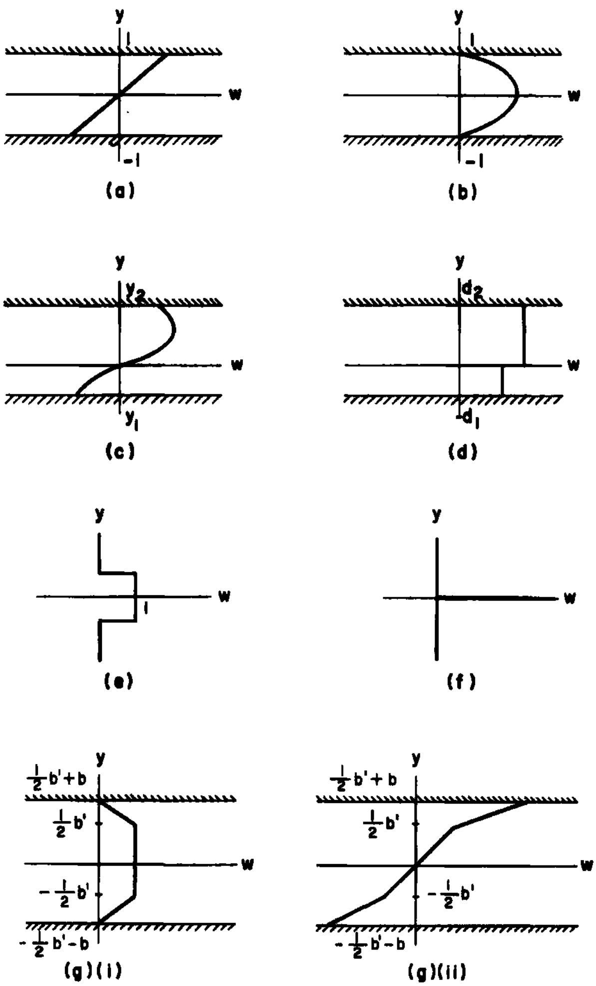

FIG. 4. (a) Plane Couette flow: $w = y$. (*b*) Plane Poiseuille flow: $w = 1 - y^2$. (c) Sinusoidal flow: $w = \sin y$. (d) Vortex sheet: $w = V_2(d_2 \geqslant y > 0)$, $w = V_1(0 > y \geqslant -d_1)$. (e) Rectangular jet: $w = 1(|y| < 1)$, $w = 0(|y| > 1)$. (f) Thin jet $w = \{\delta(y)\}^{1/2}$. (g) (i) Symmetric jet in channel. (g) (ii) Antisymmetric shear layer in channel.

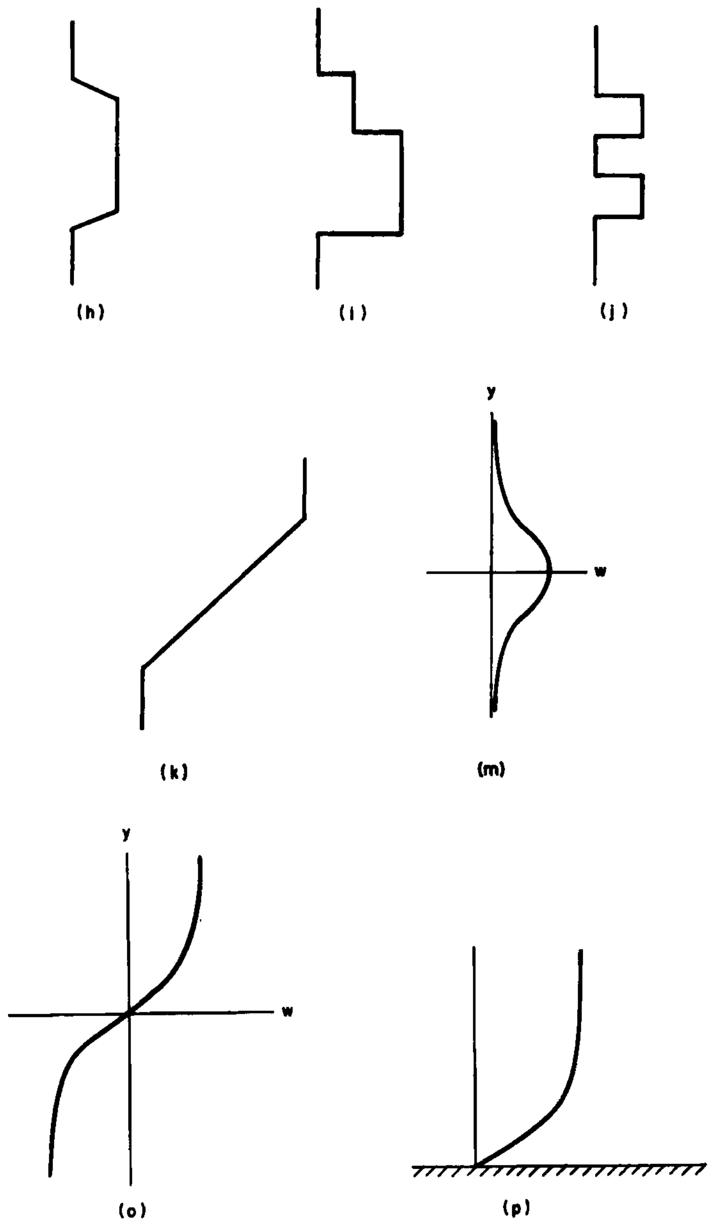

Fig. 4. (continued) (h) Unbounded symmetric-trapezium jet. (i) Double jet. (j) Symmetric separated double jet. (k) Shear layer. (m) Bickley jet: $w = \operatorname{sech}^2 y$. (o) Hyperbolic-tangent shear layer: $w = \tanh y$. (p) Boundary layer with suction: $w = 1 - e^{-y}$.

varying basic flows are partially known, and the advent of electronic computers is allowing many to be found completely in numerical terms. Diagrams of the basic flows accompany the examples below.

(a) *Plane Couette Flow*

We follow the treatment [35] of Fjørtoft and Høiland. When

$$w = y \qquad (-1 \leqslant y \leqslant 1)$$

the Rayleigh stability equation becomes

$$(y - c)(D^2 - \alpha^2)\varphi = 0.$$

Unless $c = y$ in the domain of flow, i.e. unless $-1 \leqslant c \leqslant 1$, this gives only

$$(D^2 - \alpha^2)\varphi = 0,$$

which has no solution vanishing at both $y = \pm 1$. Thus the basic flow is exceptional in that it has no eigensolutions of the discrete c-spectrum. However, when $-1 \leqslant c \leqslant 1$, the stability equation also gives

$$(D^2 - \alpha^2)\varphi = \delta(y - c),$$

where δ is the Dirac delta-function. This representation in terms of a generalized function is admissible because the typical component of wavenumber α is really only one component in an integral. It gives a solution of the continuous c-spectrum with eigenfunction

$$\varphi = \begin{cases} \dfrac{\sinh \alpha(c-1) \sinh \alpha(y+1)}{\alpha \sinh 2\alpha} & (-1 \leqslant y \leqslant c) \\[2ex] \dfrac{\sinh \alpha(c+1) \sinh \alpha(y-1)}{\alpha \sinh 2\alpha} & (c \leqslant y \leqslant 1) \end{cases}$$

for each value of α and for each value of c in the interval $(-1,1)$. The set of eigenfunctions is complete so that an arbitrary initial disturbance of the velocity field can be represented as a sum or integral of them [cf. **35**, pp. 11–12].

(b) *Plane Poiseuille Flow*

When

$$w = 1 - y^2 \qquad (-1 \leqslant y \leqslant 1)$$

there is no point of inflection, so the flow is stable (though the flow is unstable for viscous fluid at large Reynolds numbers by Heisenberg's criterion). It seems that no further examination of the stability characteristics has been made except through the viscous problem.

(c) *Sinusoidal Basic Flow*

When

$$w = \sin y \qquad (y_1 \leqslant y \leqslant y_2)$$

the stability equation becomes

$$(\sin y - c)(D^2 - \alpha^2)\varphi + \sin y\varphi = 0.$$

Now $D^2w = 0$ where $y = y_s = n\pi$ ($n = 0, \pm 1, \pm 2, \ldots$). If there is no value y_s in the interval (y_1, y_2) the flow is certainly stable by Rayleigh's criterion. If there is at least one value, we may suppose $y_s = 0$ without loss of generality, so $y_1 \leqslant 0 \leqslant y_2$. To find the s-solution we put $c = w_s = 0$. Then

$$\sin y\{D^2\varphi_s + (1 - \alpha_s^2)\varphi_s\} = 0$$

where

$$\varphi_s = 0 \qquad (y = y_1, y_2).$$

In finding the s-solution we ignore the factor $\sin y$ (and thereby discard the stable eigensolution corresponding to $c = 0$ in the continuous spectrum) to get

$$\varphi_s = \sin\{n\pi(y - y_1)/(y_2 - y_1)\},$$

$$\alpha_s = \{1 - n^2\pi^2/(y_2 - y_1)^2\}^{1/2}$$

for each positive integer $n < (y_2 - y_1)/\pi$. It follows that the flow is unstable if $(y_2 - y_1) > \pi$, but stable otherwise although the point of inflection lies at $y = 0$ in the field of flow. This counter-example to the sufficiency of Rayleigh's necessary condition for instability is due to Tollmien [**14**; see also **7**, pp. 219–220].

(d) *Vortex Sheet*

When

$$w = \begin{cases} V_2 & (d_2 \geqslant y > 0) \\ V_1 & (0 > y \geqslant -d_1) \end{cases}$$

the eigenvalue relation is [**2**; **3**, p. 379]

$$(c - V_2)^2 \coth \alpha d_2 + (c - V_1)^2 \coth \alpha d_1 = 0$$

and the eigenfunction

$$\varphi = \begin{cases} (c - V_2)\cosh \alpha d_1 \cosh \alpha(d_2 - y) & (d_2 \geqslant y > 0) \\ (c - V_1)\cosh \alpha d_2 \cosh \alpha(y + d_1) & (0 > y \geqslant - d_1). \end{cases}$$

Therefore

$$c = \{V_2 \coth \alpha d_2 + V_1 \coth \alpha d_1 + i|V_2 - V_1|[\coth \alpha d_1 \coth \alpha d_2]^{1/2}\}/\{\coth \alpha d_2 + \coth \alpha d_1\}.$$

This gives complex c and instability for each α.

When $d_1, d_2 = \infty, V_2 = 1 = -V_1$, this gives $c = i$. This is an example of an antisymmetric flow with $c_r = 0$. It also gives the limiting eigenvalue (2.29) as $\alpha \to 0$ of any smoothly-varying flow with $w(\infty) = 1 = -w(-\infty)$. The elevation of the material surface with mean level $y = 0$ is $\eta = F_0 e^{\alpha(ix+t)}$.

If $V_2 \to V_1 = V$ and $d_1, d_2 = \infty$, one finds $c = -V$ and $\eta = (A + Bt)e^{i\alpha(x - Vt)}$ for arbitrary constants A, B. This has been described as the instability of a flapping flag.

(e) *Rectangular Jet*

When

$$w = \begin{cases} 0 & (|y| > 1) \\ 1 & (|y| < 1) \end{cases}$$

there is [3, pp. 380–381] a sinuous mode with

$$c = \{1 + i(\coth \alpha)^{1/2}\}/\{1 + \coth \alpha\}$$

and a varicose mode with

$$c = \{1 + i(\tanh \alpha)^{1/2}\}/\{1 + \tanh \alpha\}.$$

Each mode is unstable for all α.

(f) *Thin Jet*

When

$$w = \{\delta(y)\}^{1/2} \qquad (-\infty < y < \infty)$$

the velocity is infinite at $y = 0$, but the total momentum flux $\int_{-\infty}^{\infty} w^2 dy = 1$. With piecewise solution of the stability equation and use of conditions (2.12) at infinity and (2.15) at $y = 0$, we find

$$c = (\tfrac{1}{2}\alpha)^{1/2} i$$

and

$$F = \begin{cases} e^{-\alpha y} & (y > 0) \\ e^{\alpha y} & (y < 0). \end{cases}$$

Although w is an even function, there is only the sinuous mode above. It is unstable for all α, and is in accord with limit (2.30) for a smoothly-varying jet of the same momentum flux.

(g) *Channel Flows*

Rayleigh [3, pp. 385–390] found eigenvalue relations for a general continuous piecewise-linear velocity profile in a channel, there being two discontinuities of the velocity derivative between the walls of the channel. In each case the eigenvalue relation is a quadratic equation for c. We give a few examples below.

(i) For a symmetric-trapezium jet with

$$w = \begin{cases} V & (|y| < \tfrac{1}{2}b') \\ Vb^{-1}(b + \tfrac{1}{2}b' - y) & (\tfrac{1}{2}b' \leqslant |y| \leqslant b + \tfrac{1}{2}b'), \end{cases}$$

Rayleigh found

$$c = V - V\{\alpha b \sinh \alpha(2b + b')\}^{-1}\{\sinh \alpha b \sinh \alpha(b + b') \pm \sinh^2 \alpha b\}.$$

Thus both the sinuous and varicose modes are stable, as was to be expected for this flow which approximates a smoothly-varying profile with curvature of one sign.

If further the middle layer is absent, then $b' = 0$ and

$$c = V - V\alpha^{-1} b^{-1} \tanh \alpha b.$$

(ii) For an antisymmetric shear layer with

$$w = \begin{cases} V + \lambda V(y - \tfrac{1}{2}b') & (\tfrac{1}{2}b' \leqslant y \leqslant b + \tfrac{1}{2}b') \\ 2Vy/b' & (|y| \leqslant \tfrac{1}{2}b') \\ -V + \lambda V(y + \tfrac{1}{2}b') & (-\tfrac{1}{2}b' \geqslant y \geqslant -b - \tfrac{1}{2}b') \end{cases}$$

Rayleigh gives [3, p. 388]

$$c^2 = V^2\{[(\lambda - 2/b') \sinh \alpha b \sinh \alpha b' + \alpha \sinh \alpha(b + b')]^2 - \alpha^2 \sinh^2 \alpha b\}/\{\alpha^2 \sinh \alpha b' \sinh \alpha(2b + b')\}.$$

Thus there is instability for some α only when $-1/b < \lambda \leqslant -1/b + 2/b'$. This result in some sense exemplifies Fjørtoft's result for smoothly-varying profiles represented in Figure 2.

(h) *Unbounded Symmetric-Trapezium Jet*

When

$$w = \begin{cases} 0 & (|y| > 1) \\ 1 - (|y| - a)/(1 - a) & (1 > |y| > a) \\ 1 & (a > |y|) \end{cases}$$

the eigenvalue relation is [3, p. 397]

$$\begin{aligned} &4(1-a)^2\alpha^2c^2 - 2(1-a)\alpha c\{2(1-a)\alpha \mp e^{-2a\alpha}(1 - e^{-2(1-a)\alpha})\} \\ &\quad + \{-1 + 2(1-a)\alpha + e^{-2(1-a)\alpha}\} \mp e^{-2a\alpha}\{1 - [1 + 2(1-a)\alpha]e^{-2(1-a)\alpha}\} \\ &\quad = 0, \end{aligned}$$

where the upper sign is for the sinuous and the lower for the varicose mode.

When $a = 1$ we get an example of the rectangular jet (Section II.4.e). When $a = 0$ we get a triangular jet, with

$$2\alpha^2c^2 + \alpha(1 - 2\alpha - e^{-2\alpha})c + \{\alpha(1 + e^{-2\alpha}) - 1 + e^{-2\alpha}\} = 0$$

for the sinuous mode, and

$$c = (2\alpha)^{-1}(1 - e^{-2\alpha})$$

for the varicose mode. Here the varicose mode is always stable, but the sinuous mode is unstable for $0 < \alpha < \alpha_s \doteqdot 1.8$. The logarithmic growth rate αc_i of the sinuous mode is greatest when $\alpha \doteqdot 1.2$.

(i) *Double Jet*

When

$$w = \begin{cases} 0 & (|y| > 1) \\ U & (-1 < y < 0) \\ 1 & (0 < y < 1) \end{cases}$$

the eigenvalue relation is [8, p. 269]

$$\begin{aligned} &(1-c)^2\{c^2 + (1-c)^2 \tanh\alpha\}\{c^2 \tanh\alpha + (U-c)^2\} \\ &\quad + (U-c)^2\{c^2\tanh\alpha + (1-c)^2\}\{c^2 + (U-c)^2\tanh\alpha\} = 0. \end{aligned}$$

This gives three modes unstable for each α, with

$$c = i\{\tfrac{1}{2}(1 + U^2)\alpha\}^{1/2} + \dots, \qquad 1 + i(\tfrac{1}{2}\alpha)^{1/2} + \dots,$$

or

$$U\{1 + i(\tfrac{1}{2}\alpha)^{1/2} + \dots\}$$

for small α.

(j) *Symmetric Separated Double Jet*

When

$$w = \begin{cases} 0 & (|y| > 2 \quad \text{or} \quad |y| < 1) \\ 1 & (1 < |y| < 2) \end{cases}$$

the eigenvalue relation is [8, p. 269]

$$\{c/(1 - c)\}^2(1 + \{c/(1 - c)\}^2 \tanh \alpha) + 1 + \{c/(1 - c)\}^2 \coth \alpha = 0$$

for the sinuous modes and

$$\{c/(1 - c)\}^2(\{c/(1 - c)\}^2 + \coth \alpha) + \{c/(1 - c)\}^2 + \tanh \alpha = 0$$

for the varicose modes. This gives four modes unstable for each α.

(k) *Shear Layer*

When

$$w = \begin{cases} y/|y| & (|y| > a) \\ y/a & (|y| < a) \end{cases}$$

the eigenvalue relation is [3, p. 393]

$$c^2 = (4a^2\alpha^2)^{-1}\{(1 - 2a\alpha)^2 - e^{-4a\alpha}\}.$$

Only two special cases are of interest: $a = 0$ and $a = 1$. The former gives Kelvin-Helmholtz instability of the vortex sheet (Section II.4.d). When $a = 1$, c is pure imaginary for $\alpha < \alpha_s \doteqdot 0.64$ and real for $\alpha \geqslant \alpha_s$. The logarithmic growth rate αc_i is greatest when $\alpha \doteqdot 0.4$.

(l) *Double Vortex Sheet*

When

$$w = \begin{cases} y/|y| & (|y| > 1) \\ 0 & (|y| < 1) \end{cases}$$

the eigenvalue relation is [8, p. 270]

$$\{c^4 + (1 - c^2)^2\} \tanh 2\alpha + 2c^2(1 + c^2) = 0.$$

There are two modes unstable for each α. For $\alpha \leqslant \frac{1}{2}\tanh^{-1}(5^{1/2}-2)$ both unstable roots c are pure imaginary; for $\alpha > \frac{1}{2}\tanh^{-1}(5^{1/2}-1)$ both are complex. Thus an antisymmetric profile does not necessarily give $c_r = 0$.

(m) *Bickley Jet*

When

$$w = \operatorname{sech}^2 y \qquad (-\infty < y < \infty)$$

the s-eigensolution for the sinuous mode is [36]

$$\varphi_s = \operatorname{sech}^2 y, \qquad w_s = \tfrac{2}{3}, \qquad \alpha_s = 2$$

and for the varicose mode [37]

$$\varphi_s = \operatorname{sech} y \tanh y, \qquad w_s = \tfrac{2}{3}, \qquad \alpha_s = 1.$$

The Lin perturbation (2.20) then gives [8, p. 281]

$$\partial c/\partial \alpha^2 = 0.0423 - i\,0.0278 \qquad (\alpha = 2 - 0)$$

for the sinuous mode, and

$$\partial c/\partial \alpha^2 = -0.0264 - i\,0.0835 \qquad (\alpha = 1 - 0)$$

for the varicose mode. For small α the sinuous mode has [8, p. 281]

$$c = \alpha + i\{\tfrac{4}{3}\alpha - \alpha^2 - \tfrac{1}{8}\alpha^2 \log(24\alpha^{-1}) - \tfrac{1}{8}\alpha^2 \pi i\}^{1/2} + \ldots$$

and the varicose [8, p. 279]

$$c = 1 + e^{(2/3)\pi i}(\tfrac{1}{4}\pi\alpha)^{2/3}\{1 + \tfrac{1}{3}e^{(2/3)\pi i}(\tfrac{1}{2}\pi^2\alpha^2)^{1/3} + O(\alpha)\}.$$

Numerical values of c_i, c_r for some α have been given by Lessen and Fox [38; cf. 8]. We give results of a somewhat more complete recent calculation in Table 1.

Stability characteristics of other jet or wake profiles have been computed by Hollingdale [39], Haurwitz and Panofsky [40], and Sato [41].

(n) *Antisymmetric Double Jets*

When

$$w = \operatorname{sech}^m y \tanh y \qquad (-\infty < y < \infty, \quad m \geqslant -1/2)$$

the s-eigensolution is [42]

$$\varphi_s = \operatorname{sech}^{m+1} y, \qquad w_s = 0, \qquad \alpha_s{}^2 = 2m + 1.$$

It seems that solutions corresponding to inflection points other than $y = 0$ are not known. Numerical results are given in Table 2 for $w = \frac{3}{2}\sqrt{3}\,\mathrm{sech}^2 y \tanh y$. Note that the "propagating mode", though associated with the

TABLE 1. INSTABILITY CHARACTERISTICS OF THE BICKLEY JET

	Sinuous Mode		Varicose Mode	
α	c_r	c_i	c_r	c_i
0.05	0.030	0.169	0.931	0.079
0.1	0.086	0.216	0.889	0.104
0.2	0.166	0.257	0.826	0.121
0.3	0.229	0.267	0.780	0.119
0.4	0.280	0.263	0.745	0.108
0.5	0.323	0.252	0.719	0.092
0.6	0.362	0.236	0.700	0.074
0.7	0.395	0.218	0.686	0.056
0.8	0.424	0.198		
0.9	0.451	0.179		
1.0	0.475	0.159	0.667	0
1.2	0.520	0.121		
1.4	0.559	0.085		
1.6	0.597	0.053		
1.8	0.632	0.024		
2.0	0.667	0		

TABLE 2. INSTABILITY CHARACTERISTICS FOR $w = \frac{3}{2}\sqrt{3}\,\mathrm{sech}^2 y \tanh y$

Standing Mode ($c_r = 0$)		Propagating Mode		
α	c_i	α	c_r	c_i
0	0	0	1.0	0
0.1	0.270	0.05	0.944	0.071
0.2	0.362	0.1	0.903	0.103
0.3	0.425	0.2	0.831	0.133
0.5	0.495	0.3	0.775	0.139
0.7	0.512	0.4	0.732	0.130
1.0	0.472	0.5	0.700	0.116
1.5	0.317	0.6	0.676	0.098
1.8	0.199	0.7	0.660	0.077
2.0	0.114	0.8	0.648	0.062
5	0	0.9	0.641	0.044

inflection point at $w = 2^{-1/2}$, does not have $c_r \to 2^{-1/2}$ as $c_i \to 0$. This is a case where the Reynolds stress for the neutral mode has two compensating jumps. There is of course another propagating mode, associated with the inflection point at $w = -2^{-1/2}$, for which the sign of c_r is reversed.

(o) *Hyperbolic-Tangent Shear Layer*

The special case $m = 0$ of the example in Section II.4.n gives the shear layer

$$w = \tanh y \qquad (-\infty < y < \infty),$$

which has better known stability characteristics [**8**, p. **281**]. The s-eigensolution is [**43**, **44**]

$$\varphi_s = \operatorname{sech} y, \qquad w_s = 0, \qquad \alpha_s = 1.$$

The Lin perturbation gives

$$c_i = (2/\pi)(1-\alpha) + O(1-\alpha)^2 \qquad \text{as} \qquad \alpha \to 1-$$

and for small α

$$c_i = 1 - 1.785\alpha + 1.526\alpha^2 + \dots .$$

Numerical calculations for this case have recently been given by Michalke [**45**].

Stability characteristics of some other smoothly-varying profiles of shear-layer type have been computed by Hollingdale [**39**], Carrier [cf. **46**] and Lessen and Fox [**38**]. Their results are similar to those for the hyperbolic tangent shear layer.

(p) *Boundary Layer with Suction*

When

$$w = 1 - e^{-y} \qquad (y \geqslant 0).$$

Lin pointed out [47, p. 90] that the Rayleigh stability equation can be transformed into the hypergeometric equation, a result used by Chiarulli and Freeman [**47**].

III. Waves and Stability of Plane Parallel Flow of Inviscid Fluid under the Actions of Various Force Fields

1. *Introduction*

The theory in this section consists chiefly of developments by Scandinavian meteorologists in the 1920's and 1930's of Kelvin's paper [2] of 1871. Their work is collected in a book by Bjernes, Bjerknes, Solberg, and Bergeron [48] of 1933. It may be more accessible to the reader in Chapter X of the book by Godske, Bergeron, Bjerknes, and Bundgaard [49], which also has a bibliography of relevant meteorological papers published by 1950. These meteorologists have considered the equations, boundary conditions, and eigensolutions for piecewise-constant velocity profiles under the influences of combinations of density variation, compressibility, and rotation. Haurwitz [50] has considered the equations and boundary conditions under the same combinations of force fields for smoothly-varying profiles also.

The very generality of these combinations and their meteorological context has obscured some of the fluid dynamics and enabled other authors to duplicate their work in ignorance of it. So we shall consider the force fields separately in order to simplify the understanding and to compare the effects of different force fields. Of course any combination of these force fields can be considered, and, indeed, there are dozens of papers dealing with various combinations. However these combinations need no special techniques in their treatment, so we shall not describe then. If the reader is interested in a physical problem of stability of parallel flow affected by some combination, he may readily adapt the methods used for the fields separately.

Again, we have ignored the effect of surface tension on an interfacial boundary condition. This may be physically important, but it is a simple matter to apply the method of Kelvin [2] to cater for surface tension.

Treating one external force field at a time, and finding its effects on the inertial instability discussed in the previous section, we first give the stability equation and boundary conditions for two-dimensional wave-disturbances, and comment on the validity or invalidity of Squire's theorem. Then we find the eigensolutions for two unbounded basic velocity profiles, that of static equilibrium ($w = 0$), and that of a vortex sheet ($w = y/|y|$). It is found that for static equilibrium a neutrally stable wave motion may occur for most of the external force fields. For example, sound waves may occur when the external "force" is due to compressibility of the fluid. Such a wave motion is important both for its own sake and for its representation of some stability characteristics of profiles with shear. The example of a vortex sheet is also important, because a vortex sheet is the simplest flow with shear, and because its stability characteristics represent those of any smoothly-varying shear layer for long-wave disturbances. Finally we summarize briefly the literature on stability problems for each force field.

Thus the section is a rapid survey of problems within our purview. Their juxtaposition emphasizes their similarities. In the next section we shall emphasize those similarities further, with dimensional and physical arguments. Then we shall choose one example of the external force fields, that of buoyancy of a fluid of variable density, as a prototype and discuss its stability characteristics in detail.

2. *Internal Gravity Waves and Stability of a Fluid of Variable Density*

Following Kelvin [2], many authors have studied the stability of parallel horizontal basic flows of incompressible fluid with piecewise-constant velocity and density distributions (i.e. with velocity and density uniform in layers but varying from layer to layer) under the action of gravity. Rayleigh [51] was the first to consider the stability of fluid of smoothly-varying density distribution $\bar{\rho}_*(y_*)$ at rest, y_* being the height. For the particular density distribution $\bar{\rho}_* = \rho_0 \exp\,(-\beta y)$ $(-\infty < y_* < \infty$, constants $\rho_0, \beta > 0)$, he found neutrally-stable *internal gravity waves* of phase velocity

$$c_* = a_* \equiv (g\beta/\alpha_*^{\,2})^{1/2}, \tag{3.1}$$

on neglect of $\alpha_*\beta$, i.e. on neglect of the variation of inertia due to the variation of density but not of the buoyancy.

Taylor [52], Goldstein [53], and Haurwitz [50] considered two-dimensional disturbances of parallel horizontal flow of incompressible fluid under gravity with smoothly-varying profiles of velocity and density. Their analysis leads to the stability equation with dimensionless form,

$$(w-c)(D^2-\alpha^2)\varphi - (D^2w)\varphi + J\varphi/(w-c) - K\{(w-c)D\varphi - (Dw)\varphi\} = 0; \tag{3.2}$$

where the local Richardson number of the basic flow is

$$J(y) \equiv -(gL^2 d\bar{\rho}_*/dy_*)/V^2\bar{\rho}_*\,, \tag{3.3}$$

a measure of the characteristic ratio of the buoyancy to inertia; and

$$K(y) \equiv -L d\bar{\rho}_*/\bar{\rho}_* dy_*\,, \tag{3.4}$$

is a measure of the characteristic ratio of the variation of inertia due to heterogeneity of the fluid to the inertia. (Note that the Froude number,

$$F \equiv gL/V^2 = J(y)/K(y), \tag{3.5}$$

is independent of y.) The stability equation can be seen to reduce to the Rayleigh stability equation for a homogeneous fluid, i.e. for $d\bar{\rho}_*/dy_* = 0$. At the walls (or infinity) we use boundary conditions (2.12) as before. At a discontinuity of w, Dw or $\bar{\rho}$ the continuity of the normal velocity component and of the pressure imply respectively that

$$[\varphi/(w-c)] = 0, \tag{3.6}$$
$$[\bar{\rho}\{(w-c)D\varphi - (Dw)\varphi + F\varphi/(w-c)\}] = 0.$$

In fact Squire's theorem may be extended to this case [54], giving for each three-dimensional wave making angle θ with the basic flow a two-dimensional wave of the same growth rate but effective Richardson number $J\cos^2\theta$ and Froude number $F\cos^2\theta$. Usually density variation with $d\bar{\rho}_*/dy_* < 0$ acts as a stabilizing agent, so we seek criteria of stability with minima of J or F. Thus two-dimensional disturbances are the most unstable usually, and anyway the stability characteristics of any three-dimensional disturbance follow at once from knowledge of the characteristics of all the two-dimensional disturbances.

The outstanding problem of this case is *Kelvin-Helmholtz instability* of the basic vortex sheet with

$$w_* = \begin{cases} V \\ -V, \end{cases} \qquad \bar{\rho}_* = \begin{cases} \rho_{1*} & (y_* > 0) \\ \rho_{2*} & (y_* < 0). \end{cases} \tag{3.7}$$

Here the eigenfunction can be shown [2] to be

$$\varphi_* = \begin{cases} (c_* - V)\exp(-\alpha_* y_*) & (y_* > 0) \\ (c_* + V)\exp(\alpha_* y_*) & (y_* < 0) \end{cases} \tag{3.8}$$

where the eigenvalue

$$c_* = V(\rho_{1*} - \rho_{2*} \pm \{g(\rho_{2*}^2 - \rho_{1*}^2)/\alpha_* V^2 - 4\rho_{1*}\rho_{2*}\}^{1/2})/(\rho_{1*} + \rho_{2*}). \tag{3.9}$$

Thus the flow is always unstable if there is heavy fluid above lighter ($\rho_{2*} < \rho_{1*}$). If there is heavy fluid below lighter, the flow is stable to those waves with

$$\alpha_* \leqslant g(\rho_{2*}^2 - \rho_{1*}^2)/4V^2\rho_{1*}\rho_{2*}, \tag{3.10}$$

but unstable to all shorter waves.

If $V = 0$, then we have

$$c_* = \pm\{g(\rho_{2*} - \rho_{1*})/\alpha_*(\rho_{1*} + \rho_{2*})\}^{1/2} \tag{3.11}$$

for neutrally-stable internal waves at the interface of the fluids. Taylor [55] recognized that the same analysis gives

$$c_* = \pm\{(g - g')(\rho_{2*} - \rho_{1*})/\alpha_*(\rho_{1*} + \rho_{2*})\}^{1/2}$$

when the interface has constant acceleration g' downwards. Thus there is instability when $(g - g')$ and $(\rho_{2*} - \rho_{1*})$ have opposite signs, i.e. when the resultant acceleration $(g' - g)$ is directed from the lighter toward the heavier fluid. This is called *Rayleigh-Taylor instability* on account of Taylor's work and the paper by Rayleigh [51] on waves in a heterogeneous fluid at rest, which he showed unstable when $d\rho_*/dy_* > 0$ somewhere. For an example, the surface of water in a bucket is subject to Rayleigh-Taylor instability when moved downwards with constant acceleration greater in magnitude than g.

If $\rho_{2*} = \rho_{1*}$, then eigenvalue (3.9) reduces to the value (Section II.4.d) for a homogeneous fluid.

In many geophysical problems the effects of variation of inertia are negligible though the buoyancy is not, so we may approximate $K = 0$ with $J \neq 0$. Then the stability equation becomes

$$(w - c)(D^2 - \alpha^2)\varphi - (D^2 w)\varphi + J(y)\varphi/(w - c) = 0. \tag{3.12}$$

When $\bar{\rho}_* = \rho_0 \exp(-\beta y_*)$, it can be seen that $J(y) = g\beta L^2/V^2$, a constant. When further $w_* = 0$, we get Rayleigh's internal gravity waves with

$$\varphi = \text{constant}, \qquad c = \pm (J/\alpha^2)^{1/2} = \pm a.$$

When $\bar{\rho}_* = \rho_0 \exp(-\beta y_*)$, $w_* = y_*/|y_*|$ it can be shown [48, 50, 56] that the eigenfunction is

$$\varphi = \begin{cases} (c-1)\exp(-\{1 - a^2/(c-1)^2\}^{1/2}\alpha y) & (y > 0) \\ (c+1)\exp(\{1 - a^2/(c+1)^2\}^{1/2}\alpha y) & (y < 0) \end{cases} \tag{3.13}$$

where

$$(c-1)^2\{1 - a^2/(c-1)^2\}^{1/2} + (c+1)^2\{1 - a^2/(c+1)^2\}^{1/2} = 0. \tag{3.14}$$

The square-roots must be chosen with non-negative real parts in order that the eigenfunction (3.13) is bounded at $y = \pm\infty$. If a square-root is pure imaginary, its sign must be chosen so that there is outward radiation of energy at infinity; however, this occurs only for real c, in which case there is stability anyway. It now follows from squaring up equation (3.14) that

$$c = 0 \quad \text{and} \quad a^2 \geqslant 1; \qquad c^2/a^2 = 1 \quad \text{and} \quad a^2 = \infty; \tag{3.15}$$

or

$$c^2 = -1 + \tfrac{1}{2}a^2.$$

The second mode represents Rayleigh's internal gravity wave with $V = 0$ and $c_* = \pm a_*$. The first and second modes are isolated from one another, and from the third mode, which is the only one that can give instability. It can be seen that there is stability to all waves only when

$$a^2 \geqslant 2, \qquad \text{i.e.} \qquad \alpha^2 \leqslant \tfrac{1}{2}g\beta/V^2. \tag{3.16}$$

We shall not discuss the stability characteristics when $K = 0$ further now, because we shall take them up in detail in Section V.

The case $J = 0$, $K \neq 0$ is also not without interest, and it may represent instability of vertical flames. Here the stability equation becomes

$$(w - c)(D^2 - \alpha^2)\varphi - (D^2w)\varphi + K(y)\{(w - c)D\varphi - (Dw)\varphi\} = 0.$$

For a vortex sheet with exponential density,

$$w = y/|y| \quad \text{and} \quad \bar{\rho} = \exp(-Ky) \qquad (\text{constant } K > 0),$$

the eigenvalue can be shown to be

$$c = (-K^2/2\alpha^2 + i)/(1 + K^2/2\alpha^2)^{1/2},$$

giving instability for all K,α.

Menkes [57] has considered such a problem for the smoothly-varying shear layer $w = \tanh y$.

3. *Sound Waves and Stability of Compressible Fluid*

Stability of a basic parallel flow of compressible perfect gas with piecewise-constant temperature and velocity was first studied by Bjerknes *et al.* [48] and Haurwitz [50]. Haurwitz also found the stability equation for basic flows with smoothly-varying temperature and velocity. He in fact considered external fields due to buoyancy and rotation as well as compressibility, but in our special case for two-dimensional waves in adiabatic motion it has the dimensionless form

$$D(\{(w - c)D\varphi - (Dw)\varphi\}/\{a^2 - (w - c)^2\}) - \alpha^2 a^{-2}(w - c)\varphi = 0. \tag{3.17}$$

Here φ is defined by the equation for the lateral velocity,

$$v' = i\alpha\varphi(y) \exp\{i\alpha(x - ct)\},$$

because two-dimensional motion of a compressible fluid has no stream function. Also the local inverse Mach number of the basic flow is $a(y) \equiv a_*(y_*)/V$, a_* being the local speed of sound. In general a_* varies with the basic temperature $T_*(y_*)$ of the perfect gas so that $a_* = (\gamma RT_*)^{1/2}$, where γ is the ratio of its specific heat at constant pressure to that at constant volume, and R is the gas constant. Note that the stability equation (3.17) above reduces to the Rayleigh stability equation as $a \to \infty$, i.e. as the fluid tends to be incompressible.

The boundary conditions (2.12), at the walls are valid for compressible fluid as well as incompressible fluid. The boundary conditions at a discontinuity of w, Dw, a, or Da can be shown in the usual way to be

$$[\varphi/(w - c)] = 0, \tag{3.18}$$

$$[\{(w - c)D\varphi - (Dw)\varphi\}/\{a^2 - (w - c)^2\}] = 0.$$

A generalization of Squire's theorem for this case is valid, giving [58] each three-dimensional disturbance of the basic flow with $w_*(y_*)$, $a_*(y_*)$ the same growth rate as a two-dimensional one for a basic flow with $w_*\cos\theta$, a_*, i.e. with velocity scale $V\cos\theta$ and therefore Mach number $V\cos\theta/a_* = a^{-1}\cos\theta < a^{-1}$. Thus to each two-dimensional disturbance there corresponds a three-dimensional one of the same growth rate but higher Mach number. It follows that if a flow of slightly compressible fluid is unstable to some two-dimensional disturbances then the same flow is unstable at all Mach numbers to some three-dimensional disturbances. Thus, although we shall find the cushioning effect of compressibility a stabilizing one by and large, it can never stabilize waves nearly perpendicular to the basic flow. However, it is fruitful to examine the stability characteristics of compressible fluids, and it is again sufficient to consider two-dimensional disturbances only, because their characteristics trivially imply those of all three-dimensional disturbances.

The important problem of a vortex sheet has been treated by Landau [59], Hatanaka [60], and Miles [27]. With

$$w = \begin{cases} 1 \\ -1 \end{cases} \qquad a = \begin{cases} a_1 & (y > 0) \\ a_2 & (y < 0) \end{cases}$$

the stability equation (3.17) solved piecewise with boundary conditions (2.12), (3.18) gives eigenfunction

$$\varphi = \begin{cases} (c-1)\exp\left(-\alpha\{1-(c-1)^2/a_1^2\}^{1/2}y\right) & (y>0) \\ (c+1)\exp\left(\alpha\{1-(c+1)^2/a_2^2\}^{1/2}y\right) & (y<0); \end{cases} \tag{3.19}$$

$$\begin{aligned} &a_1^{-2}(c-1)^2\{1-(c-1)^2/a_1^2\}^{-1/2} \\ &\quad + a_2^{-2}(c+1)^2\{1-(c+1)^2/a_2^2\}^{-1/2} = 0, \end{aligned} \tag{3.20}$$

where the square-roots have non-negative real parts, etc.

For illustration, let us take the special case of uniform basic temperature. Then $a_2 = a_1, = a$ say. Therefore

$$(c-1)^2\{a^2-(c+1)^2\}^{1/2} + (c+1)^2\{a^2-(c-1)^2\}^{1/2} = 0 \tag{3.21}$$

and it follows that

$$\begin{aligned} &c^2/a^2 \geqslant 1 \quad \text{and} \quad a^2 = \infty; \\ &c = 0 \quad \text{and} \quad a^2 \leqslant 1; \end{aligned} \tag{3.22}$$

or

$$c^2 = 1 + a^2 - a^2(1+4/a^2)^{1/2}.$$

The first mode represents a sound wave with $c_* = \pm a_*$ when $V = 0$. The second mode $c = 0$ gives a root of the unsquared equation (3.21) only when the square-roots are pure imaginary and have opposite signs. In fact, it represents the steady flow of supersonic streams on both sides of a thin wavy rigid wall coincident with the stationary interface between the two streams (cf. Liepmann and Roshko [61], art. 8.6); all streamlines have the same shape and size, but wave crests are out of phase, lying on Mach lines. These stable modes are isolated from one another and from the unstable third mode, which exists for all values of a. As $a \to \infty$ (which may be effected by letting $V \to 0$ or $a_* \to \infty$) the third mode gives $c \to i$ or $\pm 2^{1/2}a$. The former limiting root corresponds to the instability of a vortex sheet in an incompressible fluid. As $a \to 0$, $c = \pm 1 \pm a + O(a^2)$; these four roots correspond to stable sound waves superposed on the upper and lower streams of the basic flow. The vortex sheet is stable to two-dimensional disturbances if and only if c is real for all α, i.e.

$$a \leqslant 2^{-1/2}. \tag{3.23}$$

Küchemann [62] has studied the stability of a piecewise-linear profile representing a boundary layer in a compressible fluid. Lin [63] has considered general stability characteristics, and particular ones of a shear layer. Eckart [64] has generalized Howard's semicircle theorem for compressible fluid.

4. *Planetary Waves and Stability in a Rotating System*

If the equations of motion of inviscid fluid are referred to a frame rotating with constant angular velocity $\mathbf{\Omega}$, the Coriolis acceleration must be added to Euler's equations, but the centrifugal force may be put with the pressure, giving

$$\partial \mathbf{u}_*/\partial t_* + \mathbf{u}_* \cdot \nabla \mathbf{u}_* + 2\mathbf{\Omega} \times \mathbf{u}_* = -\nabla P_*,$$

where $\rho_* P_*$ is the pressure plus the centrifugal potential. Thus the vorticity with respect to a non-rotating (inertial) frame is $2\mathbf{\Omega}$ plus the relative vorticity $\boldsymbol{\omega}_* = \nabla \times \mathbf{u}_*$. Using the modified vorticity equation and the usual methods of normal modes, Johnson [65] has found the stability equation of a three-dimensional disturbance,

$$\begin{aligned}(w\cos\theta - c)^2(D^2 - \alpha^2)\varphi - \cos\theta(w\cos\theta - c)(D^2 w)\varphi \\ + R_0^{-1}\cos\chi(R_0^{-1}\cos\chi + \sin\theta Dw)\varphi = 0;\end{aligned} \tag{3.24}$$

where φ is defined by $v' = i\alpha\varphi(y)\exp\{i(\alpha x + \gamma z - \alpha ct)\}$, χ is the angle between $\mathbf{\Omega}$ and the wave-number vector $\tilde{\boldsymbol{\alpha}} = (\alpha, 0, \gamma)$, and the Rossby number is

$$R_0 \equiv V/2L\Omega, \tag{3.25}$$

a characteristic ratio of the inertial to Coriolis forces of the basic flow. When $\cos\chi/R_0$ and θ are zero this equation becomes the Rayleigh stability equation for two-dimensional disturbances in a non-rotating system. When $\theta = 0$, $\cos\chi/R_0 \neq 0$, the equation has the same form as equation (3.12) for a heterogeneous fluid. However, Squire's theorem is invalid in this case, the above equation for $\theta \neq 0$ differing essentially from its form for $\theta = 0$.

The boundary conditions (2.12) at a wall are still valid. At a discontinuity of w or Dw, continuity of normal velocity and pressure at the material interface give respectively,

$$\begin{gathered}[\varphi/(w\cos\theta - c)] = 0, \\ [(w\cos\theta - c)D\varphi - \cos\theta(Dw)\varphi + R_0^{-1}\cos\chi\tan\theta\varphi] = 0.\end{gathered} \tag{3.26}$$

When $w = 0\,(-\infty < y < \infty)$ the solution is [cf. **49**, p. **336**]

$$\varphi = \text{constant}, \qquad c = \pm\cos\chi/\alpha R_0 \equiv \pm a, \tag{3.27}$$

say, giving *planetary* (or *inertial*) *waves* with phase speed

$$a_* = 2\Omega\cos\chi/\alpha_*.$$

For the vortex sheet $w = y/|y|\,(-\infty < y < \infty)$ the eigenfunction is

$$\varphi = \begin{cases}(c - \cos\theta)\exp(-\alpha y\{1 - a^2/(c - \cos\theta)^2\}^{1/2}) & (y > 0) \\ (c + \cos\theta)\exp(\alpha y\{1 - a^2/(c + \cos\theta)^2\}^{1/2}) & (y < 0)\end{cases} \tag{3.28}$$

and the eigenvalue relation is

$$\begin{aligned}&(c - \cos\theta)^2\{1 - a^2/(c - \cos\theta)^2\}^{1/2} \\ &\qquad + (c + \cos\theta)^2\{1 - a^2/(c + \cos\theta)^2\}^{1/2} = 2a\sin\theta.\end{aligned} \tag{3.29}$$

When $\theta = 0$ this eigenvalue relation has the same form as that of (3.13), (3.14) for a heterogeneous fluid in a non-rotating system. On squaring up (3.29) etc. for general θ, we get the cubic equation in c^2,

$$\begin{aligned}0 = f(c^2) &\equiv 4\cos^2\theta c^6 + \{8\cos^4\theta - a^2(1 + 3\cos^2\theta)\}c^4 \\ &+ \{4\cos^6\theta - 2a^2\cos^2\theta(3 - \cos^2\theta) + a^4\}c^2 + a^2\sin^2\theta(a^2 - \cos^4\theta).\end{aligned} \tag{3.30}$$

As $a \to 0$, the three roots are: $c^2 = \frac{1}{4}a^2\tan^2\theta + O(a^4)$ or $-\cos^2\theta \pm 2ia\sin\theta + O(a^2)$. This gives only two admissible roots of the unsquared equation (3.30) with square-roots having non-negative real parts, namely $c = \pm i\cos\theta + a\sin\theta + O(a^2)$. As $a \to \infty$, $c^2 \to -\sin^2\theta$, or

$$(a^2/8\cos^2\theta)\{1 + 3\cos^2\theta \pm \sin\theta(1 - 9\cos^2\theta)^{1/2}\}.$$

The first root is inadmissible. The latter pair give only a complex conjugate pair of roots c when $\cos^2\theta > 1/9$, implying instability. When $\cos^2\theta \leqslant 1/9$ there is stability provided $a \sin\theta \geqslant 0$ but not otherwise.

The Scandinavian meteorologists [cf. **48**, **49**] have considered the stability characteristics of various piecewise constant velocity profiles and Johnson [**65**] has treated the shear layer $w = \tanh y$.

5. *Rossby Waves and Stability of Fluid in a Rotating System with Variable Coriolis Parameter*

For large-scale ($\approx 10^3$ km) motions of the earth's atmosphere or oceans it is customary to neglect the kinematic effects of the earth's curvature and use rectangular Cartesian coordinates, but to retain the more important dynamic effects of the variation of the Coriolis parameter $f \equiv 2\Omega \sin\lambda$ with latitude λ. This is done in the β-plane method of Rossby [cf. **66**]. With this approximation it can be shown that the only modification to the stability of an eastward horizontal basic flow $\bar{\mathbf{u}}_* = w_*(y_*)\mathbf{i}$ relative to the earth is the addition of $f\mathbf{k}$ to the relative vorticity $D_*w_*\mathbf{k}$, the earth having angular velocity $\Omega(\cos\lambda\mathbf{j} + \sin\lambda\mathbf{k})$. Kuo [**66**] has shown that this leads to the stability equation,

$$(w - c)(D^2 - \alpha^2)\varphi - (D^2w - \beta)\varphi = 0$$

by the usual methods for two-dimensional disturbances, where $\beta_* \equiv D_*f$ is usually approximated by a constant and y_* by the product of the local value of λ and the radius of the earth.

The boundary conditions (2.12) at a wall hold in this case. At a discontinuity of w or Dw conditions (2.13), (2.14) still hold.

Squire's theorem gives each three-dimensional disturbance of the basic flow with $w_*(y_*), \beta_*$ the same growth rate as some two-dimensional one of the basic flow with $w_* \cos\theta, \beta_* \cos\theta$. Hence it is sufficient to consider two-dimensional disturbances only as in Section II.

When $w = 0$ $(-\infty < y < \infty)$, we get the solution

$$\varphi = \text{constant}, \qquad c = -\beta/\alpha^2 \equiv -a, \tag{3.31}$$

say. This represents a *Rossby wave* of phase speed $a_* \equiv \beta_*/\alpha_*^2$. Rossby waves travel westwards and are dispersive. They are really a form of neutrally-stable inertial oscillation on the rotating earth.

For the vortex sheet $w = y/|y|$ $(-\infty < y < \infty)$ the eigenfunction is [**67**]

$$\varphi = \begin{cases} (c-1)\exp\left(-\alpha y\{1 + a/(c-1)\}^{1/2}\right) & (y > 0) \\ (c+1)\exp\left(\alpha y\{1 + a/(c+1)\}^{1/2}\right) & (y < 0) \end{cases} \tag{3.32}$$

and the eigenvalue relation is

$$(c-1)^2\{1+a/(c-1)\}^{1/2}+(c+1)^2\{1+a/(c+1)\}^{1/2}=0. \tag{3.33}$$

On squaring up, etc., it follows that $c/a \geqslant -1$ and $a=\infty$ or

$$0=f(c)\equiv c(c^2+1)+a(3c^2+1)/4. \tag{3.34}$$

The former mode is isolated, giving the Rossby wave with $c=-a$ when $w_*=0$ (i.e. $V=0$). The cubic has one real root, admissible only if $2\leqslant a<\infty$, for which there is stability with $-1\geqslant c\geqslant -\frac{3}{4}a$. However, there is also an unstable mode with complex conjugate pair of roots of the cubic, for which $c\to \pm 3^{-1/2}i$ as $a\to\infty$ and $c\to\pm i$ as $a\to 0$. Thus the rotation is a weakly stabilizing influence.

Kuo [**66**], Lipps [**68**] and Howard and Drazin [**67**] have considered other problems of this case.

6. *Magnetohydrodynamic Waves and Stability of an Electrically-Conducting Fluid in a Magnetic Field*

Many problems of stability of parallel flow of an inviscid incompressible electrically-conducting fluid in a magnetic field have been considered. They may be classified by use of the magnetic Reynolds number,

$$\mathrm{R_M}\equiv VL/\lambda,$$

an overall measure of the ratio of the convection of the magnetic field to its diffusion, where λ is the magnetic diffusivity of the fluid. Thus stability problems may be specified by $\mathrm{R_M}$ as well as the variation and magnitude of the basic magnetic field $\bar{\mathbf{H}}$.

We shall restrict our attention to problems for which

(a) the basic magnetic field is uniform and steady, so that the variables may be separated to yield a tractable stability equation;

(b) $\mathrm{R_M}$ is zero or infinite, so that the stability equation is of second order, like the other stability equations discussed in this paper;

(c) the basic magnetic field is directed in the (x,y)-plane of flow, because Squire's theorem is invalid otherwise.

With these restrictions we may state three eigenvalue problems typical of magnetohydrodynamic stability of parallel flow of inviscid fluid.

(1) When $\mathrm{R_M}=\infty$ (i.e. the fluid is a perfect conductor) and the basic magnetic field $\bar{\mathbf{H}}=(H_0,0,0)$ is parallel to the flow, the stability equation can be shown [**69**, **70**, **71**] to be

$$D(\{(w-c)^2-a^2\}D\{\varphi/(w-c)\})-\alpha^2\{(w-c)^2-a^2\}\{\varphi/(w-c)\}=0, \tag{3.35}$$

where

$$a^2 \equiv \mu H_0{}^2/4\pi V^2\rho_* , \tag{3.36}$$

a characteristic ratio of the magnetic to kinetic energy of the basic flow, μ being the magnetic permeability of the fluid.

The boundary conditions (2.12) at a wall hold as usual. At a discontinuity of w, Dw, or a the conditions are [71]

$$\begin{gathered} [\varphi/(w-c)] = 0, \\ [\{1 - a^2/(w-c)^2\}\{(w-c)D\varphi - (Dw)\varphi\}] = 0. \end{gathered} \tag{3.37}$$

When $w = 0$ $(-\infty < y < \infty)$ the solution is

$$\varphi = \text{constant}, \qquad c = \pm a. \tag{3.38}$$

This represents *Alfvén* (or *magnetohydrodynamic*) *waves* of phase speed $a_* \equiv (\mu H_0{}^2/4\pi\rho_*)^{1/2}$.

For a vortex sheet $w = y/|y|$ $(-\infty < y < \infty)$ the eigenvalue is [71]

$$c = \pm (a^2 - 1)^{1/2}. \tag{3.39}$$

Therefore the flow is stable if and only if $a^2 \geqslant 1$.

(2) When $R_M = 0$ and the basic magnetic field is parallel to the flow, the stability equation can be shown to be [70]

$$(w-c)(D^2 - \alpha^2)\varphi - (D^2w)\varphi + i\alpha N\varphi = 0, \tag{3.40}$$

where

$$N \equiv \mu H_0{}^2 L/4\pi\rho_*\lambda V. \tag{3.41}$$

The boundary conditions are the same as with no magnetic field. There is no progressive wave possible when $w = 0$, and the vortex sheet is unstable for all values of H_0, however large, although the magnetic field reduces the instability [cf. 72].

(3) When $R_M = 0$ and the basic field $\bar{\mathbf{H}} = H_0\mathbf{j}$ perpendicular to the basic flow, the stability equation can be shown [73] to be

$$(w-c)(D^2 - \alpha^2)\varphi - (D^2w)\varphi - iND^2\varphi = 0.$$

However, in this case the chief effects of the magnetic field on the stability characteristics occur through change of the basic flow itself rather than through change in the mechanism of instability.

IV. Heuristic Theory of Instability

1. *Dimensional Analysis*

The instability we have described is essentially a manifestation of three mechanisms:

(a) the inertial instability of the basic flow, whereby the basic balance of vorticity is upset;

(b) the kinematic constraints of the boundaries, which by and large reduce instability;

(c) the external force field, such as buoyancy or the Coriolis force.

In Section II we discussed mechanisms (a), (b) extensively in our review of inertial instability of parallel flow. The action of mechanism (c) alone is also well known, for it gives wave motions, such as sound. In this section we shall discuss qualitatively the interaction of mechanisms (a), (c). We shall exclude mechanism (b) because it is subsidiary and complicates the discussion. To understand the interaction better we shall relate the stability characteristics under both mechanisms to those under each separately by use of dimensional analysis.

To illustrate the use of dimensional analysis it seems clearest to consider one specific case, and we have chosen that of the stability of parallel flow of a fluid of variable density under the action of buoyancy, with stability governed by equation (3.12),

$$(w - c)(D^2 - \alpha^2)\varphi - (D^2 w)\varphi + J(y)\varphi/(w - c) = 0. \tag{4.1}$$

The methods we shall use for this problem can be readily applied to the other stability problems of Section III, which have a similar form. The stability equation above shows that gravity occurs only in the product $-gD_*\rho_*/\rho_*$, for $J(y) \equiv -g(D_*\rho_*)L^2/\rho_* V^2$. Therefore the eigenvalue problem (3.12), (2.12) gives eigenvalues of the form

$$c_* = c_*(\alpha_*, J_{0*}, V, L),$$

for the class of similar profiles $w(y)$, $\rho(y)$, where J_{0*} is the value of $J_* \equiv -gD_*\rho_*/\rho_*$ at any specified point y. Now dimensional analysis implies that

$$c = c(\alpha, J_0) \tag{4.2}$$

where $J_0 \equiv J_{0*}L^2/V^2$ acts as a characteristic value of $J(y)$. To solve a problem we find this relation explicitly, and, in particular, find the values $J_0(\alpha)$ for which $c_i(\alpha, J_0) = 0$ but for which $c_i(\alpha, J_0) > 0$ nearby. These values $J_0(\alpha)$ define ***the curve of neutral stability*** (or ***neutral curve*** or ***stability boundary***) in the (α, J_0)-plane. This neutral curve is important, because it separates the unstable from the stable disturbances and hence shows at once whether there is instability or not for any given wave and flow defined by the point (α, J_0).

Now consider the limit as $J_0 \to 0$ for fixed $\alpha \neq 0$. Then (some of) the eigenvalues

$$c \to c(\alpha, 0), \tag{4.3}$$

which limit we suppose to exist and to equal the eigenvalues $c(\alpha)$ of the Rayleigh stability equation (2.11), which is equation (4.1) with $J_0 \equiv 0$. It should be borne in mind that internal gravity waves exist for all $J_0 > 0$, however small, but not for $J_0 \equiv 0$; so we cannot expect there to be an eigenvalue with $J_0 = 0$ for each eigenvalue as $J_0 \to 0$. However, knowledge of $c(\alpha)$ from the theory of section II now tells us the behavior of some of the branches of the eigenvalues $c(\alpha, J_0)$ as $J_0 \to 0$, i.e. as the buoyancy becomes small, as the velocity scale becomes large, or as the length scale becomes small.

To consider mechanism (c) alone let us take the limit as $w_* \to 0$ for an unbounded flow, i.e. as $V \to 0$ for fixed $w(y)$, $\rho(y)$, L, J_{0*}, a_*. In this limit we suppose that c_* tends to a function which is independent of w_*, and therefore of both L and V, because as w_* vanishes its length scale and shape cannot be relevant physically. Therefore c_* is some function of α_*, J_{0*} which has dimensions of velocity. This implies that

$$c_* \sim kJ_{0*}^{1/2}/a_*, \qquad \text{as} \qquad V \to 0,$$

where k is a (many-valued) dimensionless constant independent of $w(y)$ but dependent on $\rho(y)$. But when $V = 0$ and the flow is unbounded it is well known that there are internal gravity waves whose speeds do depend on $\rho(y)$. These speeds will give k. For example, when

$$\rho_* = \rho_{0*} \exp(-\beta y_*),$$

it can be shown [51] that

$$c_* = (g\beta)^{1/2}/\alpha_*;$$

therefore $k = 1$ if we choose $J_{0*} = g\beta$. (Of course the arbitrary multiplicative constant in J_{0*} affects k, because it is only the product $kJ_{0*}^{1/2}$ that is determined physically.)

To determine mechanisms (a), (c) together let us again suppose that the flow is unbounded. We can now let the length scale L of the velocity profile tend to zero without altering the infinite domain of flow. Thus we let $L \to 0$ while α_*, V, $w(y)$, g, $\rho(y)$ are fixed. Then $\alpha = \alpha_* L \to 0$ and $J_0 = -gL(D\rho/\rho)_0/V^2 \to 0$, although J_0/α is fixed. Thus if we write

$$c = c(\alpha, J_0/\alpha)$$

and let $L \to 0$ we find

$$c \to c(0, J_0/\alpha) \qquad \text{as} \qquad \alpha \to 0,$$

for fixed smoothly-varying profiles $w(y)$, $\rho(y)$ and for fixed J_0/α. In this same limit we find

$$\begin{aligned} w_*(y_*) &= Vw(y_*/L) \qquad (-\infty < y_* < \infty) \\ &\to \begin{cases} Vw(\infty) \\ Vw(0) \\ Vw(-\infty) \end{cases} = \begin{cases} w_*(\infty) & (y_* > 0) \\ w_*(0) & (y_* = 0) \\ w_*(-\infty) & (y_* < 0) \end{cases} \\ &= \begin{cases} Vy_*/|y_*| & \text{(for shear layers)} \\ 0 & \text{(for jets)}, \end{cases} \end{aligned}$$

on ignoring the isolated point $y_* = 0$ which can have no physical significance. It is understood that

$$w_*(-\infty) = -w_*(\infty) = -V$$

for profiles of shear-layer type and that

$$w_*(-\infty) = 0 = w_*(\infty)$$

for profiles of jet type, as can be effected without loss of generality by a Galilean transformation if necessary. Thus for a shear layer w_* represents a vortex sheet in the limit as $L \to 0$ and for a jet w_* represents no flow in the limit. Similarly we find

$$\rho_*(y_*) \to \begin{cases} \rho_{*\infty} & (y_* > 0) \\ \rho_{*0} & (y_* = 0) \\ \rho_{*-\infty} & (y_* < 0) \end{cases}$$

as $L \to 0$ if these limits exist.

Now let us review what happens in the limit as $L \to 0$ for a profile of shear-layer type. We have found that then $c \to c(0, J_0/\alpha)$ as $\alpha \to 0$ for fixed $w(y)$, $\rho(y)$ and also $w_* \to Vy_*/|y_*|$, $\rho_* \to \rho_{*\infty}(y_* > 0)$ or $\rho_{*-\infty}(y_* < 0)$ for fixed α_*. However, we know the value of c_* for the vortex sheet from Kelvin-Helmholtz instability (3.9), which gives

$$c_* = \pm V(J_0/\alpha - 1)^{1/2}$$

on choosing $J_0 = (Lg/V^2)[\rho_{*-\infty} - \rho_{*\infty}]/[\rho_{*-\infty} + \rho_{*\infty}]$ and neglecting the variation of density (but not buoyancy) of the fluid in the inertia. We conclude that for any profile of shear-layer type

$$c \to c(0,J_0/\alpha) = \pm (J_0/\alpha - 1)^{1/2} \qquad \text{as} \qquad \alpha \to 0$$

for fixed J_0/α. We shall confirm this result analytically in the next section.

For a profile of jet type we similarly identify $c(0,J_0/\alpha)$ as the speed of internal waves when $w_* = 0$ and $\rho_* = \rho_{*\infty}(y_* > 0)$ or $\rho_{*-\infty}(y_* < 0)$, as given in equation (3.11). Thus

$$c \to c(0,J_0/\alpha) = \pm (J_0/\alpha)^{1/2} \qquad \text{as} \qquad \alpha \to 0.$$

We shall also confirm this result analytically in the next section.

Similar dimensional arguments for each of the force fields discussed in the last section can be used to apply the results for zero basic flow and for a vortex sheet to profiles of jet and shear-layer type respectively with long waves. In each case, on the large scale of a long wave ($L \ll \alpha_*^{-1}$) every shear layer looks like a vortex sheet and every jet like no flow; so their stability characteristics should correspond as $\alpha \to 0$. Unfortunately these arguments do not seem quantitatively correct for all force fields. For example, in the case (Section III.5) of a rotating system with variable Coriolis parameter, we expect that for each smoothly-varying shear layer $w(y)$ the eigenvalues $c(\alpha,a) \to c(0,a)$ as given by equation (3.33) for the vortex sheet, where $a = \beta/\alpha^2$, the Rossby wave speed. Thus it would seem that each shear layer is unstable as $\alpha \to 0$ for fixed a, as a vortex sheet is. However, an exact solution for the shear layer $w = \tanh y$ seems [**67**] to imply that the neutral curve touches $a = 1$ as $\alpha \to 0$, i.e. that there is stability for $a > 1$ as $\alpha \to 0$. This type of inconsistency occurs for some other force fields and has not been satisfactorily resolved. Possibly the resolution may come from there being more than one mode of instability for a smoothly-varying shear layer, yet only one for a vortex sheet; again the limits $c_i \to 0$, $\alpha \to 0$ may not be uniform.

2. *Physical Arguments*

The mechanism of instability of a vortex sheet $w_* = Vy_*/|y_*|$ in a compressible fluid at uniform temperature will now be described, essentially in the way attributed to Ackeret [cf. **74**, p. 240]. Consider a small irrotational two-dimensional disturbance of the velocity field in which the interface between the two streams of speeds V, $-V$ is distorted. Thus the interface has small bends. If the streams are subsonic ($V < a_*$), then by continuity the speed on the convex side of a bend has a small increase over its basic value, and the flow on the concave side a small decrease. Now Bernoulli's theorem for irrotational unsteady flow of barotropic inviscid fluid plausibly suggests that the pressure decreases on the convex side and increases on the concave side of the bend. This pressure difference induced across the bend increases the curvature of the bend and thus causes instability of the interface. By the theory of the Laval nozzle [cf. **74**, § **3.6**] the speed is decreased on the convex and increases on the concave side of a bend if the streams are supersonic ($a_* < V$). Hence the trend is reversed and the flow is stable. This heuristic argument indicates that there is instability for all $a > 1$.

This sufficient condition for instability of a vortex sheet is confirmed by an analytic argument of Lin [63], which gave this condition for a smoothly-varying shear layer subject to two-dimensional disturbances. However the condition apparently contradicts the result (3.23) that the vortex sheet is stable for all $a \leqslant 2^{-1/2}$ and unstable for all $a > 2^{-1/2}$. The disturbances considered for equation (3.22) are in fact irrotational on either side of the vortex sheet, the rotational disturbances being part of the continuous spectrum. Any contradiction may be due to the difference between the modes of instability of a vortex sheet and of a smoothly-varying shear layer in the limit as $\alpha \to 0$, a difference similar to that for flow with variable Coriolis parameter discussed at the end of the last section.

Another physical argument may be applied to jets. We take the qualitative argument of Backus [cf. 8, p. 264] for inertial instability of a homogeneous fluid, make it quantitative, and generalize it for a fluid of variable density under gravity. Let us suppose the jet has profile $w_*(y_*)$ where $w_*(\pm\infty) = 0$ in a fluid of basic density $\bar\rho_*(y_*)$ such that $(D_*\bar\rho_*)_{\pm\infty} = 0$. We shall consider only long-wave disturbances of this jet.

For long waves the effective width L of the jet is much less than a wavelength $2\pi/\alpha_*$. Thus the jet oscillates sinusoidally like a string. Far away from the core of the jet the flow is irrotational, because the basic flow is uniform and the disturbance of finite origin receives no vorticity. Therefore the amplitude F_* of the oscillation of a particle path dies away exponentially at $y_* = \pm\infty$ with scale height $1/\alpha_*$; the height of a material surface above its basic level is $\eta_* = F_*(y_*)\exp\{i\alpha_*(x_* - c_*t_*)\}$ where $F_* \sim F_{0*}e^{-\alpha_*|y_*|}$ as $y_* \to \pm\infty$. The density is $\bar\rho_{*-\infty}$ below the jet. Close to the jet, within a distance of order of magnitude L from the jet, i.e. much closer than a wavelength, the long waves seem locally like a vertical translation of the jet. Thus the jet oscillates like a string with form

$$\eta_{0*} = F_{0*}\exp\{i\alpha_*(x_* - c_*t_*)\},$$

for η_* is approximately constant on any vertical line near the jet.

In this motion the vertical mass-acceleration of the fluid on both sides of the jet is in balance with buoyancy and the centrifugal force due to the (small) curvature of the jet. The buoyancy comes from lifting fluid below the jet a height η_* into space previously occupied by the lighter fluid above the jet and *vice versa*. In this way the buoyancy gives rise to a pressure disturbance across the jet

$$= g\{\bar\rho_{*-\infty} - \bar\rho_{*\infty}\}\eta_{0*},$$

to first order for small η_*.

The centrifugal force exerted by a volume element of the jet of vertical thickness dy_* and unit horizontal area is the product of its mass, curvature, and the square of its horizontal velocity, namely $\bar{\rho}_* dy_*(-\partial^2\eta_*/\partial x_*^2)w_*^2$ to first order in η_*. This builds up the pressure difference across the jet

$$= \alpha_*^2 \int_{-\infty}^{\infty} \bar{\rho}_* w_*^2 \eta_* dy_*.$$

Now over the effective width of the jet $\eta_* = \eta_{0*}$, because α_* is small; in the distant regions where $\eta_* \to 0$ exponentially w_* is small anyway. Therefore this pressure difference

$$= \alpha_*^2 \eta_{0*} \int_{-\infty}^{\infty} \bar{\rho}_* w_*^2 dy_*$$

for small α_*, η_*.

The vertical mass-acceleration of the flow over unit horizontal area

$$= \int_{-\infty}^{\infty} \bar{\rho}_* \partial^2\eta_*/\partial t_*^2 dy_* = -\alpha_*^2 c_*^2 \left\{ \int_0^{\infty} \bar{\rho}_* \eta_* dy_* + \int_{-\infty}^{0} \bar{\rho}_* \eta_* dy_* \right\}.$$

Now $\bar{\rho}_*$ changes rapidly from its value at the origin to its values $\bar{\rho}_{*\pm\infty}$ at infinity, whereas η_* changes slowly like $F_{0*}e^{-i\alpha_*|y_*| + i\alpha_*(x_* - c_* t_*)}$. Therefore, for small α_*, this expression for the mass-acceleration

$$= -\alpha_*^2 c_*^2 \eta_{0*} \left\{ \bar{\rho}_{*\infty} \int_0^{\infty} e^{-\alpha_* y_*} dy_* + \bar{\rho}_{*-\infty} \int_{-\infty}^{0} e^{\alpha_* y_*} dy_* \right\}$$

$$= -\alpha_* c_*^2 \{\bar{\rho}_{*\infty} + \bar{\rho}_{*-\infty}\} \eta_{0*}.$$

Finally the balance of pressure and mass-acceleration per unit area gives

$$-\alpha_* c_*^2 \{\bar{\rho}_{*-\infty} + \bar{\rho}_{*-\infty}\}\eta_{0*} = g\{\bar{\rho}_{*-\infty} - \bar{\rho}_{*\infty}\}\eta_{0*} + \alpha_*^2 \eta_{0*} \int_{-\infty}^{\infty} \bar{\rho}_* w_* dy_*, \tag{4.4}$$

i.e.

$$c_*^2 \sim \frac{g}{\alpha_*} \frac{\bar{\rho}_{*-\infty} - \bar{\rho}_{*\infty}}{\bar{\rho}_{*-\infty} + \bar{\rho}_{*\infty}} - \alpha_* \int_{-\infty}^{\infty} \frac{\bar{\rho}_*(y_*) w_*^2}{\bar{\rho}_{*\infty} + \bar{\rho}_{*-\infty}} dy_*$$

as $\alpha_* \to 0$ for fixed

$$g\{\bar{\rho}_{*-\infty} - \bar{\rho}_{*\infty}\}/V^2\alpha_*\{\bar{\rho}_{*-\infty} + \bar{\rho}_{*\infty}\}.$$

This result will be verified analytically for a sinusoidal disturbance in the next section. In particular, we now see that long waves are stable when

$$g\{\bar{\rho}_{*-\infty} - \bar{\rho}_{*\infty}\} \geqslant \alpha_*^2 \int_{-\infty}^{\infty} \bar{\rho}_* w_*^2 dy_*.$$

In this argument we have approximated the buoyancy force with only the change of density from one side of the jet to the other. Thus we have neglected the modification of the buoyancy due to the density structure of the jet, which should be of order of magnitude $\alpha_* L$ times our first approximation $g(\bar{\rho}_{*-\infty} - \bar{\rho}_{*\infty})\eta_{0*}$. This modification may lead to the addition to equation (4.4) of a term larger than the last included term unless the first term of the right-hand side of (4.4) is of comparable or lesser magnitude than the second term in the limit as $\alpha_* L \to 0$.

Holmboe [75] has given other physical descriptions of the instability of parallel flow of fluid of variable density under gravity. In particular, he has looked at the development of symmetric waves in terms of real variables rather than in the usual way with normal modes.

V. Instability of an Incompressible Fluid of Variable Density

1. *General Stability Characteristics*

In this section we consider the instability of a basic steady plane parallel flow of an inviscid incompressible fluid of variable density under the action of gravity. We take the basic velocity $\bar{\mathbf{u}}_* = w_*(y_*)\mathbf{i}$ and density $\bar{\rho}_* = \bar{\rho}_*(y_*)$ as before, y_* being the height. Also we neglect the variation of inertia due to the variation of density of the fluid, i.e. we take $K(y) \equiv -L d\bar{\rho}_*/\bar{\rho}_* dy_* \doteqdot 0$ but retain $J(y) \equiv -L^2 g(d\bar{\rho}_*/dy_*)/\bar{\rho}_* V^2 \neq 0$. This is similar to the Boussinesq approximation and can be justified for many practical applications of the theory in which K is small and J of order one. We have shown in Section III, by the usual methods of hydrodynamic stability with normal modes, that the instability is described by the dimensionless eigenvalue problem:

$$(w - c)(D^2 - \alpha^2)\varphi - (D^2 w)\varphi + J(y)\varphi/(w - c) = 0; \tag{5.1}$$

$$\varphi = 0 \qquad (y = y_1, y_2). \tag{5.2}$$

We shall consider general and particular properties of the eigensolutions in the two subsections of this section, following the methods of Section II. As in Section II, we have dynamically independent two-dimensional waves, each having a stream function of the form $\psi' = \varphi(y) \exp\{i\alpha(x - ct)\}$. The eigenvalue problem is again invariant under complex conjugation, so there is stability only when c is real, and instability when c is complex, one of the conjugate solutions growing like $\exp(\alpha c_i t)$.

We can proceed to generalize Rayleigh's theorem and some other results of Section II.2 as follows. Assuming that $c_i > 0$, let $W \equiv w - c$, $H \equiv W^{n-1}\varphi$, some definite branch being chosen when n is not an integer. Then the stability equation becomes

$$D(W^{2(1-n)}DH) - (\alpha^2 W^{2(1-n)} + nW^{1-2n}D^2w + W^{-2n}\{n(1-n)(Dw)^2 - J\})H = 0. \tag{5.3}$$

Multiply this equation by H^* and integrate from y_1 to y_2 to get

$$\int_{y_1}^{y_2} W^{2(1-n)}(|DH|^2 + \alpha^2|H|^2) + nW^{1-2n}(D^2w)|H|^2 + W^{-2n}\{n(1-n)(Dw)^2 - J\}|H|^2 dy = 0. \tag{5.4}$$

This result of Howard [23] can lead to various properties according to the value of n chosen.

When $n = 1$ we have

$$\int_{y_1}^{y_2} |D\varphi|^2 + \alpha^2|\varphi|^2 + W^{-1}(D^2w)|\varphi|^2 - W^{-2}J|\varphi|^2 dy = 0. \tag{5.5}$$

The imaginary part of this gives

$$c_i \int_{y_1}^{y_2} \{D^2w - 2(w - c_r)J|W|^{-2}\}|W|^{-2}|\varphi|^2 dy = 0. \tag{5.6}$$

Therefore

$$D^2w = 2(w - c_r)J(y)/\{(w - c_r)^2 + c_i^2\} \tag{5.7}$$

somewhere in the field of flow. If $D^2w \neq 0$ in the field of flow we further have

$$c_i \leqslant |W| \leqslant \max\{(w - c_r)J/|W||D^2w|\} \leqslant \max|2J/D^2w|. \tag{5.8}$$

These results (5.7), (5.8) are due to Synge [76]. When $J = 0$ they give Rayleigh's necessary condition for instability that $D^2w = 0$ somewhere in the field of flow. Unfortunately when $J \neq 0$ they are not so simple, because they involve the unknown c.

When $n = 0$ we have $H = \varphi/W = F$, and

$$\int_{y_1}^{y_2} (w-c)^2(|DF|^2 + \alpha^2|F|^2) - J|F|^2 dy = 0. \tag{5.9}$$

This leads to the proof of the semicircle theorem, as in Section II.2. The extra term in the present case only strengthens the inequalities used provided $J \geqslant 0$ everywhere. Thus, when $c_i > 0$ and $J(y) \geqslant 0$ in the field of flow,

$$\{c_r - \tfrac{1}{2}(w_{\min} + w_{\max})\}^2 + c_i^2 \leqslant \{\tfrac{1}{2}(w_{\max} - w_{\min})\}^2. \tag{5.10}$$

Howard proved this result [23] for a heterogeneous fluid originally. Even when $J < 0$ somewhere, it follows that $w_{\min} < c_r < w_{\max}$. When $J(y) \leqslant 0$ everywhere, equation (5.9) shows that no non-singular neutral mode can exist, i.e. that either $c_i \neq 0$ or c lies within the range of w and F is therefore singular. However, when $J(y) > 0$ somewhere, it is possible that non-singular neutral modes exist with c outside the range $[w_{\min}, w_{\max}]$ of $w(y)$; these isolated neutral modes in fact occur as internal gravity waves.

When $n = \tfrac{1}{2}$ we have

$$\int_{y_1}^{y_2} (w-c)\{|DH|^2 + \alpha^2|H|^2\} + \tfrac{1}{2}(D^2w)|H|^2 + W^{-1}\{\tfrac{1}{4}(Dw)^2 - J\}|H|^2 dy = 0. \tag{5.11}$$

The imaginary part of this gives

$$-c_i \int_{y_1}^{y_2} |DH|^2 + \alpha^2|H|^2 + |W|^{-2}\{J - \tfrac{1}{4}(Dw)^2\}|H|^2 dy = 0. \tag{5.12}$$

Therefore, when $c_i > 0$,

$$0 > -\int_{y_1}^{y_2} |DH|^2 dy = \int_{y_1}^{y_2} (\alpha^2 + \{J - \tfrac{1}{4}(Dw)^2\}/|W|^2)|H|^2 dy. \tag{5.13}$$

Therefore $J(y) < \tfrac{1}{4}(Dw)^2$ somewhere in the field of flow. This gives Miles' [77] sufficient condition of stability that $J - \tfrac{1}{4}(Dw)^2$ should be everywhere non-negative. Further from inequality (5.13) we have

$$\alpha^2 c_i^2 \leqslant \alpha^2|W|^2 \leqslant \max\{\tfrac{1}{4}(Dw)^2 - J(y)\}, \tag{5.14}$$

a result due to Howard [23].

The results of Section II.2 about s-eigensolutions in the case $J = 0$ have no simple generalizations for the present case $J \neq 0$, because the singularities of the stability equations for the two cases differ. We cannot now form a Sturm-Liouville problem to find φ_s after choosing some particular value $c = w_s$. However the neutral s-solutions which occur for $J = 0$ will be modified as J increases and give some stability boundary of the form $J_0 = J_s(\alpha)$; where J_0 is some characteristic value of $J(y)$, such as

$$gL\{\bar{\rho}_*(y_{1*}) - \bar{\rho}_*(y_{2*})\}/\{\bar{\rho}_*(y_{1*}) + \bar{\rho}_*(y_{2*})\},$$

for dynamically similar basic velocity and density distributions. If $w(y)$, $J(y)$ are analytic functions in the real interval $[y_1,y_2]$, then the solution $\varphi(y;\alpha,J_0,c)$ of the stability equation (5.1) will be an integral function of α, J_0, c over any fixed domain of y within $[y_1,y_2]$ which excludes a neighborhood of the singularity $w(y) = c$, if any. It follows that the eigenvalue c is a continuous function of α and J_0 [78, p. 211]. This result, together with the semicircle theorem in the limit as $c_i \to 0$, implies that a stability boundary consists of singular neutral modes, i.e. modes for which $c_i = 0$ and $w = c$ in (y_1,y_2) [77, p. 505]. Further, for a certain class of basic velocity and density distributions at any rate, every singular neutral mode has a contiguous unstable mode in the (α,J_0)-plane [78, § 4]. In general the stability boundary $J_0 = J_s(\alpha)$ is both many-valued and has many branches, as will be indicated by examples in Section V.2. As yet there is no general theory to find this stability boundary, but it has been found for many special velocity and density profiles.

However, supposing the stability boundary to be known, Howard [25] gave a heuristic method to perturb it and find neighboring unstable solutions. The method generalizes the argument leading to equation (2.20), which was for the case $J = 0$. Proceeding as in that argument but with fixed $J(y) \neq 0$, we find

$$d\alpha^2/dc = \int_{y_1}^{y_2} (2J - WD^2w)\varphi^2/W^2 dy \Big/ \int_{y_1}^{y_2} \varphi^2 dy, \tag{5.15}$$

where φ is any eigenfunction with eigenvalue c. When c is real, care must be taken in evaluating these integrals because φ is singular. Examination of equation (5.3) shows that φ behaves like W^{1-n} when $W \to 0$, i.e. near $y = y_c$, n being a root of the equation $n(1-n)(Dw)^2 = J$ evaluated at $y = y_c$. We have found that $J/(Dw)^2 < \frac{1}{4}$ somewhere in (y_1,y_2) in order that a singular neutral mode should exist; in that event there are two roots n between 0 and 1, which coincide at $n = \frac{1}{2}$ when $J/(Dw)^2 = \frac{1}{4}$ at $y = y_c$. Thus

$$(d\alpha^2/dc)_s = \lim_{\alpha \to \alpha_s} \int_{y_1}^{y_2} \{2JW^{-1-2n} - (D^2w)W^{-2n}\}H^2 dy \div \int_{y_1}^{y_2} W^{2(1-n)}H^2 dy, \tag{5.16}$$

where $H = W^{n-1}\varphi$ behaves smoothly at $y = y_c$ as $\alpha \to \alpha_s$, $c \to w_s$ for a fixed function $J(y)$. For definiteness let us suppose that $-\pi < \arg(w - c) < 0$ for $c_i > 0$, so that

$$(5.17) \qquad (w - w_s)^{-2n} = \begin{cases} |w - w_s|^{-2n} & (w > w_s) \\ e^{2\pi n i}|w - w_s|^{-2n} & (w < w_s). \end{cases}$$

Now it follows that the denominator of equation (5.16),

$$(5.18) \qquad \lim_{\alpha \to \alpha_s} \int_{y_1}^{y_2} W^{2(1-n)} H^2 dy = \int_{y_1}^{y_2} W_s^{2(1-n)} H_s^2 dy,$$

which we are supposing to be known. The integral of the numerator of (5.16) diverges at $y = y_c$ in the limit, so care is need to approach the limit with $c_i \to 0$ through positive values. In this way Howard [25] was able to evaluate the right-hand side of equation (5.16) and thence find

$$(dc/d\alpha)_s = (d\alpha/dc)_s^{-1} = 2\alpha_s(d\alpha^2/dc)_s^{-1}.$$

We can use this result to find further stability characteristics. In general we seek the function $c = c(\alpha, J_0)$ and thence criteria of stability. But by partial differentiation we see

$$(5.19) \qquad \left(\frac{\partial c}{\partial J_0}\right)_\alpha = \frac{\partial(c,\alpha)}{\partial(J_0,\alpha)} = \frac{\partial(c,J_0)}{\partial(J_0,\alpha)} \Big/ \frac{\partial(c,J_0)}{\partial(c,\alpha)} = -\left(\frac{\partial c}{\partial \alpha}\right)_{J_0} \Big/ \left(\frac{\partial J_0}{\partial \alpha}\right)_c.$$

We suppose that the stability boundary $c_i(\alpha, J_0) = 0$ is known and gives $J_0 = J_s(\alpha)$. Thus, on that boundary,

$$(5.20) \qquad \left(\frac{\partial c}{\partial J_0}\right)_\alpha = -\left(\frac{\partial c}{\partial \alpha}\right)_s \Big/ \left(\frac{dJ_s(\alpha_s)}{d\alpha_s}\right).$$

In the previous paragraph we have shown how to determine $(\partial c/\partial \alpha)_s$ from knowledge of the neutral s-eigensolution, so we now can find $c(\alpha, J_0)$ on the unstable side of the neutral curve. Howard [25] derived these results and has applied them to two examples of shear layers.

The dimensionless Reynolds stress is $\tau = \frac{1}{4}\alpha\bar{\rho}(\varphi^* D\varphi - \varphi D\varphi^*)e^{2\alpha c_i t}$, as for a homogeneous fluid, where here $\bar{\rho}$ varies with y but its derivative is neglected except in the buoyancy term, i.e. $K(y) \doteqdot 0$. Whence one finds from the stability equation (5.1) that

$$(5.21) \qquad \frac{\partial \tau}{\partial y} = -c_i\alpha^{-1}\bar{\rho}\{2(w - c_r)J/|W|^4 - (D^2 w)/|W|^2\}\overline{v'^2}.$$

The boundary conditions imply that τ vanishes at $y = y_1, y_2$. Therefore $\partial\tau/\partial y$ vanishes somewhere in between, which when $c_i \neq 0$ gives Synge's generalization (5.7) for heterogeneous fluid of Rayleigh's necessary condition for instability. Equation (5.21) also shows that, when $c_i = 0$, τ is constant except for possible discontinuities where $w = c$. These discontinuities do not occur for non-singular modes with c outside the range of $w(y)$. For monotonic profiles $w(y)$ only one discontinuity $y = y_c$ is possible, but the boundary conditions give $\tau = 0$ on either side of the possible discontinuity, so $\tau = 0$ everywhere.

The occurrence of modes when $J \not\equiv 0$ is somewhat like that for the case when $J \equiv 0$ discussed in Section II. Unstable modes that exist for $J_0 = 0$ continue to exist as J_0 increases from zero. By and large, increase of J_0 decreases their instability, as would be anticipated from the physical effects of buoyancy. When $J(y)/(Dw)^2 > \frac{1}{4}$ everywhere all modes are stable. In addition to modification of the modes present when $J = 0$, variation of density gives rise to new modes. These are the internal gravity waves, which are isolated modes not associated with instability. Profiles with even functions $w(y)$, $J(y) \not\equiv 0$ have sinuous and varicose modes as when $J \equiv 0$. For odd functions $w(y)$ with even function $J(y)$ there is often exchange of stabilities with $c_r = 0$ when $c_i > 0$. However, there may be exceptions when the unstable mode is not unique [25]. Similar arguments to those valid when $J \equiv 0$ may be applied to problems when $J \not\equiv 0$. We shall illustrate them by examples in the next subsection.

Drazin and Howard [79] have considered the stability characteristics of unbounded flow for long waves. Their method is a natural generalization of that for a homogeneous fluid with $J \equiv 0$. The eigenfunction must be such that $\varphi \sim$ constant $\times$ exp $(\mp \alpha y)$ as $y \to \pm\infty$, when $D\bar{\rho} \to 0$ smoothly at infinity. Proceeding for this case $J \not\equiv 0$ in the manner of Section II.2, one can show that the eigenvalue relation for small α is

$$(5.22)\quad \begin{aligned} 0 = \alpha(W_{-\infty}^2 + W_\infty{}^2) - 2J_0 + \int_{-\infty}^{\infty} \{\alpha(W^2 - W_\infty{}^2) \\ + J_0(1-\lambda)\}\{\alpha(W^2 - W_{-\infty}^2) + J_0(1+\lambda)\}W^{-2}dy + \ldots; \end{aligned}$$

where

$$(5.23)\quad \lambda(y) \equiv \{\bar{\rho}_{-\infty} + \bar{\rho}_\infty - 2\bar{\rho}(y)\}/\{\bar{\rho}_{-\infty} - \bar{\rho}_\infty\}$$

and

$$J_0 \equiv gL(\bar{\rho}_{-\infty} - \bar{\rho}_\infty)/V^2(\bar{\rho}_{-\infty} + \bar{\rho}_\infty),$$

so $J(y) = J_0 D\lambda$. For profiles of shear-layer type with $w_{\pm\infty} = \pm 1$, this relation gives

$$(5.24)\quad c^2 = J_0/\alpha - 1 + \ldots \qquad \text{as } \alpha \to 0 \text{ for fixed } J_0/\alpha.$$

This result in the limit as $\alpha \to 0$ agrees with that of (3.9) for Kelvin-Helmholtz instability of a vortex sheet when $(\bar\rho_{-\infty} - \bar\rho_\infty)/(\bar\rho_{-\infty} + \bar\rho_\infty) \ll 1$, i.e. when $K(y) \equiv 0$. For profiles of jet type with $w_{\pm\infty} = 0$, relation (5.22) gives

$$c^2 = J_0/\alpha - \tfrac{1}{2}\alpha \int_{-\infty}^{\infty} (W^2 - c^2)^2/W^2 + 2J_0(W^2 - c^2)/\alpha W^2 + J_0{}^2(1 - \lambda^2)/\alpha^2 W^2 dy + \dots. \tag{5.25}$$

When $w \equiv 0$ everywhere this relation in turn gives the speeds of the internal gravity waves. For $w(y)$ not identically zero, these internal gravity waves are modified and become isolated stable modes. There are also at least two unstable modes for sufficiently small α, J_0. When J_0 is of order α^2, equation (5.25) gives the sinuous mode with

$$c^2 \sim J_0/\alpha - \frac{1}{2}\alpha \int_{-\infty}^{\infty} w^2 dy \qquad \text{as} \qquad \alpha \to 0, \tag{5.26}$$

in agreement with the physically deduced result (4.4) when

$$(\bar\rho_{-\infty} - \bar\rho_\infty)/(\bar\rho_{-\infty} + \bar\rho_\infty) \ll 1.$$

So the sinuous mode has stability boundary with

$$J_0 \sim \frac{1}{2}\alpha^2 \int_{-\infty}^{\infty} w^2 dy \qquad \text{as} \qquad \alpha \to 0, \tag{5.27}$$

there being stability to long waves when J_0 is greater than this value.

It is physically realistic to have a model for which $\bar\rho(y)$ tends to constants as $y \to \pm\infty$. However, in many circumstances it is practical to suppose $\bar\rho$ tends to zero or infinity. For example, one might suppose that $\bar\rho = \rho_0 \exp(-\beta y)$ for constant $\beta > 0$ to represent the atmospheric density in a problem of instability that occurs effectively in a finite range of y; in this case $J(y) = g\beta L^2/V^2$ is independent of y. We have shown (3.1) that then internal gravity waves have speed $a = J^{1/2}/\alpha$. It can also be seen from the stability equation (5.1) and boundary conditions (5.2) at infinity that

$$\varphi \sim \text{constant} \times \exp(\mp \alpha L_\pm y) \qquad \text{as} \qquad y \to \pm\infty, \tag{5.28}$$

where $L_\pm \equiv +(1 - a^2/W^2_{\pm\infty})^{1/2}$, provided that $w(y) \to w_{\pm\infty}$ smoothly as $y \to \pm\infty$. Now a direct generalization of the method of Drazin and Howard [79] described above gives for small α and fixed a the eigenvalue relation

$$0 = L_+ w_\infty^2 + L_- w_{-\infty}^2 + a\left\{\int_0^\infty W^2 - W_\infty^2 dy\right.$$

$$(5.29) \qquad + \int_{-\infty}^0 W^2 - W_{-\infty}^2 dy - L_+ L_- W_{-\infty}^2 \int_0^\infty 1 - W_\infty^2/W^2 dy$$

$$\left. - L_+ L_- W_\infty^2 \int_{-\infty}^0 1 - W_{-\infty}^2/W^2 dy \right\} + \dots .$$

For a shear layer with $w_{-\infty} = -w_\infty = -1$, this relation in the first approximation gives the same stability characteristics (3.15) as a vortex sheet. Higher approximations are obscured by the difficulty discussed at the end of Section IV.2 for a compressible fluid. For a jet with $w_{\pm\infty} = 0$, relation (5.29) gives

$$(5.30) \qquad 0 = 2Lc^2 + \alpha \int_{-\infty}^\infty (W^2 - c^2)(1 - L^2c^2/W^2)dy + \dots ,$$

where $L = +(1 - a^2/c^2)^{1/2}$. In the first approximation for small α this gives an unstable mode with

$$(5.31) \qquad c \sim \frac{1}{2} i(\alpha/a) \int_{-\infty}^\infty w^2 dy \qquad \text{as} \qquad \alpha \to 0,$$

and an internal gravity wave with

$$(5.32) \qquad c^2 - a^2 \sim \frac{1}{4}(\alpha^2/a^2)\left(\int_{-\infty}^\infty w^2 - 2awdy\right)^2 \qquad \text{as} \qquad \alpha \to 0.$$

In this special case with constant J and unbounded flow, we can find a sufficient condition for stability. We have required that the real parts of $L_\pm$ be non-negative in formula (5.29) in order that φ does not exponentially increase at infinity. In fact the solution of the initial-value problem must die down as $y \to \pm\infty$, so isolated modes may have non-zero or even unbounded eigenfunctions at infinity, but any dense set of waves must have eigenfunctions which tend to zero there. Therefore, when c is real, only isolated waves may be not exponentially damped as $y \to \pm\infty$. It follows that in the limit as $c_i \to 0$, L_+ and L_- are real and non-negative. Therefore

$1 - a^2/(w_\infty - c)^2$, $1 - a^2/(w_{-\infty} - c)^2 \geqslant 0$ for eigenvalues c on the stability boundary. Therefore $a^2 \leqslant (w_\infty - c)^2, (w_{-\infty} - c)^2$ on the stability boundary. Therefore a sufficient condition for stability is that

$$a^2 > \max \{(w_\infty - c)^2, (w_{-\infty} - c)^2\}. \tag{5.33}$$

Now the semicircle theorem gives $w_{\min} < c < w_{\max}$. Therefore another sufficient condition for stability is that

$$a^2 > (w_{\max} - w_{\min})^2. \tag{5.34}$$

2. *Stability Characteristics of Various Basic Flows*

In Section II.4 we have given some stability characteristics of several basic flows of homogeneous fluid. In Section III.2 we gave the stability characteristics of heterogeneous fluid in a state of rest and in the motion of a vortex sheet. In this subsection we shall exemplify the interaction of the inertial instability of some other flows of Section II.4 and the effects of buoyancy due to various basic density distributions.

(a) *Internal Gravity Waves*

The stability of heterogeneous fluid in a state of rest ($w = 0$) is governed by buoyancy alone, there being no shear in the basic flow. This problem is simpler, being a regular Sturm-Liouville problem. If the basic density anywhere increases with height there is instability. Otherwise neutrally-stable internal gravity waves occur. Their structure and speeds depend on their wavelength and the distribution $\bar{\rho}(y)$ of basic density. These waves also occur as isolated modes for velocity profiles with shear, and are treated in detail in some of the papers we refer to. However, by and large, we shall exclude any detailed treatment of internal gravity waves from this review. Here we shall merely refer the reader to the classic paper of Fjeldstad [**80**] and to a more recent survey of the literature by Davis and Patterson [**81**].

(b) *Plane Couette Flow*

The stability of a heterogeneous fluid in the basic flow with velocity $w = y(y_1 \leqslant y \leqslant y_2)$ was first considered by Taylor [**52**]. He took $\bar{\rho} = \rho_0 \exp(-\beta y)$ so that $J(y)$ is a constant, and then showed that the stability equation was essentially Bessel's equation of order $\nu \equiv (\frac{1}{4} - J)^{1/2}$. With one of the boundaries at infinity, he found there were no eigensolutions when $0 < J < \frac{1}{4}$ and only stable ones when $J > \frac{1}{4}$. Taylor [**52**, § 4] considered also the stability of plane Couette flow with three and four layers of homogeneous fluids of different densities.

In a more complete investigation Eliassen *et al.* [35] have shown that when $\bar{\rho} = \rho_0 \exp(-\beta y)$ plane Couette flow, whether bounded, semi-bounded, or unbounded, is stable if $J \geqslant 0$ and unstable if $J < 0$. They considered the initial-value problem as well as normal modes. When $-\frac{3}{4} < J < \frac{1}{4}$ there is no discrete spectrum of normal modes. These results seem natural in view of the known stability of plane Couette flow when $J = 0$ and of the anticipated stabilizing influence of buoyancy.

Also Høiland [82] has considered stability of plane Couette flow when J is a quadratic function of y.

(c) *Sinusoidal Flow*

When

$$w = \sin y, \qquad J = \text{constant} \qquad (y_1 \leqslant y \leqslant y_2) \tag{5.35}$$

certain exact neutral parts of the solution are known [83] for $0 \leqslant \alpha \leqslant 1$. The stability equation (5.1) has the following solutions.

(5.36i) $c = 0, \quad J = (1-\alpha^2)^{1/2} - 1 + \alpha^2, \quad \varphi = |\sin y|^{(1/2)+\nu};$

(5.36ii) $c = 0, \quad J = 3(1-\alpha^2)^{1/2} - 3 + \alpha^2, \quad \varphi = \cos y\,|\sin y|^{(1/2)+\nu};$

(5.36iii) $c = 0, \quad J = \frac{1}{4}(\alpha^2 - \frac{3}{4}), \quad \varphi = |\sin \frac{1}{2}y|^{(1/2)+\nu}|\cos \frac{1}{2}y|^{1-(1/2)\nu};$

where $\nu \equiv (\frac{1}{4} - J)^{1/2}$.

For $y_1 = 0, y_2 = \pi$ the flow is known to be stable when $J = 0$. However, solutions (i)–(iii) are all eigensolutions. It is presumed [83] that the associated neutral curves in the (α, J)-plane are not stability boundaries.

For $y_1 = -\pi, y_2 = \pi$ there is instability for $J = 0$ when $0 < \alpha < 3^{1/2}/2$; the limiting eigensolutions being

$$\begin{aligned} &c = 0, \quad \alpha = 0, \quad \varphi = \sin y \quad \text{and} \\ &c = 0, \quad \alpha = 3^{1/2}/2, \quad \varphi = \cos \tfrac{1}{2}y. \end{aligned} \tag{5.37}$$

Again (5.36 i–iii) are eigensolutions, but it seems [83] that none is a stability boundary.

(d) *Thin Jet*

Applying the method of derivation of "jump" conditions (2.15) to the stability equation (5.1), one can show that when

$$w = \{\delta(y)\}^{1/2} \qquad (-\infty < y < \infty) \tag{5.38}$$

in a heterogeneous fluid, the conditions at $y = 0$ are that

$$c^2[DF] + J_0[\lambda]F = \alpha^2 F, \qquad [F] = 0, \tag{5.39}$$

where

$$F \equiv \varphi/W, \qquad J(y) \equiv J_0 D\lambda, \qquad \lambda \equiv \{\bar{\rho}_{-\infty} + \bar{\rho}_\infty - 2\bar{\rho}(y)\}/\{\bar{\rho}_{-\infty} - \bar{\rho}_\infty\}.$$

With these conditions one can solve the stability equation (5.1) piecewise for $y > 0$ and $y < 0$, join up the solutions at $y = 0$, and find the eigenvalue relation. When

$$\bar{\rho} = \begin{cases} \bar{\rho}_\infty & (y > 0) \\ \bar{\rho}_{-\infty} & (y < 0), \end{cases} \tag{5.40}$$

the eigenvalue relation can be shown to be

$$c^2 = -\tfrac{1}{2}\alpha + J_0/\alpha. \tag{5.41}$$

It can be seen that this result also follows exactly from relation (5.25), and gives stability of the thin jet when $J_0 \geqslant \frac{1}{2}\alpha^2$. This result is typical of the stability of jets to long waves, and will have to serve for other results on jets, which the literature lacks.

(e) *Shear Layer*

The stability of heterogeneous fluid with basic flow

$$w = \begin{cases} -1 & (y < -1) \\ y & (-1 < y < 1) \\ 1 & (1 < y) \end{cases} \tag{5.42}$$

was first considered by Taylor [52, § 3] and Goldstein [53, §§ 3, 5]. They took essentially

$$\bar{\rho} = \begin{cases} \bar{\rho}_{-\infty} & (y < -1) \\ \bar{\rho}_0 & (-1 < y < 1) \\ \bar{\rho}_\infty & (1 < y) \end{cases} \tag{5.43}$$

and found the eigenvalue relation. When $\bar{\rho}_0 = \frac{1}{2}(\bar{\rho}_{-\infty} + \bar{\rho}_\infty)$ and $(\bar{\rho}_{-\infty} - \bar{\rho}_\infty)/(\bar{\rho}_{-\infty} + \bar{\rho}_\infty) \ll 1$, there is instability if and only if

$$2\alpha/(1 + e^{-2\alpha}) - 1 < J_0 < 2\alpha/(1 - e^{-2\alpha}) - 1 \tag{5.44}$$

where $J_0 \equiv gL(\bar{\rho}_{-\infty} - \bar{\rho}_\infty)/V^2(\bar{\rho}_{-\infty} + \bar{\rho}_\infty)$. When $J_0 = 0$ this reduces to Rayleigh's result [cf. Section II.4.k] that there is instability if $0 < \alpha < \alpha_s \doteqdot 0.64$. For general values of J_0, α the stability boundaries etc. are shown in Figure 5, after Goldstein. It can be seen that any wave unstable when $J_0 = 0$ becomes stable when J_0 is sufficiently large. However, other waves are made unstable as J_0 increases, some narrow band of waves being unstable however large J_0 is.

Goldstein [53, § 5] further considered the shear layer with the continuous density distribution

$$\bar{\rho} = \begin{cases} \bar{\rho}_{-\infty} & (y < -1) \\ \bar{\rho}_{-\infty} e^{-\beta(y+1)} & (-1 < y < 1) \\ \bar{\rho}_{-\infty} e^{-2\beta} & (1 < y), \end{cases} \tag{5.45}$$

where again β is negligibly small except when multiplied by gravity. Goldstein's solution, involving Bessel functions, is complicated. In brief,

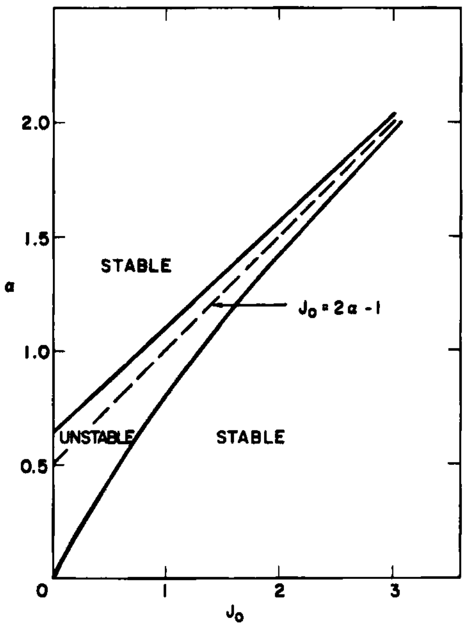

FIG. 5. Stability characteristics of the shear layer with $w = y/|y|(|y| > 1)$, $w = y(|y| < 1)$ and $\bar{\rho} = \bar{\rho}_{-\infty}\ (y < -1)$, $\bar{\rho} = \frac{1}{2}(\bar{\rho}_{-\infty} + \bar{\rho}_{\infty})(|y| < 1)$, $\bar{\rho} = \bar{\rho}_{\infty}(y > 1)$.

it gives stability to all waves if and only if $J_0 \equiv g\beta_* L^2/V^2 \geqslant \frac{1}{4}$. This condition is in agreement with Miles' sufficient condition $J/(Dw)^2 \geqslant \frac{1}{4}$ for stability. When $\alpha \ll 1$, it can be shown from Goldstein's work that is there is stability when

$$J_0 \geqslant \alpha - \frac{2}{3}\alpha^2 - \frac{4}{9}\alpha^3 - \frac{16}{45}\alpha^4 - \dots, \tag{5.46}$$

in agreement with relation (5.22). Recently Miles and Howard [84] have clarified an obscure point in Goldstein's paper and given some numerical results for this example. The principal stability characteristics are shown in Figure 6.

Holmboe [75] studied the model of Taylor and Goldstein with density distribution (5.43) by his method of symmetric waves. Holmboe also considered density distribution (5.40) with a single discontinuity at $y = 0$.

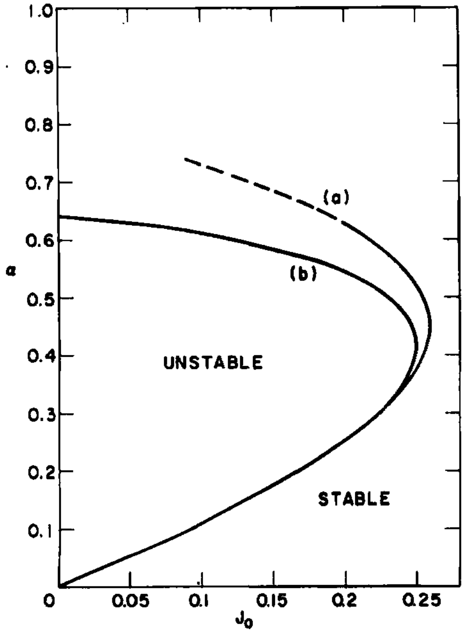

FIG. 6. Stability characteristics of the shear layer with $w = y/|y|(|y| > 1)$, $w = y(|y| < 1)$, and $\bar\rho = \bar\rho_{-\infty}(y < -1)$, $\bar\rho = \bar\rho_{-\infty}e^{-\beta(y+1)}(|y| < 1)$, $\bar\rho = \bar\rho_{-\infty}e^{-2\beta}\cdot(y > 1)$. (a) $J_0 = \alpha - \frac{2}{3}\alpha^2 - \frac{4}{9}\alpha^3 - \frac{16}{45}\alpha^4$. (b) Stability boundary.

Then the eigenvalue relation [75, equation (7.6) essentially] can be shown to be

$$4\alpha^2c^4 - c^2\{(2\alpha - 1)^2 - e^{-4\alpha} + 4\alpha J_0\} + (J_0/\alpha)(2\alpha - 1 + e^{-2\alpha})^2 = 0. \tag{5.47}$$

It follows that c^2 is complex only when

$$\begin{aligned}&(2\alpha - 1 + 3e^{-2\alpha}) - 2\{2e^{-2\alpha}(2\alpha - 1 + e^{-2\alpha})\}^{1/2}\\ &\quad < 4\alpha J_0/(2\alpha - 1 + e^{-2\alpha})\\ &\quad < (2\alpha - 1 + 3e^{-2\alpha}) + 2\{2e^{-2\alpha}(2\alpha - 1 + e^{-2\alpha})\}^{1/2}.\end{aligned} \tag{5.48}$$

The curves representing equalities above are shown in Figure 7. Both curves touch $J_0 = \alpha$ at the origin and $J_0 = \alpha - 1$ at infinity. For values of J_0, α between these curves, c^2 is complex and therefore $c_r, c_i \neq 0$, i.e. there is overstability. On the curves, $c^2 = \pm (J_0/2\alpha^2)(2\alpha - 1 + e^{-2\alpha})$. It follows that there is stability ($c^2 > 0$) between each curve and an axis except between the upper curve and the α-axis for $0 < \alpha < \alpha_s \doteqdot 0.64$. In that region $c_r = 0$, $c_i \neq 0$, there being exchange of stabilities at $\alpha = 0$, $\alpha = \alpha_s$ when $J_0 = 0$. This is accordingly an example of a flow with odd function $w(y)$

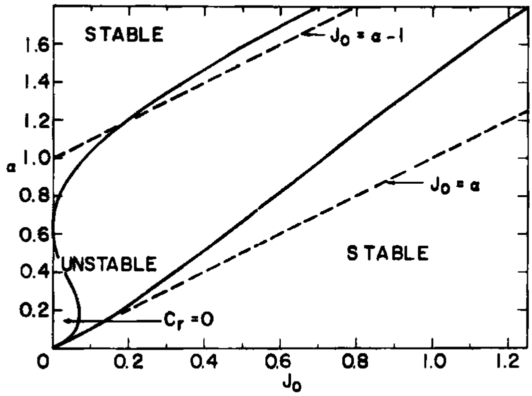

FIG. 7. Stability characteristics of the shear layer with $w = y/|y|(|y| > 1)$, $w = y(|y| < 1)$ and $\bar{\rho} = \bar{\rho}_{\infty}(y > 0)$, $\bar{\rho} = \bar{\rho}_{-\infty}(y < 0)$.

and even $J(y)$ where there is not everywhere exchange of stabilities, this being possible because there is not a unique mode of instability.

(f) *Double Shear Layer*

Another flow for which w is an odd function and J even, yet for which there is not exchange of stabilities has been pointed out by Howard [25]. When

$$(5.49) \qquad w = \begin{cases} -1 & (y < -1) \\ 0 & (-1 < y < 1) \\ 1 & (y > 1) \end{cases} \qquad \bar{\rho} = \begin{cases} \rho_0(1+\varepsilon) \\ \rho_0 \\ \rho_0(1-\varepsilon) \end{cases}$$

and $J_0 \equiv gL\varepsilon/V^2, \varepsilon \ll 1$, it can be shown that

$$(5.50) \qquad (2c^2 + 1 - J_0/\alpha)^2 - 4c^2(1 - e^{-4\alpha}) = e^{-4\alpha}(1 - J_0/\alpha).$$

Howard [25] found stability with four real roots c when

$$(5.51) \qquad \frac{J_0}{\alpha} \geqslant \frac{e^{-4\alpha} + (1 - e^{-4\alpha})^{1/2}}{1 + (1 - e^{-4\alpha})^{1/2}}.$$

On the stability boundary, the curve with equality in the above,

$$c = \pm (1 - e^{-4\alpha})\{2(1 - e^{-4\alpha})^{1/2} + 2 - 2e^{-4\alpha}\}^{-1/2}.$$

There are waves with $c = 0$; in fact they occur when $J_0 = \alpha$; however, this locus lies within the stable region and is not adjacent to parts corresponding to instability. These results are illustrated in Figure 8.

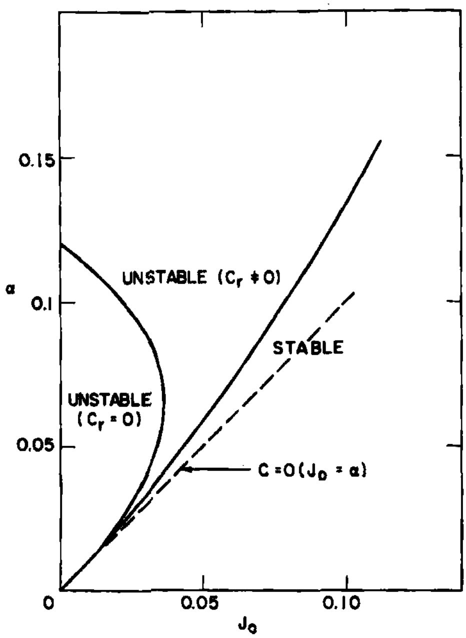

FIG. 8. Stability characteristics for the double shear layer $w = y/|y|(|y| > 1)$, $w = 0(|y| < 1)$ with density $\bar{\rho} = \rho_0(1 + \varepsilon)(y < -1)$, $\bar{\rho} = \rho_0(|y| < 1)$, $\bar{\rho} = \rho_0(1 - \varepsilon)(y > 1)$.

(g) *Bickley Jet*

Howard and Drazin [83] have found various exact parts of the neutral eigensolutions for

$$w = \operatorname{sech}^2 y \qquad (-\infty < y < \infty), \tag{5.52}$$

with various density distributions.

When $J(y)$ is constant, the following solutions may be verified.

(5.52i) *Sinuous mode* $c = (6 + \alpha^2)/15, \qquad J = \alpha^2(4 - \alpha^2)(9 - \alpha^2)/225,$

$$\varphi = (\operatorname{sech}^2 y - c)^k (\operatorname{sech} y)^m \qquad (0 \leqslant \alpha^2 \leqslant 4)$$

where

$$k \equiv 3(4 - \alpha^2)/2(6 + \alpha^2), \qquad m \equiv 5\alpha^2/(6 + \alpha^2).$$

(5.52ii) *Varicose mode* $c = (3 + \alpha^2)^2/3(3 + 5\alpha^2),$

$$J = \alpha^2(1 - \alpha^2)(9 - \alpha^2)(3 + \alpha^2)^2/9(3 + 5\alpha^2)^2,$$

$$\varphi = \tanh y(\operatorname{sech}^2 y - c)^k (\operatorname{sech} y)^m \qquad (0 \leqslant \alpha^2 \leqslant 1)$$

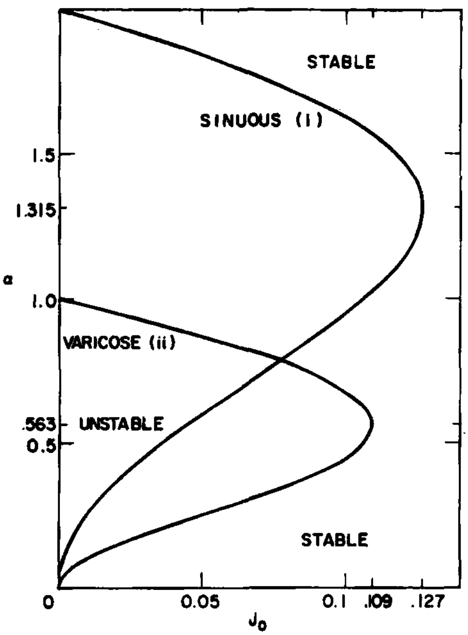

Fig. 9a.

where

$$k \equiv 3(1-\alpha^2)/2(3+\alpha^2),$$

$$m \equiv 4\alpha^2/(3+\alpha^2).$$

The neutral curves are sketched in Figure 9(a). Howard and Drazin [**83**] argued that the curve for the varicose mode (ii) is a stability boundary, but that at most the upper part of the curve for the sinuous mode (i) is a stability boundary.

When $J = J_0 \operatorname{sech}^2 y$, eigensolutions may be verified as follows.

(5.53i) *Sinuous mode* $c = \dfrac{1}{3}\alpha,$

$$J_0 = \frac{1}{9}\alpha^2(2-\alpha)(3-\alpha),$$

$$\varphi = (\operatorname{sech} y)^{\alpha}(\operatorname{sech}^2 y - \tfrac{1}{3}\alpha)^{1-(1/2)\alpha}$$
$$(0 \leqslant \alpha \leqslant 2).$$

(5.53ii) *Varicose mode*

$$c = (3+\alpha^2)/3(1+\alpha),$$
$$J_0 = \alpha(1-\alpha)$$
$$\cdot (3-\alpha)(3+\alpha^2)/9(1+\alpha),$$
$$\varphi = \tanh y(\operatorname{sech} y)^{\alpha}$$
$$\cdot (\operatorname{sech}^2 y - c)^{(1/2)(1-\alpha)} \; (0 \leqslant \alpha \leqslant 1).$$

The neutral curves, shown in Figure 9(b), are both thought [**83**]

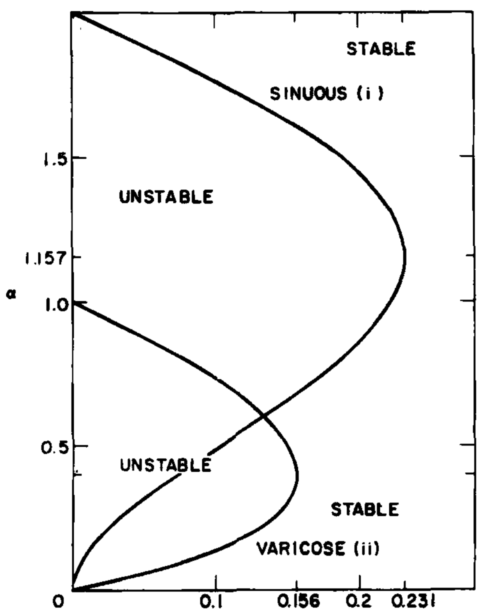

Fig. 9b.

FIG. 9 (a). Stability characteristics for Bickley jet $w = \operatorname{sech}^2 y$ when $J = \text{const.}$ (i) Neutral curve for sinuous mode: $J = \alpha^2(4-\alpha^2)(9-\alpha^2)/225$. (ii) Neutral curve for varicose mode: $J = \alpha^2(1-\alpha^2)(9-\alpha^2) \cdot (3+\alpha^2)^2/9(3+5\alpha^2)^2$. (b). Stability characteristics for Bickley jet $w = \operatorname{sech}^2 y$ when $J = J_0 \operatorname{sech}^2 y$. (i) Stability boundary for sinuous mode: $J_0 = \alpha^2(2-\alpha)(3-\alpha)/9$. (ii) Stability boundary for varicose mode: $J_0 = \alpha(1-\alpha) \cdot \cdot (3-\alpha)(3+\alpha^2)/9(1+\alpha)$.

to be stability boundaries for their respective modes. Further information on this example, and others, can be found by use of the formula (5.25) for small α and the perturbation (5.20).

(h) *Antisymmetric Double Jet*

When

$$w = \operatorname{sech} y \tanh y, \qquad J = J_0 \operatorname{sech}^2 y \qquad (-\infty < y < \infty), \tag{5.54}$$

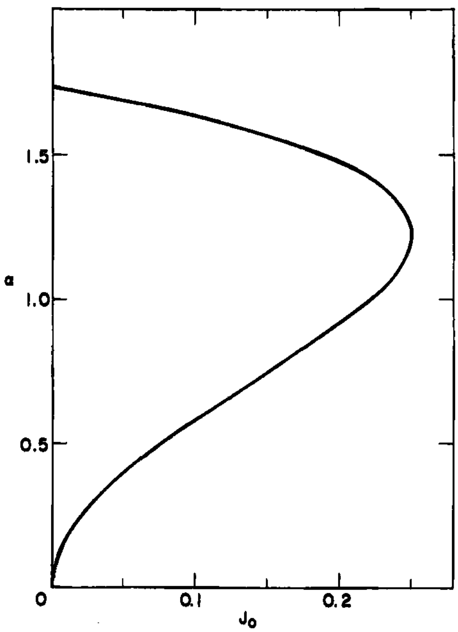

FIG. 10. Stability boundary of one mode for $w = \operatorname{sech} y \tanh y$, $J = J_0 \operatorname{sech}^2 y(-\infty < y < \infty)$: $J_0 = \alpha^2(3 - \alpha^2)/9$.

it can be seen [83] that a neutral eigensolution is

$$c = 0, \qquad J_0 = \frac{1}{9}\alpha^2(3 - \alpha^2), \qquad \varphi = (\tanh y)^{1-(1/3)\alpha^2}(\operatorname{sech} y)^{1+(1/3)\alpha^2} \qquad (0 \leqslant \alpha^2 \leqslant 3). \tag{5.55}$$

This seems to be a stability boundary of one of the modes. It is shown in Figure 10.

(i) *Hyperbolic-Tangent Shear Layer*

Various exact neutral solutions have been found for various density distributions with velocity profile $w = \tanh y(-\infty < y < \infty)$.

When J is constant, corresponding to basic density of the form $\bar{\rho} = \rho_0 \exp(-\beta y)$, Drazin [85] verified that an eigensolution is given by

$$c = 0, \qquad J_0 = \alpha^2(1-\alpha^2), \qquad \varphi = (\operatorname{sech} y)^{\alpha^2} |\tanh y|^{1-\alpha^2} \quad (0 \leqslant \alpha \leqslant 1). \tag{5.56}$$

It gives the stability boundary, shown in Figure 11(a).

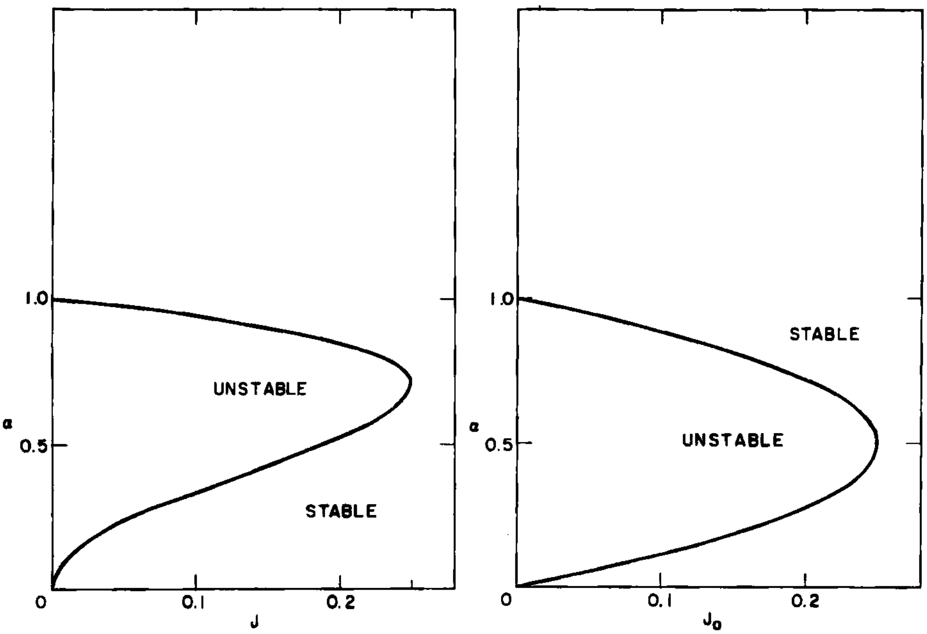

FIG. 11(a). Stability boundary for $w = \tanh y$, $J =$ constant $(-\infty < y < \infty)$: $J = \alpha^2(1-\alpha^2)$ (b). Stability boundary for $w = \tanh y$, $J = J_0 \operatorname{sech}^2 y (-\infty < y < \infty)$: $J_0 = \alpha(1-\alpha)$.

When $J = J_0 \operatorname{sech}^2 y$, corresponding to $\bar{\rho} = \rho_0 \exp(-\beta \tanh y)$, Holmboe [cf. 78] found the eigensolution

$$c = 0, \qquad J_0 = \alpha(1-\alpha), \qquad \varphi = (\operatorname{sech} y)^{\alpha} |\tanh y|^{1-\alpha} \qquad (0 < \alpha < 1) \tag{5.57}$$

for the stability boundary shown in Figure 11(b). It is somewhat similar to the boundary of the example in Section V.2.e shown in Figure 6.

When $J = 3J_0 \operatorname{sech}^2 y \tanh^2 y$, corresponding to density $\bar{\rho} = \rho_0 \cdot \exp(-\beta \tanh^3 y)$, Garcia [cf. 75, 78] found the eigensolutions,

$$\begin{aligned} c &= 0, \qquad J_0 = \tfrac{1}{3}\alpha(\alpha+3), \qquad \varphi = \tanh y (\operatorname{sech} y)^{\alpha}; \\ c &= 0, \qquad J_0 = \tfrac{1}{3}(\alpha-1)(\alpha+2), \qquad \varphi = (\operatorname{sech} y)^{\alpha}. \end{aligned} \tag{5.58}$$

These define a stability boundary, shown in Figure 11(c). It is somewhat similar to the boundaries of the example in Section V.2.e shown in Figures 5 and 7. Some waves are unstable for each value of J_0, however large. Thus the

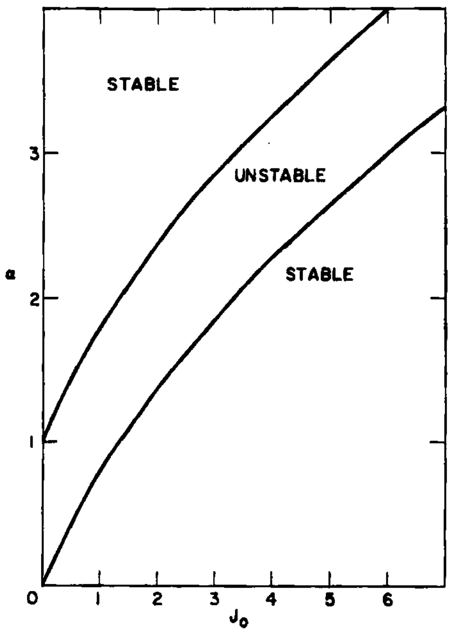

FIG. 11(c). Stability boundary for $w = \tanh y$, $J = 3J_0 \operatorname{sech}^2 y \tanh^2 y(-\infty < y < \infty)$: $J_0 = \alpha(\alpha + 3)/3$ and $J_0 = (\alpha - 1)(\alpha + 2)/3$.

flow cannot be stabilized. This occurs because $J(y)/(Dw)^2$ vanishes where $w = c$, i.e. at $y = 0$, and therefore cannot be everywhere larger than $\frac{1}{4}$ for sufficiently large J_0.

Miles [78] has considered combinations of the above two density distributions, with local Richardson number $J(y) = J_0(1 - r + 3r \tanh^2 y) \operatorname{sech}^2 y$. Thus $r = 0$ corresponds to Holmboe's case above, and $r = 1$ to Garcia's. Miles showed that, when $r > \frac{1}{2}$, the relation $J_0 = J_0(\alpha)$ on the stability boundary is no longer single valued, because the neutral curve turns away from the α-axis. Further, when $0.895 < r < 0.968$, there are two distinct branches of the stability boundary. As $r \to 1$, the number of distinct branches increases to infinity.

(j) *Free Surface Flows*

A class of flows with important applications comprises those with a free surface, i.e. with $\bar{\rho} = 0$ for $y \geqslant 0$, say. Esch [86] has considered a few examples of this class with further variation of $\bar{\rho}$.

VI. Stability of Other Parallel Flows

1. *Discussion*

So far we have considered the stability of *plane* parallel flows, with basic velocity $\bar{u}_*(y_*)\mathbf{i}$, to which in fact the literature is largely confined. There is a little work on the more general parallel flows, with basic velocity $\bar{u}_*(y_*, z_*)\mathbf{i}$, partly on the basis of known properties of plane parallel flow [**87**], [**88**], partly by use of general properties like the energy and vorticity integrals [**89**], [**64**]. We give an example of this (the proof of the semicircle theorem) in Section VI.2. However it seems essential to reduce the linearized partial differential equations of motion to ordinary ones in order to analyze them thoroughly. For plane parallel flow, this is possible by the techniques of transforms and normal modes, discussed in Section II.1. It seems that the only other class of parallel flows for which this is possible is those with an *axisymmetric* basic velocity $\bar{u}_*(r_*)\mathbf{i}$, where $r_*^2 = y_*^2 + z_*^2$ —the round jets. Although their instability has some significant differences from that of plane parallel flow, both the mechanisms and the mathematics are similar in the two cases. It thus seems sufficient for our purposes to present only a brief account of the instability of axisymmetric jets in an inviscid incompressible fluid. This is given in Section VI.3.

2. *The Semicircle Theorem for General Parallel Flow*

The most general discussion of the semicircle theorem appears to be that given by Eckart [**64**], who has derived it for compressible flow, with gravity, which is parallel ($w(y,z)\mathbf{i}$) or circular ($w(r,z)\boldsymbol{\theta}_1$). However since Eckart's notation is rather personal, we give here a sketch of another version of the proof with more traditional terminology, restricting ourselves to parallel incompressible flow though the compressible case is almost as easy. Our proof is essentially the same as one constructed by H. Schade and Howard (**1963**), and independently by Hocking [**88**]. We mention, also that while Eckart's proof seems to cover about as general a case as one might expect to find, there is at least one other case in which the semicircle theorem holds: non-parallel flow which is parallel and uniform in layers, but varies both in magnitude and direction from layer to layer—for example the Ekman boundary layer flow.

We assume that the flow is in a cylindrical region $\{-\infty < x < \infty, (y,z)$ in $S\}$, where S is some connected region in the (y, z)-plane with a sufficiently smooth boundary. The basic flow is $w(y,z)\mathbf{i}$; since the coefficients of the stability equation are independent of x and t we look for normal modes of the form $f(y,z) \exp i\alpha(x - ct)$, just as in the plane case. The stability equations become:

$$i\alpha(w-c)u' + \mathbf{v}' \cdot \nabla_2 w + i\alpha p' = 0 \tag{6.1}$$

$$i\alpha(w-c)\mathbf{v}' + \nabla_2 p' = 0 \tag{6.2}$$

$$i\alpha u' + \nabla_2 \cdot \mathbf{v}' = 0 \tag{6.3}$$

where ∇_2 is the transverse gradient operator, p' is the perturbation pressure divided by density, and u' and $\mathbf{v}'$ are the longitudinal and transverse parts of the perturbation velocity vector. The boundary conditions are $\mathbf{v}' \cdot \mathbf{n} = 0$,$\mathbf{n}$ being the normal to the boundary B of S; from (6.2) this can be expressed instead as

$$\frac{\partial p'}{\partial n} = 0 \qquad \text{on} \qquad B. \tag{6.4}$$

Using (6.2) to eliminate $\mathbf{v}'$ and (6.3) to eliminate u', (6.1) is readily transformed into:

$$\nabla_2 \cdot [(w-c)^{-2}\nabla_2 p'] - \alpha^2(w-c)^{-2}p' = 0. \tag{6.5}$$

Multiplying by $\bar{p}'$, integrating over S, and using the boundary condition (6.4) we obtain

$$\iint_S (w-c)^2[|\nabla_2 p'|^2 + \alpha^2|p'|^2]|w-c|^{-4}dydz = 0. \tag{6.6}$$

This is of the same form $\int(w-c)^2 Q = 0$ with $Q > 0$ as in the plane parallel case, and the semicircle theorem thus follows immediately, as before.

3. *Inertial Instability of Axisymmetric Jets*

The work of this subsection is analogous to that of Section II on inertial instability of *plane* parallel flow. By reference to the motivation and methods of that section, we may state results briefly here. We take the basic axisymmetric parallel flow of inviscid incompressible fluid, with velocity $\bar{\mathbf{u}}_* = U_*(r_*)\mathbf{i}$ $(r_{1*} \leqslant r_* \leqslant r_{2*})$. This represents a jet between the rigid coaxial cylinders $r_* = r_{1*}, r_{2*}$, where r_{1*} may be zero and r_{2*} infinite. It is again convenient to choose dimensional scales V of $U_*(r_*)$ and L of its variation, and to render all variables dimensionless by scaling. Then the basic flow of the jet is

$$\bar{\mathbf{u}} = U(r)\mathbf{i} \qquad (r_1 \leqslant r \leqslant r_2). \tag{6.7}$$

With use of cylindrical polar coordinates (x,r,φ) and associated velocity components (u_x,u_r,u_φ), the equations of motion may be linearized much as before to give the perturbation equations,

$$\partial u_x'/\partial t + U\partial u_x'/\partial x + u_r' dU/dr = -\partial p'/\partial x,$$
$$\partial u_r'/\partial t + U\partial u_r'/\partial x = -\partial p'/\partial r,$$
$$\partial u_\varphi'/\partial t + U\partial u_\varphi'/\partial x = -\partial p'/r\partial\varphi,$$
$$\partial u_x'/\partial x + \partial(r u_r')/r\partial r + \partial u_\varphi'/r\partial\varphi = 0.$$

With the method of normal modes one assumes that

$$u_x', u_r', u_\varphi', p' = \operatorname{Re}\,[\{F(r), iG(r), H(r), P(r)\}e^{in\varphi + i\alpha(x-ct)}],$$

where α is a real wave-number, c a complex velocity, n any integer which represents the azimuthal Fourier component, and F, G, H, P are eigen-functions to be determined. Then the linearized equations reduce to the ordinary differential equations,

$$\alpha(U-c)F + (DU)G = -\alpha P,$$
$$\alpha(U-c)G = DP,$$
$$\alpha(U-c)H = -nP/r,$$
$$\alpha F + DG + G/r + nH/r = 0,$$

where now $D \equiv d/dr$. On elimination of F, H, P one may get the single linear ordinary differential equation for G:

$$D\{rD(rG)/(n^2+\alpha^2 r^2)\} - G - \{rG/(U-c)\}D\{rDU/(n^2+\alpha^2 r^2)\} = 0. \tag{6.8}$$

The boundary conditions are that the normal velocity u_r' vanishes on the coaxial cylinders $r = r_1, r_2$. Therefore, in general,

$$G = 0 \qquad (r = r_1, r_2). \tag{6.9}$$

However, when $r_2 = \infty$ we require that all perturbations vanish there in order that the energy of the disturbance of finite origin be bounded. For this it is sufficient that $G \to 0$ and is well behaved as $r \to \infty$; and thus we may use condition (6.9) at infinity. When $r_1 = 0$ the continuity of $\mathbf{u}, p$ implies that u_x', p' are independent of φ and that u_r', u_φ' vanish at $r = 0$ (except for $n = 1$, when u_r' and u_φ' need only be bounded, but $G(0) + H(0) = 0$, from the continuity equation). Therefore $F(0) = P(0) = 0$ ($n \neq 0$) and $G(0) = H(0) = 0$; and thus we may use condition (6.9) at $r = 0$ except when $n = 0$. In fact it can be seen that in general equation (6.8) gives $G \sim \text{constant} \times r^{n-1}$ ($n \neq 0$) and $G \sim \text{constant} \times r$ ($n = 0$) as $r \to 0$.

If U or DU is discontinuous, at r_0 say, then the pressure must be continuous at the material interface with mean position $r = r_0$. It follows that

$$[(U-c)D(rG) - (DU)(rG)] = 0 \qquad (r = r_0). \tag{6.10}$$

Also the normal velocity must be continuous at this material interface. Therefore

$$[G/(U - c)] = 0 \qquad (r = r_0). \tag{6.11}$$

Note that when U is piecewise constant it is easier to work with the amplitude $\Phi = \int G dr$ of the velocity potential rather than with G directly [90].

The eigenvalue problem (6.8), (6.9) is essentially due to Rayleigh [91]. It has prompted surprisingly little later work, in view of the scores of papers on the analogous problem (2.11), (2.12) of plane parallel flow. Perhaps the similarity of the two problems and their methods of solution has discouraged duplication of work, perhaps the greater physical importance of plane parallel flows has overshadowed that of round jets. However, there is one important difference between the two eigenvalue problems, namely the essentially three-dimensional nature of instability of a round jet. Experience of plane parallel flows suggests that varicose instability (axisymmetric disturbances with $n = 0$) of a round jet should be less than sinuous instability ($n = 1$), so it comes as no surprise to find that there is no analogue of Squire's theorem. In fact, in a recent examination of non-axisymmetric disturbances, Batchelor and Gill [90] found that a certain jet is most unstable to the mode $n = 1$.

The eigenvalue problem (6.8), (6.9) is symmetric in α and $(-\alpha)$, so we can again take $\alpha \geqslant 0$ without loss of generality. There is also a symmetry in G, c and G^*, c^* for the same α, so c is real for stability and complex for instability. Again, we write $c_i > 0$ when there is instability, bearing in mind the initial-value problem and the inviscid limit of the viscous problem [92], [90, § 2].

Rayleigh [91] found a necessary condition for instability, analogous to there being a point of inflexion in the velocity profile of a plane parallel basic flow. Essentially by multiplying the stability equation (6.8) by $rG^*/(U - c)$, integrating from r_1 to r_2, and taking the imaginary part, he found that

$$c_i \int_{r_1}^{r_2} |g|^2 DQ dr = 0, \tag{6.12}$$

where $g \equiv rG/(U - c)$, $Q \equiv r(DU)/(n^2 + \alpha^2 r^2)$. Therefore a necessary condition for instability ($c_i > 0$) is that $DQ = 0$ somewhere in the field of flow. This is equivalent to U having a point of inflection with respect to the variable $\rho = \int (n^2 + \alpha^2 r^2)/r dr = n^2 \log r + \frac{1}{2}\alpha^2 r^2$. This reduces to Rayleigh's condition for plane parallel flow if one regards the round jet as being plane parallel flow locally when $r_1, r_2 \to \infty$ and $r_2 - r_1 = y_2 - y_1$ is fixed.

The following general stability characteristics are due to Batchelor and Gill [90], who give details of the proofs.

On multiplying the stability equation (6.8) by $rG^*/(U-c)$, integrating from r_1 to r_2, and taking the real part, they found that

$$\int_{r_1}^{r_2} |g|^2(U-c_r)DQdr \leqslant 0. \tag{6.13}$$

Therefore, when $c_i > 0$, equation (6.12) gives

$$\int_{r_1}^{r_2} |g|^2(U-U_f)DQdr \leqslant 0, \tag{6.14}$$

U_f being the value of $U(r)$ at $r = r_f$, where $DQ = 0$. Therefore, when DQ changes sign only once, a necessary condition for instability is that $(U - U_f)DQ \leqslant 0$ throughout the flow.

The semicircle theorem follows much as in Section II.2, giving

$$\{c_r - \tfrac{1}{2}(w_{\min} + w_{\max})\}^2 + c_i^2 \leqslant \{\tfrac{1}{2}(w_{\max} - w_{\min})\}^2 \qquad (c_i > 0). \tag{6.15}$$

It can be shown that the Reynolds stress tensor, averaged over one period $2\pi/\alpha$ of x and one $2\pi/n$ of φ, has orthogonal components

$$\tau_{1r} \equiv \overline{u_1'u_r'} \equiv (n^2 + \alpha^2r^2)^{-1/2}\overline{(nu_\varphi' + \alpha ru_r')u_r'} \tag{6.16}$$

$$= \tfrac{1}{4}r^{-1}(n^2 + \alpha^2r^2)^{1/2}We^{2\alpha c_it}, \tag{6.17}$$

where

$$W \equiv ir(n^2 + \alpha^2r^2)^{-1}\{(rG^*)D(rG) - (rG)D(rG^*)\};$$

and

$$\tau_{r3} \equiv \overline{u_r'u_3'} \equiv (n^2 + \alpha^2r^2)^{-1/2}\overline{u_r'(\alpha ru_\varphi' - nu_x')} \tag{6.18}$$

$$= \frac{nc_i(DU)|G|^2}{2\alpha(n^2 + \alpha^2r^2)^{1/2}\{(U-c_r)^2 + c_i^2\}}. \tag{6.19}$$

The stability equation (6.8) gives

$$DW = -\frac{2c_ir^2|G|^2DQ}{(U-c_r)^2 + c_i^2}. \tag{6.20}$$

It follows that, as $c_i \to 0$ through positive values, W is piecewise constant and

$$[W] = -2\pi(r^2|G|^2DQ/DU)_{r=r_c}, \tag{6.21}$$

where $U(r_c) = c_r$ in the limit, provided that $(DU)_{r=r_c} \neq 0$. Now $W = 0$ at $r = r_1, r_2$. Therefore, in the limit as $c_i \to 0, W = 0$ at $r = r_1, r_2$. Therefore, in the limit as $c_i \to 0, W = 0$ everywhere if $U = c_r$ at only one point $r = r_c$, or, in particular, if $U(r)$ is monotonic. In that event $[W] = 0$, and therefore either $DQ = 0$ or $G = 0$ at $r = r_c$. The latter equality is compatible with the stability equation (6.8) only if $G \equiv 0$. Therefore $DQ = 0$ at $r = r_c$, i.e. $r_c = r_f$ and $c_r = U_f$.

If we put $c = U_f$ and look for neutral solutions that are limits of unstable solutions, the task is more difficult than that of (2.18). However, Batchelor and Gill [90] showed as follows that there is no such singular neutral solution for sufficiently large n. Equation (6.8) now can be written as

$$(6.22) \qquad rD\left\{\frac{rD(rG)}{n^2+\alpha^2r^2}\right\} = \left\{1 - \frac{r}{U_f - U} D\left(\frac{rDU}{n^2+\alpha^2r^2}\right)\right\} rG.$$

Therefore, if

$$1 \geqslant \max_{r_1 \leqslant r \leqslant r_2} \left\{\frac{r}{U_f - U} D\left(\frac{rDU}{n^2+\alpha^2r^2}\right)\right\} = \max\left\{\frac{rDQ}{U_f - U}\right\},$$

the solution (rG) of the stability equation will be monotonic and cannot satisfy both boundary conditions. Thus a necessary condition for the existence of the singular neutral solution is that n is not so large that $\max\{rDQ/(U_f - U)\} < 1$. In fact this condition is quite restrictive.

Very few examples have been treated in the literature. First we take the exact solution of the Navier-Stokes equations for a viscous fluid as our basic flow, namely Poiseuille flow in a pipe with

$$(6.23) \qquad U = Ar^2 + B\log r + C \qquad (r_1 \leqslant r \leqslant r_2).$$

Rayleigh [91] investigated the stability of this basic flow in an inviscid fluid. It gives $Q = (2Br^2 + C)/(n^2 + \alpha^2r^2)$, which varies monotonically with r. Therefore the flow is always stable.

For the cylindrical vortex sheet,

$$(6.24) \qquad U = \begin{cases} 1 & (r < 1) \\ 0 & (r > 1), \end{cases}$$

Batchelor and Gill [90] used the velocity potential on each side of the discontinuity to deduce that the eigenvalue is

$$(6.25) \qquad c = \{1 + iL_n^{1/2}(\alpha)\}/\{1 + L_n(\alpha)\},$$

where $L_n(\alpha) \equiv -K_n(\alpha)I_n'(\alpha)/K_n'(\alpha)I_n(\alpha)$ in terms of the modified Bessel functions I_n, K_n of the first and second kinds and their derivatives. This flow is unstable for each pair of values n, α. As $\alpha \to \infty$ (i.e. as the radius

L of the vortex sheet $\to \infty$ for fixed α_*), $L_n(\alpha) \to 1$ and the result $c = \frac{1}{2}(1+i)$ for a plane vortex sheet is recovered.

For the profile of a realistic round jet,

$$U = (1 + r^2)^{-2} \qquad (0 \leqslant r < \infty) \tag{6.26}$$

DQ does not vanish anywhere when $n = 0$, so the axisymmetric disturbances are stable [90]. Further, $\max\limits_{0 \leqslant r < \infty} \{rDQ/(U_f - U)\} < 1$ when $n \geqslant 2$, so the only possibility of instability occurs when $n = 1$. In fact there is instability when $n = 1$ [90], the singular neutral mode occurring for $\alpha = \alpha_f = 1.46$ and $(\alpha r_c)^2 = 0.57$; i.e. $c = U_f = 0.62$. Thus there is instability only for the sinuous mode with $\alpha < 1.46$.

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