SPACE–FILLING CURVES GENERATED BY FRACTAL INTERPOLATION FUNCTIONS

V. DRAKOPOULOS
Department of Informatics, Theoretical Informatics, University of Athens, Panepistimioupolis 157 71 Athens, Greece
E-mail: vasilios@di.uoa.gr

L. DALLA
Department of Mathematics, University of Athens, Panepistimioupolis 157 84 Athens, Greece
E-mail: ldalla@eudoxos.dm.uoa.gr

Interpolation formulas are the starting points in the derivations of many methods in several areas of Numerical Analysis. The goal is always the same: to represent the data by a classical geometrical entity. Fractal interpolation functions provide a new means for fitting experimental data and their graphs can be used to approximate natural scenes. We show how one can construct space–filling curves using hidden variable linear fractal interpolation functions. These curves result from the projection of the attractor of an iterated function system.

1 Introduction

Interpolation lies in the heart of classical Numerical Analysis. Almost all the classical methods of numerical differentiation, numerical quadrature and numerical integration of ordinary differential equations are directly derivable from interpolation formulas. Because we are especially interested in computer applications, our approach to interpolation will not emphasize traditional interpolation methods. We focus attention on a special class of continuous functions, referred to as *fractal functions*, since their graphs usually have non–integral dimension. These functions may be used for interpolation purposes and are in this way analogous to splines and polynomial interpolations.

Based on a theorem of J. E. Hutchinson ([6], p.730) and using IFS theory, M. F. Barnsley introduced a class of functions in [1] which he called *fractal interpolation functions* or *FIF's* for short. He worked basically with *linear FIF's* in the sense that they are obtained using affine transformations. These functions have in common with elementary functions that they are of a geometrical character and that they can be computed rapidly. The main difference is their fractal character. The graphs of these functions can be used to approximate image components such as the profiles of mountain ranges, the tops of clouds and horizons over forests, to name but a few.

Considering the projections of the graphs of higher–dimensional self–affine functions, M. F. Barnsley et al. in [4] have extend the class of one–dimensional interpolation functions to the *hidden variable FIF's* (see also [2] and [7]).

We use these functions to construct some *space-filling curves* in the plane. These curves are projections of the graphs of some continuous functions. Finally, with the aid of microcomputer–generated plots, we examine these graphs in the three–dimensional space using orthogonal and other projections. The IFS code for this class of functions is given as well.

2 Iterated Function Systems

Within Fractal Geometry, the method of iterated function systems introduced by J. E. Hutchinson in [6] and popularized by M. F. Barnsley and S. Demko in [3], is a relatively easy way to generate fractal images.

A function $f: X \to Y$ is called a *Hölder function of exponent a* if

$$|f(x) - f(y)| \leq c\,|x - y|^a$$

for $x, y \in X$, $a \geq 0$ and for some constant c. Obviously, $c \geq 0$. The function f is called a *Lipschitz function* if a may be taken to be equal to 1. A Lipschitz function is called *contractive* with *contractivity factor c*, if $c < 1$. An *iterated function system* or *IFS* for short, is a collection of a complete metric space (X, ρ) together with a finite set of mappings $w_n: X \to X$, $n = 1, 2, \ldots, N$, where ρ is a distance between elements of X. If w_n are contractive with respective contractivity factors s_n for $n = 1, 2, \ldots, N$, then the IFS is termed *hyperbolic*. It is often convenient to write an IFS formally as $\{X; w_1, w_2, \ldots, w_N\}$ or, somewhat more briefly, as $\{X; w_{1-N}\}$.

We introduce the associated map of subsets $W: \mathcal{H}(X) \to \mathcal{H}(X)$, given by

$$W(E) = \bigcup_{n=1}^{N} w_n(E), \text{ for all } E \in \mathcal{H}(X),$$

where $\mathcal{H}(X)$ is the metric space of all nonempty compact subsets of X with respect to the Hausdorff distance. The map W itself is contractive with ratio $s = \max\{s_1, s_2, \ldots, s_N\}$. (Ref. [2], Theorem 7.1, p.81). The map W is called the *collage map* to alert us to the fact that $W(E)$ is formed as a union or collage of sets. Sometimes $\mathcal{H}(X)$ is referred to as the 'space of fractals in X' (but note that not all members of X are fractals). In what follows we abbreviate f^k the k–fold composition $f \circ f \circ \cdots \circ f$.

The *attractor* of an IFS is the unique set $\mathcal{A}$ for which $\lim_{k\to\infty} W^k(E_0) = \mathcal{A}$ for every starting set E_0. The term attractor is chosen to suggest the movement of E_0 towards $\mathcal{A}$ under successive applications of W. By contrast, $\mathcal{A}$ is also the unique set in $\mathcal{H}(X)$ which is not changed by W, so $W(\mathcal{A}) = \mathcal{A}$, and from this important perspective it is often called the *invariant set* of the IFS.

A transformation w is *affine* if it may be represented by a matrix A and translation $\mathbf{t}$ as $w(\mathbf{x}) = A\mathbf{x} + \mathbf{t}$, or (if $X = \mathbf{R}^3$)

$$w \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & g \\ h & k & s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} l \\ m \\ r \end{bmatrix}. \tag{1}$$

The *code* of w is the 12–tuple $(a, b, c, d, e, g, h, k, s, l, m, r)$ and the *code of an IFS* is a table whose rows are the codes of $w_1, w_2, \ldots, w_N$. We refer the interested reader to [2] or [5].

3 Fractal Interpolation Functions

Let the continuous function f be defined on a real closed interval $I = [x_0, x_N]$ and with range a complete metric space (Y, ρ_Y), where $x_0, x_1, \ldots, x_N$ be $N+1$ distinct points and $x_0 < x_1 < \cdots < x_N$. It is not assumed that these points are equidistant. We shall write for brevity $f(x_i) = f_i$, $i = 0, 1, \ldots, N$. The function f is called an *interpolation function* corresponding to the generalized set of data

$$\{(x_i, f_i) \in I \times Y : i = 0, 1, \ldots, N\}.$$

The points (x_i, f_i) are called the *interpolation points.* We say that the function f *interpolates* the data and that (the graph of) f passes through the interpolation points. We focus on the existence and construction of such functions whose graphs are attractors of IFS. Throughout this and next section we will work in the complete metric space $K = I \times Y$ with respect to the Euclidean, or to some other equivalent, metric.

Set $I_n = [x_{n-1}, x_n]$ and let $L_n : I \to I_n$, for $n = 1, 2, \ldots, N$, be contractive homeomorphisms such that

$$L_n(x_0) = x_{n-1}, \quad L_n(x_N) = x_n, \tag{2}$$

$$|L_n(b_1) - L_n(b_2)| \le l\,|b_1 - b_2| \tag{3}$$

whenever $b_1, b_2 \in I$, for some $l \in [0, 1)$.

Furthermore, let mappings $M_n : K \to Y$ be continuous such that

$$M_n(x_0, f_0) = f_{n-1}, \quad M_n(x_N, f_N) = f_n, \tag{4}$$

$$\rho_Y(M_n(x, b_1), M_n(x, b_2)) \le s\,\rho_Y(b_1, b_2), \tag{5}$$

for all $x \in I$, $b_1, b_2 \in Y$ and for some $s \in [0,1)$. Condition (5) means that M_n are contractive in the second variable, for $n = 1, 2, \ldots, N$.

Now define functions $w_n: K \to K$ by

$$w_n(x, y) = (L_n(x), M_n(x, y)) \tag{6}$$

for all $(x, y) \in K$ and $n = 1, 2, \ldots, N$.

Theorem 1 *The IFS $\{K; w_{1-N}\}$ defined above has a unique attractor $G \in \mathcal{H}(K)$. Furthermore, G is the graph of a continuous function $f: I \to Y$ which obeys*

$$f(x_i) = f_i, \quad i = 0, 1, \ldots, N.$$

Proof. See [1], Theorem 1, p.306. □

Definition 1 *The function f whose graph is the attractor of an IFS as described in Theorem 1, is called a* fractal interpolation function *or* FIF *for short.*

Notice that the IFS $\{K; w_{1-N}\}$ may not be hyperbolic. To construct a hyperbolic IFS whose attractor is the graph of a function, we assume that the mappings M_n, $n = 1, 2, \ldots, N$ not only satisfy Condition (5) but also

$$\rho_Y(M_n(b_1, y), M_n(b_2, y)) \leq c\,|b_1 - b_2| \tag{7}$$

for all $y \in Y$, $b_1, b_2 \in I$, $n = 1, 2, \ldots, N$ and for some $c > 0$. This condition means that M_n are uniformly Lipschitz in the first variable, for $n = 1, 2, \ldots, N$.

Since the completeness depends on the choice of metric we have the following

Theorem 2 *There is a metric ρ_ϕ on K, equivalent to the Euclidean metric, such that the IFS $\{K; w_{1-N}\}$ is hyperbolic with respect to ρ_ϕ.*

Proof. See [9]. □

4 Hidden variable Fractal Interpolation Functions

Now, we will restrict our attention to affine transformations. Let N be a positive integer greater than 1. Define $L_n: I \to I_n$ by

$$L_n(x) = a_n x + k_n,$$

where the real numbers a_n, k_n, for $n = 1, 2, \ldots, N$, are chosen to ensure that Condition (2) holds, i.e. $L_n(I) = I_n$. Thus, for $n = 1, 2, \ldots, N$,

$$a_n = \frac{x_n - x_{n-1}}{x_N - x_0}, \tag{8}$$

$$k_n = \frac{x_N x_{n-1} - x_0 x_n}{x_N - x_0}. \tag{9}$$

Since $N \geq 2$, $|a_n| < 1$, so L_n are contractive homeomorphisms, for $n = 1, 2, \ldots, N$, as they obey Condition (3) with $l = \max\{|a_n| : n = 1, 2, \ldots, N\}$.

Let $Y = \mathbf{R}^2$ and define $M_n : K \to Y$ by

$$M_n \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_n \\ g_n \end{bmatrix} x + A_n \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} l_n \\ m_n \end{bmatrix},$$

where

$$A_n = \begin{bmatrix} d_n & e_n \\ h_n & s_n \end{bmatrix}$$

and $c_n, g_n, d_n, e_n, h_n, s_n, l_n, m_n$ are real numbers, for $n = 1, 2, \ldots, N$. Let us replace f_n in Condition (4) by (f_n, H_n). The real constants c_n, g_n, l_n, m_n, depending on the adjustable real parameters d_n, e_n, h_n, s_n, are chosen to ensure that Condition (4) holds. That is,

$$c_n = \frac{f_n - f_{n-1}}{x_N - x_0} - d_n \frac{f_N - f_0}{x_N - x_0} - e_n \frac{H_N - H_0}{x_N - x_0}, \tag{10}$$

$$g_n = \frac{H_n - H_{n-1}}{x_N - x_0} - h_n \frac{f_N - f_0}{x_N - x_0} - s_n \frac{H_N - H_0}{x_N - x_0}, \tag{11}$$

$$l_n = \frac{x_N f_{n-1} - x_0 f_n}{x_N - x_0} - d_n \frac{x_N f_0 - x_0 f_N}{x_N - x_0} - e_n \frac{x_N H_0 - x_0 H_N}{x_N - x_0}, \tag{12}$$

$$m_n = \frac{x_N H_{n-1} - x_0 H_n}{x_N - x_0} - h_n \frac{x_N f_0 - x_0 f_N}{x_N - x_0} - s_n \frac{x_N H_0 - x_0 H_N}{x_N - x_0}, \tag{13}$$

for $n = 1, 2, \ldots, N$. Let us define

$$c = \max\{\max\{c_n, g_n\} : n = 1, 2, \ldots, N\}.$$

Then Condition (7) is true. Assume that the linear transformation $A_n : \mathbf{R}^2 \to \mathbf{R}^2$ is contractive with contractivity factor $s \in [0, 1)$. Then Condition (5) is true. Define functions w_n as in Eq. 6. Then the IFS is of the form $\{K; w_{1-N}\}$, where the maps are affine transformations as in (1) and, in particular, of the special structure

$$w \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_n & 0 & 0 \\ c_n & d_n & e_n \\ g_n & h_n & s_n \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} k_n \\ l_n \\ m_n \end{bmatrix},$$

where $a_n, c_n, d_n, e_n, g_n, h_n, s_n, k_n, l_n, m_n$ are real numbers. These transformations, constrained by Conditions (2) and (4), are giving

$$w_n \begin{bmatrix} x_0 \\ f_0 \\ H_0 \end{bmatrix} = \begin{bmatrix} x_{n-1} \\ f_{n-1} \\ H_{n-1} \end{bmatrix} \text{ and } w_n \begin{bmatrix} x_N \\ f_N \\ H_N \end{bmatrix} = \begin{bmatrix} x_n \\ f_n \\ H_n \end{bmatrix}, \tag{14}$$

for $n = 1, 2, ..., N$. Then we have the following

Theorem 3 *The attractor of the IFS $\{K; w_{1-N}\}$ defined above is the graph of a continuous function $f: I \to Y$ which obeys*

$$f(x_i) = (f_i, H_i), \quad i = 0, 1, \ldots, N.$$

Proof. The proof is analogous to the proof of Theorem 1. □

Now write

$$f(x) = (F_1(x), F_2(x)).$$

Then $F_1: I \to \mathbf{R}$ is a continuous function such that

$$F_1(x_i) = f_i, \quad i = 0, 1, \ldots, N.$$

Definition 2 *The function $F_1(x)$ constructed above is called a* hidden variable linear fractal interpolation function *or* HLFIF *for short.*

5 Space–filling curves

G. Peano in 1890 constructed the first curve that passes through every point of the unit square $[0, 1] \times [0, 1]$. Continuous mappings from $[0, 1]$ (or any other interval) into the plane (or space) with this property are called *space–filling* or *Peano curves.* Further examples by D. Hilbert (in 1891), E. H. Moore (in 1900), H. Lebesgue (in 1904), E. Cesàro (in 1905), W. Sierpiński (in 1912), G. Pólya (in 1913), and others followed. For more examples of such curves the interested reader is referred to [8] and [7] .

We can apply Theorem 3 to construct some space–filling curves. Notice that we now require equidistant points $x_0, x_1, \ldots, x_N$, with $x_0 = f_0 = H_0 = 0$, $x_N = f_N = 1$ and $H_N = 0$. Then the attractor of the IFS $\{\mathbf{R}^3; w_{1-N}\}$, where the maps w_n, $n = 1, 2, \ldots, N$ are the affine transformations

$$w_n \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/N & 0 & 0 \\ 0 & a_n & b_n \\ 0 & c_n & s_n \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} (n-1)/N \\ d_n \\ e_n \end{bmatrix}, \qquad (15)$$

is the graph G of a continuous function $f: [0, 1] \to \mathbf{R}^2$ such that $f([0, 1]) = \mathcal{A}$, where $\mathcal{A}$ is a nonempty pathwise–connected compact set. The range of this function is the projection of G into the (y, z) plane. The projections of G into (x, y) and (x, z), are graphs of hidden variable fractal functions. The easiest method for computing the graph of a hidden variable fractal interpolation function is with the aid of the *Random Iteration Algorithm* or *RIA* for short.

In the RIA we calculate at each stage just one new point x_{n+1} from its predecessor x_n, by $x_{n+1} = w_i(x_n)$ with a randomly chosen i from $1, 2, \ldots, N$.

Given a large number of iterations–say 40,000, it works. Beneath the dancing iterated point the attractor emerges miraculously from the mist. See [2] or [5] for more details on this subject. We must first calculate the coefficients in the three–dimensional affine transformations from Eq. 8 to Eq. 13 and then apply the RIA to the resulting IFS. Finally, we render each point in a color that depends on its z–coordinate. This helps us to visualize the 'hidden' three–dimensional character of the curve. In all of the examples below the interpolation points are marked with a circle.

w	a	b	c	s
1	0.25	0.375	0.5	-0.25
2	0.5	0	0	0.5
3	0.25	-0.375	-0.5	-0.25

(a)

w	a	b	c	s
1	0.5	0.5	0.5	-0.5
2	0.5	-0.5	-0.5	-0.5

(b)

Table 1: The IFS code for the space–filling curves constructed (a) in Example 1 and (b) in Example 2.

Example 1 Let us take $N = 3$ and the set of data

$$\{(0,0,0),(0.33,0.25,0.5),(0.66,0.75,0.5),(1,1,0)\}.$$

The maps w_n, $n = 1,2,\ldots,N$ are of the form (15) and chosen so that (14) holds. The solution of (15) and (14) yields the constants a_n, c_n, d_n and e_n. For the other constants we require that also the following conditions hold:

$$w_1\begin{bmatrix}1/2\\1/2\\1\end{bmatrix}=\begin{bmatrix}1/6\\1/2\\0\end{bmatrix}, w_2\begin{bmatrix}1/2\\1/2\\1\end{bmatrix}=\begin{bmatrix}3/6\\1/2\\1\end{bmatrix}, w_3\begin{bmatrix}1/2\\1/2\\1\end{bmatrix}=\begin{bmatrix}5/6\\1/2\\0\end{bmatrix}.$$

The resulting constants are given in Table 1(a). Several views of the attractor G for the IFS defined above are illustrated in Figure 1. We see that G is a Sierpiński triangle when viewed from the yz–plane.

Example 2 Let us take $N = 2$ and the set of data

$$\{(0,0,0),(0.5,0.5,0.5),(1,1,0)\}.$$

The solution of (15) and (14) yields the constants a_n, c_n, d_n and e_n, whereas for the other constants we set $b_n = c_n$ and $s_n = -a_n$ for $n = 1,2,\ldots,N$. The resulting constants are given in Table 1(b). Several views of the attractor G for the IFS defined above are illustrated in Figure 2. We see that G is a Peano curve when viewed from the yz–plane.

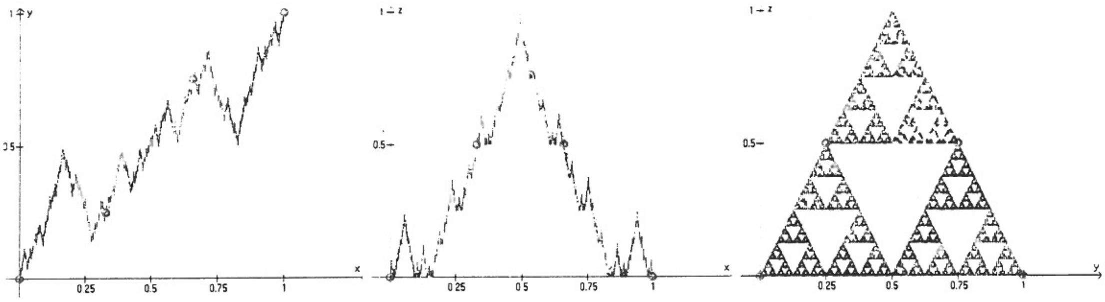

Figure 1: Three orthogonal projections of the attractor for the IFS in Example 1.

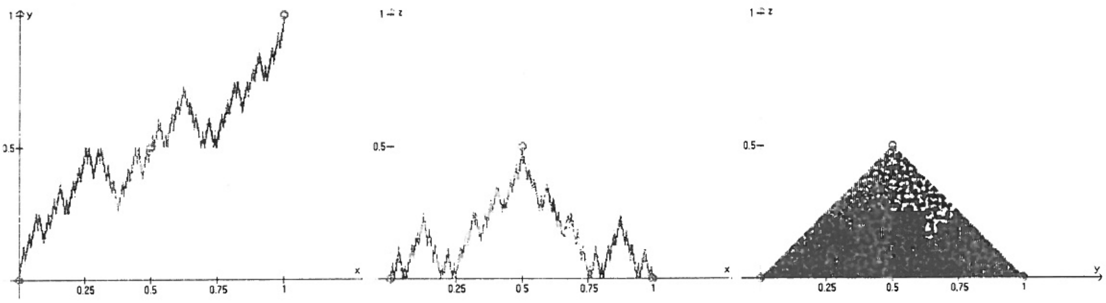

Figure 2: Three orthogonal projections of the attractor for the IFS in Example 2.

Example 3 Let us take $N = 4$ and the set of data

$$\{(0,0,0),(0.25,0.33,0),(0.5,0.5,0.43),(0.75,0.66,0),(1,1,0)\}.$$

The solution of (15) and (14) yields the constants a_n, c_n, d_n and e_n, whereas for the other constants we set $b_n = -c_n$ and $s_n = a_n$ for $n = 1, 2, \ldots, N$. The resulting constants are given in Table 2. Several views of the attractor G for the IFS defined above are illustrated in Figure 3. We see that G is a Koch curve when viewed from the yz–plane.

References

1. Barnsley M. F., *Fractal functions and interpolation*, Constr. Approx. **2** (1986), 303–329.
2. Barnsley M. F., *Fractals everywhere*, 2nd ed., Academic Press Professional, San Diego, 1993.

w	a	b	c	s
1	0.33	0	0	0.33
2	0.16	-0.433	0.433	0.16
3	0.16	0.433	-0.433	0.16
4	0.33	0	0	0.33

Table 2: The IFS code for the space–filling curve constructed in Example 3.

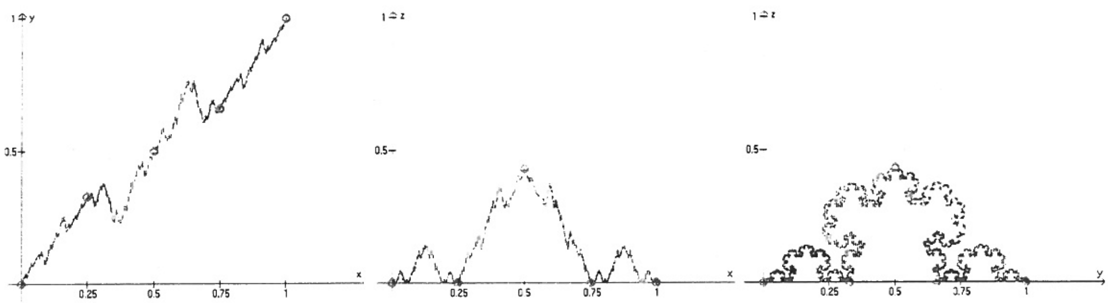

Figure 3: Three orthogonal projections of the attractor for the IFS in Example 3.

3. Barnsley M. F. and Demko S., *Iterated function systems and the global construction of fractals*, Proc. Roy. Soc. London, Ser. A **399** (1985), 243–275.
4. Barnsley M. F., Elton J. H., Hardin D. and Massopust P., *Hidden variable fractal interpolation functions*, SIAM J. Math. Anal. **20** (1989), 1218–1242.
5. Hoggar, S. G., *Mathematics for Computer Graphics*, Cambridge Univ. Press, London and New York, 1992.
6. Hutchinson J. E., *Fractals and self similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.
7. Massopust P. R., *Fractal functions, fractal surfaces and Wavelets*, Academic Press, San Diego, CA, 1994.
8. Sagan H., *Space-filling curves*, Springer–Verlag, New York, 1994.
9. Tziovaras A., *Fractal Interpolation Functions*, Master Thesis, Univ. of Athens, 1996, (in greek).