AN ESTIMATION OF THE BOX DIMENSION OF GREATER ATHENS

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Abstract

The need for describing complex structures that appear in Nature and Mathematics resulted in the definition of the box dimension, a generalization of the usual idea of dimension. The most common are the Hausdorff and box-dimensions. Here the box dimension of Greater Athens is estimated using stratified two stage cluster sampling. The estimated parameter is the total number of cubes of side 3m required to cover the area of Greater Athens.

1. INTRODUCTION

In this paper we attempt to estimate the box-dimension of the city of Athens.

The box-dimension of a set $S \subseteq \mathbf{R}^3$ is defined by

$$D_B(s)=\lim_{d\to 0}\frac{\log N_d(S)}{-\log d}$$

where $N_d(S)$ is the minimum number of 3-dimensional cubes of edge *å* required to cover *S*.

The choice of box-dimension was suggested by the profile of the city where the buildings are mainly rectangular parallelepipeds (boxes).

In the estimation we select *d=3* metres, which is the average height of the appartments and flats of Athens and at the same time is practically zero in comparison with the diameter of the city.

The box-dimension of greater Athens was estimated and it was compared with the box-dimension of the city of Athens in 1842, which was estimated from a three dimensional anaglyphic map found in the Museum of the Town.

Extensive literature on measurements for fractals in $\mathbf{R}^2, \mathbf{R}^3$ may be found in the review article by B. Lea Cox and J.S.Y. Wang[1].

2. METHOD

It is well known in sampling surveys that stratified random sampling provides estimators with small variance when there is little variation among elements within each group. Cluster sampling, on the other hand, does well when the elements within each group are highly

variable, and all groups are quite similar to one another. With this in mind and taking into account the morphology of the area of greater Athens, we used the two-stage cluster sampling combined with stratification.

Applying the Dalenius-Hodges[2] rule we created four strata for optimum allocation in the sense of Neyman, using as criterion the density populated of 48 different districts of greater Athens. In each stratum we have performed a two-stage cluster sampling. The first, we selected a sample of municipalities, the primary units, using again the Dalenius-Hodges rule, and the second we selected a random sample of five blocks, the subunits, from each chosen primary unit. The four strata and selected municipalities are given in Table 1.

The following notation is used.

L =total number of strata

Y_h =total number of cubes of side 3m in stratum *h*, *h=1,2,...,L*

E_h =total area (m^2) in stratum *h*

M_h =total number of blocks in stratum *h*,

N_h =total number of primary units (municipalities) in stratum *h*

M_{hi} =total number of blocks in cluster *i* of stratum *h*

n_h =number of clusters (primary units) selected from stratum *h*

m_{hi} =number of blocks (secondary units) selected from cluster *i* of stratum *h*, $i = 1,2,\ldots,n_h$, *h=1,2,...,L*

y_{hij} =number of cubes of side 3m of *j*-th block in cluster *i* of statum *h*, $i = 1,2,\ldots,n_h$, $j = 1,2,\ldots,m_{hi}$, *h=1,2,...,L.*

The total number Y_h of cubes of side 3m of stratum *h*, *h=1,2,...,L* is unbiased estimated[3,4] by

$$\hat{Y}_h = \frac{N_h}{n_h} \sum_{i=1}^{n_h} M_{hi}\, \overline{y}_{hi}$$

(2)

with an unbiased estimated variance

$$v(\hat{Y}_h) = \frac{N_h(N_h - n_h)}{n_h} s_{hb}^2 + \frac{N_h}{n_h} \sum_{i=1}^{n_h} \frac{M_{hi}(M_{hi} - m_{hi})}{m_{hi}}\, s_{hi}^2 \qquad (3)$$

where

$$\overline{y}_{hi} = \frac{1}{m_{hi}} \sum_{j=1}^{m_{hi}} y_{hij}$$

$$s_{hb}^2 = \frac{1}{n_h - 1} \sum_{i=1}^{n_h} (M_{hi}\overline{y}_{hi} - \hat{Y}_h / N_h)^2$$

and

$$s_{hi}^2 = \frac{1}{m_{hi} - 1} \sum_{j=1}^{m_{hi}} (y_{hij} - \overline{y}_{hi})^2 .$$

3. RESULTS

Results for the estimated totals of cubes and their variances in different strata are given in Table 2. In the same table is also given the total area E_h in m^2 for each stratum based on the corresponding municipalities as they are given by the National Statistical Service of Greece.

An unbiased estimator of total number *Y* of cubes in the area of greater Athens is given by

$$\hat{Y}_{tot} = \sum_{h=1}^{L} \hat{Y}_h = 106.682.023$$

(4)

and the extended estimated variance to stratified sampling can be made by summing the appropriate variance formulas over the strata[3], i.e.

$$v(\hat{Y}_{tot}) = \sum_{h=1}^{L} v(\hat{Y}_h) = 5{,}138997 \times 10^{14} \,. \qquad (5)$$

Under assumption of normality, a 95% confidence interval for total number of cubes of side 3m is given by substituting the results in Eqs. (4), (5),

$$\hat{Y}_{tot} \pm 1{,}96\sqrt{v(\hat{Y}_{tot})} \text{ or } (62.250.085 \ , \ 151.113.961).$$

(6)

From definition of box dimension in Eq. (1), we obtain the estimated box dimension of Greater Athens

$$D = -\frac{\ln(total\ number\ of\ cubes\ of\ side\ 3m)}{\ln(3/\sqrt{total\ area\ in\ m^2})}$$

$$= -\frac{\ln(\hat{Y}_{tot})}{\ln(3/\sqrt{E_{tot}})}$$

(7)

from which, after substituting the totals $\hat{Y}_{tot}$, E_{tot} from Table 2, we obtain

$$D = -\frac{\ln(106.682.023)}{\ln(3/\sqrt{398.686.000})} = 2{,}10.$$

In Table 2 are also given the estimated box dimensions D_h in different strata, from which we can conclude that greater Athens "resemples" as stratum 2.

Using the 95% confidence interval in Eq. (6) for the total number of cubes of side 3m, a corresponding 95% confidence interval for the box dimension of greater Athens is given, by (2, 04 , 2, 14).

3.1. Box dimension of Athens in 1842.

The box dimension of the city of Athens in 1842 was estimated from a three dimensional anaglyphic map found in the Museum of the Town with scale 1:1000. Five blocks were selected in random from which 1535 cubes of side 3m required to cover the corresponding area of 11.160 m^2. Since the area of the city of Athens in 1842 was 308.700 m^2, the estimated total number of cubes are 42.460. Thus, using Eq. (7) the estimated box dimension is

$$D = -\frac{\ln(41.460)}{\ln(3/\sqrt{308.700})} = 2,04$$

from which we conclude that the city of Athens in 1842 was like municipalities in the above stratum one, see Table 2.

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Table 1. ***The strata of greater Athens with the corresponding number of blocks and selected municipalities for the estimation of box dimension.***

Stratum	Municipality-Communes	Density populated	Blocks M_i a	Municipalities Selected
1	C of Pendeli	1,23	235	
	C of Ekali	1,58	182	
	C of Elinico	1,90	200	*
	C of Voula	2,38	461	
	M of Glyfada	2,62	1088	
	M of Kifisia	3,58	1072	*
	M of Kamateron	3,75	735	
	C of Melissia	3,82	297	
	C of Psykhikon	3,92	156	
	C of Nea Pendeli	3,97	132	
	M of Khaidari	4,35	531	
2	C of Vrilissia	5,05	307	
	C of Papagou	5,42	287	
	M of Alimos	5,47	569	
	M of Metamorphosis	5,60	474	*
	M of Tavros	7,18	238	
	M of Khalandri	7,68	1102	
	M of Perama	8,07	402	
	M of Maroussi	8,16	926	
	M of Drapetsona	8,21	211	
	M of A. Paraskevi	9,08	550	
	M of Moskhato	9,53	277	*

	M of A. Anargyri	9,88	616	*
	M of Nea Liossia	9,99	1616	

Table 1. (continued)

	M of Petroupolis	10,31	491	
	M of Ilioupolis	10,37	931	
	M of Argyroupolis	10,47	291	
	M of Nea Philadelphia	10,78	431	
	M of Byron	11,43	646	
	M of Hymittos	11,55	215	
3	M of A. Dimitrios	11,95	1138	*
	M of Kholargos	13,61	243	
	M of Nea Ionia	13,72	721	
	M of Galatsi	14,07	471	
	M of Aegaleo	14,20	1054	*
	M of Peristeri	14,44	2240	
	M of Phaliron	14,52	506	
	M of Keratsini	14,66	1011	
	M of Piraeus	15,56	2141	*
	M of A. Varvara	15,80	393	
	M of Dafni	17,35	336	
	M of Korydalos	17,99	675	
	M of Zographou	18,71	534	
4	M of Nikea	19,11	1351	
	M of Nea Smyrni	19-80	526	*
	M of Athens	19,53	6069	*
	M of Kesariani	22,60	287	
	M of Kallithea	23,56	617	*

Source: National Statistical Service of Greece.

M = Municipality, C = Commune.

Table 2. ***Estimated cubes of side 3m required to cover the area of greater Athens and estimated box dimensions.***

Stratum h	Estimated Cubes of side 3m $\hat{Y}_h$	Estimated variance $v(\hat{Y}_h)$	Area in m^2 E_h *	Estimated Box Dimension D_h
1	19.890.682	$4{,}968831\times10^{13}$	139.980.000	2,03
2	20.965.476	$8{,}926456\times10^{11}$	82.685.000	2,10
3	24.807.209	$3{,}365931\times10^{12}$	86.788.000	2,12
4	41.018.656	$4{,}599505\times10^{14}$	89.233.000	2,18
Total	$\hat{Y}_{tot}$ =106.682.023	$5{,}138997\times10^{14}$	E_{tot} =398.686.000	$D_{tot}=2{,}10$

* ***Source***: National Statistical Service of Greece.