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Closed fractal interpolation surfaces

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Abstract

Based on the construction of bivariate fractal interpolation surfaces, we introduce closed spherical fractal interpolation surfaces. The interpolation takes place in spherical coordinates and with the transformation to Cartesian coordinates a closed surface arises. We give conditions for this construction to be valid and state some useful relations about the Hausdorff and the Box counting dimension of the closed surface.

Keywords: Fractal interpolation; Iterated function systems; Bivariate fractal interpolation surfaces; Closed fractal surfaces

1. Introduction

Fractal surfaces have gained widespread consideration in scientific areas such as metallurgy, physics, image processing, computer graphics, etc. Massopust [10,11] was the first who considered self-affine fractal interpolation surfaces (FISs) on triangular domains in the special case, where the interpolation points on the boundary of the domain are coplanar. This construction, however, lacks the flexibility which is most necessary in modelling complex surfaces. Thus, Geronimo and Hardin [8] generalized this construction to allow more general boundary data. Zhao [16], gave an even more general construction which involved affine and nonaffine FIS with arbitrarily selected contraction factors on triangular domains. Both constructions used consistent triangulations to overcome the problems arising when noncoplanar boundary data are used.

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A few years later Xie and Sun [14,15] used bivariate functions on rectangular grids with arbitrary contraction factors and without any condition on the boundary points. They used their studies to model rock surfaces. Their construction led to attractors that are not (in general) graphs of continuous functions as Dalla in [5] demonstrated. She used colinear boundary data and proved that in this case the attractor is a continuous surface. Malysz in [9] presented a construction that generalized Dalla's approach, using arbitrary boundary data, but he used the same contraction factors for each map of the iterated function system.

Nevertheless, all the constructions mentioned above lead to self-similar attractors, which means that any small part of the surface looks like the whole. A more general approach was introduced in [2], where recurrent iterated function systems (RIFS) were used to address the problem and the box-counting dimension of the constructed surfaces was computed. This method is flexible enough to allow its use in the approximation of any natural surface. In particular, in [3] this method is used in image compression.

In this paper we use the theory presented in [2] to construct closed fractal interpolation surfaces. We give the conditions needed for this construction and prove some theorems about their Hausdorff and box-counting dimension.

2. Fractal interpolation surfaces on rectangular grids

Let $X = [0,1] \times [0,p] \times \mathbb{R}$ and $\Delta = \{(x_i, y_j, z_{ij})\colon\ i = 0,1,\ldots,N;\ \ j = 0,1,\ldots,M\}$ be an interpolating set with $(N+1)\times(M+1)$ interpolation points such that $x_0 = 0$, $x_N = 1$, $x_i - x_{i-1} = \delta$, $y_0 = 0$, $y_N = p$, $y_i - y_{i-1} = \delta$, where $\delta \in (0,1)$ and $M, N \in \mathbb{N}$, $M = pN$, $p \in \mathbb{N}$. Furthermore, consider a subset of Δ, $Q = \{(\hat{x}_i, \hat{y}_j, \hat{z}_{ij})\colon\ i = 0,1,\ldots,K;\ \ j = 0,1,\ldots,L\}$, consisting of $(K+1)\times(L+1)$ points, such that $\hat{x}_0 = 0$, $\hat{x}_N = 1$, $\hat{x}_i - \hat{x}_{i-1} = \psi$, $\hat{y}_0 = 0$, $\hat{y}_N = p$, $\hat{y}_i - \hat{y}_{i-1} = \psi$, where $\psi = a\delta$, $a \in \mathbb{N}$ and $K, L \in \mathbb{N}$, $L = pK$. We will define a recurrent iterated function system (RIFS) associated with the set of data Δ and the set Q.

We will use mappings of the form

$$w_{ij}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{ij}x + b_{ij} \\ c_{ij}y + d_{ij} \\ e_{ij}x + f_{ij}y + g_{ij}xy + s_{ij}z + k_{ij} \end{pmatrix} = \begin{pmatrix} \phi_{ij}(x) \\ \psi_{ij}(y) \\ F_{ij}(x,y,z) \end{pmatrix} \tag{1}$$

which are called *bivariate* maps. Define the function T_{ij} by

$$T_{ij}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{ij}x + b_{ij} \\ c_{ij}y + d_{ij} \end{pmatrix} = \begin{pmatrix} \phi_{ij}(x) \\ \psi_{ij}(y) \end{pmatrix}, \tag{2}$$

so as $w_{ij} = (T_{ij}, F_{ij})$. If the vertical scaling factors obey $|s_{ij}| < 1$, then there is a metric d on X equivalent to the Euclidean metric, such that w_{ij} is a contraction with respect to d (i.e., $\exists \hat{s}_{ij}\colon\ 0 \leqslant \hat{s}_{ij} < 1\colon\ d(w_{ij}(\bar{x}), w_{ij}(\bar{y})) \leqslant \hat{s}_{ij} d(\bar{x}, \bar{y})$, $\forall \bar{x}, \bar{y} \in X$). One such metric d is given by (see [2,5]):

$$d\big((x_1,y_1,z_1),(x_2,y_2,z_2)\big) = |x_1 - x_2| + |y_1 - y_2| + \theta|z_1 - z_2|,$$

where

$$\theta = \min\left\{\frac{\min_{i,j}\{1 - a_{ij}\}}{\max_{i,j}\{2(|e_{ij}| + p|g_{ij}|)\}}, \frac{\min_{i,j}\{1 - c_{ij}\}}{\max_{i,j}\{2(|f_{ij}| + |g_{ij}|)\}}\right\}.$$

The number s_{ij} is often called the *contraction factor* or the *vertical scaling factor* of the map w_{ij}.

The interpolation points divide $[0,1]\times[0,p]$ into $N \cdot M$ rectangles $I_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $i = 1,\ldots,N$ and $j = 1,\ldots,M$, which we call *sections*, while the points of Q divide

$[0,1] \times [0,p]$ to $K \cdot L$ rectangles $J_{kl} = [\hat{x}_{k-1}, \hat{x}_k] \times [\hat{y}_{l-1}, \hat{y}_l]$, $k = 1, \dots, K$ and $l = 1, \dots, L$ which we simply call *intervals*. The number

$$a = \frac{\psi}{\delta} = \frac{N}{K},$$

the square of which expresses how many sections lie inside any interval, is an integer greater than one.

Furthermore, let $\mathbb{J}$ be a labelling map such that

$$\mathbb{J} : \{1, 2, \dots, N\} \times \{1, 2, \dots, M\} \to \{1, 2, \dots, K\} \times \{1, 2, \dots, L\}$$

with $\mathbb{J}(i, j) = (k, l)$. The mappings $w_{ij} : X \to X$ are constrained by the data according to

$$w_{ij}\begin{pmatrix} \hat{x}_{k-1} \\ \hat{y}_{l-1} \\ \hat{z}_{k-1,l-1} \end{pmatrix} = \begin{pmatrix} x_{i-1} \\ y_{j-1} \\ z_{i-1,j-1} \end{pmatrix}, \qquad w_{ij}\begin{pmatrix} \hat{x}_{k} \\ \hat{y}_{l-1} \\ \hat{z}_{k,l-1} \end{pmatrix} = \begin{pmatrix} x_{i} \\ y_{j-1} \\ z_{i,j-1} \end{pmatrix},$$

$$w_{ij}\begin{pmatrix} \hat{x}_{k-1} \\ \hat{y}_{l} \\ \hat{z}_{k-1,l} \end{pmatrix} = \begin{pmatrix} x_{i-1} \\ y_{j} \\ z_{i-1,j} \end{pmatrix} \quad \text{and} \quad w_{ij}\begin{pmatrix} \hat{x}_{l} \\ \hat{y}_{l} \\ \hat{z}_{kl} \end{pmatrix} = \begin{pmatrix} x_{i} \\ y_{j} \\ z_{i,j} \end{pmatrix} \tag{3}$$

for $i = 1, \dots, N$ and $j = 1, \dots, M$. The functions w_{ij} map the vertices of the interval $J_{kl} = J_{\mathbb{J}(i,j)}$ onto the vertices of the section I_{ij}. One can solve Eq. (3) and express $a_{ij}, b_{ij}, c_{ij}, d_{ij}, g_{ij}, e_{ij}, f_{ij}, k_{ij}$ in terms of the coordinates of the interpolation points and the contraction factor s_{ij} (see [2,5]).

Finally, let $\Phi(i, j) = (i - 1)M + j$, $i = 1, \dots, N$ and $j = 1, \dots, M$, (then $\Phi^{-1}(n) = ((n-1) \operatorname{div} M + 1, (n-1) \bmod M + 1)$, $n = 1, \dots, N \cdot M$) be an enumeration of the set $\{(i, j)\colon i = 1, \dots, N;\ j = 1, \dots, M\}$. The $NM \times NM$ stochastic matrix $P = (p_{nm})$ is defined by

$$p_{nm} = \begin{cases} \frac{1}{a^2}, & \text{if } I_{\Phi^{-1}(n)} \subseteq J_{\mathbb{J}(\Phi^{-1}(m))}, \\ 0, & \text{otherwise.} \end{cases}$$

The RIFS is described as the IFS $\{X, w_{1-N,1-M}\}$ together with the stochastic matrix P. It has a unique attractor $A = \lim_{k\to\infty} W^k(A_0)$ for every starting set $A_0 \in \mathcal{H}(X)$, where

$$W(A) = \bigcup_{i=1,j=1}^{N,M} w_{i,j}(A),$$

$W^k = W \circ W \circ \cdots \circ W$ and $\mathcal{H}(X)$ denotes the space whose points are the compact subsets of X, other than the empty set (see [1]). If this unique compact set A is the graph of a continuous function $f : [0,1] \times [0,p] \to \mathbb{R}$, then it is called a fractal interpolation surface (or FIS for short). The following proposition gives conditions that are needed to construct such a surface. Its proof and more general results may be found in [2].

Proposition 1. *With the same notation as above, assume that for every interval J_{kl}, $k = 1, 2, \dots, K$, $l = 1, 2, \dots, L$, the points of each of the sets*

$$\bigl\{(x_{(k-1)a} = \hat{x}_{k-1}, y_{(l-1)a+\nu}, z_{(k-1)a,(l-1)a+\nu})\colon \nu = 0, 1, 2, \dots, a\bigr\},$$
$$\bigl\{(x_{ka} = \hat{x}_{k}, y_{(l-1)a+\nu}, z_{ka,(l-1)a+\nu})\colon \nu = 0, 1, 2, \dots, a\bigr\},$$
$$\bigl\{(x_{(k-1)a+\nu}, y_{(l-1)a} = y_{l-1}, z_{(k-1)a+\nu,(l-1)a})\colon \nu = 0, 1, 2, \dots, a\bigr\},$$
$$\bigl\{(x_{(k-1)a+\nu}, y_{la} = y_{l}, z_{(k-1)a+\nu,la})\colon \nu = 0, 1, 2, \dots, a\bigr\}$$

are collinear. Then there exists a continuous function $f:[0,1]\times[0,p]\to\mathbb{R}$ *that interpolates the given data* $\Delta=\{(x_i,y_j,z_{ij}):\ i=1,2,\dots,N,\ \ j=1,2,\dots,M\}$ *and its graph* $\{(x,y,f(x,y)):\ (x,y)\in[0,1]\times[0,p]\}$ *is the attractor A of the RIFS.*

In this case one can compute the box-counting dimension of the attractor. We define the connection matrix of the respective RIFS as

$$C_{nm}=\begin{cases}1, & \text{if } p_{mn}>0,\\ 0, & \text{if } p_{mn}=0,\end{cases}$$

where $n,m=1,2,\dots,N\cdot M$. If the matrix P is irreducible, then C is irreducible and the following theorem applies.

Theorem 1. *Let the RIFS be defined as above with irreducible connection matrix* C*. Let* S *be the* $N\cdot M\times N\cdot M$ *diagonal matrix*

$$S=\operatorname{diag}\big(|s_{11}|,|s_{12}|,\dots,|s_{1M}|,|s_{21}|,|s_{22}|,\dots,|s_{2M}|,\dots,|s_{N1}|,|s_{N2}|,\dots,|s_{NM}|\big)$$

with $0<|s_{ij}|<1$*,* $i=1,\dots,N$*,* $j=1,\dots,M$*. Suppose that the attractor* A *of the RIFS is the graph of a continuous function f, that interpolates* Δ *and that the interpolation points of every interval are not* x*-collinear or are not* y*-collinear. The box counting dimension of* A *is given by*

$$D(A)=\begin{cases}1+\log_a\lambda, & \text{if } \lambda>a,\\ 2, & \text{if } \lambda\leqslant a,\end{cases}$$

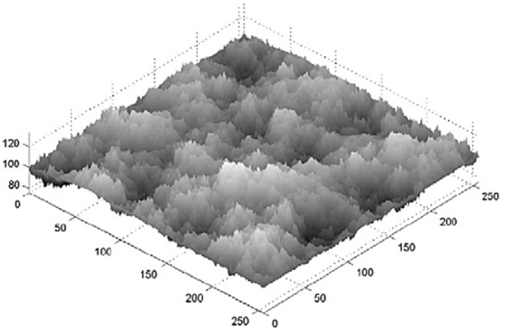
where $\lambda=\rho(SC)>0$*, the spectral radius of the irreducible matrix* $S\cdot C$*.*

We call the points of Δ *x-collinear* iff all the points with the same x coordinate are collinear. Similarly, we call the points of Δ *y-collinear* iff all the points with the same y coordinate are collinear. The proof of Theorem 1 is given in [2]. In Fig. 1 two FISs are shown. In both cases the interpolation points satisfy the conditions of Proposition 1.

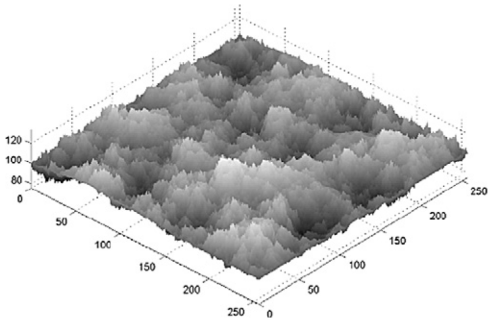
One can use this construction to approximate any given surface. We choose δ and ψ a priori, pick some points of the surface (forming a rectangular grid as described above) and construct the interpolation set Δ and the set Q. Then, for each section I_{ij} we compute the interval J_{kl} (where $(k,l)=\mathbb{J}(i,j)$) and the contraction factor s_{ij} (thus we form the map w_{ij}) that are "best mapped" (through w_{ij}) to I_{ij}. Storing the interpolation points and the parameters describing the RIFS we formed, we are able to reconstruct a fractal set which approximates the original surface. Details on this algorithm can be found in [3] where this method was used to approximate images with very good results (there the coordinate z gives the gray level at each point (x,y)). We were able to compress pictures by a factor of 70 without significant loss of the quality of the reconstructed (fractal) image.

3. Parameter identification problem

In many cases one must ensure that the attractor of an IFS or RIFS is contained in a given rectangle R. For example, in image compression, the surface should not take negative values. In [4] and [13] this problem is examined in the case of the affine fractal interpolation functions. Here, we give conditions on the contraction factors ensuring that the attractor of the RIFS defined in Section 2 is contained in R.



(a)



(b)

Fig. 1. Two fractal surfaces that interpolate given points. In both cases 9×9 interpolation points were used.

Theorem 2. *Consider the RIFS defined in Section* 2*. The graph of the attractor of this RIFS remains within a given parallelepiped* $R = [0,1] \times [0,p] \times [a,b]$ $(\Delta \subset R)$ *if the contraction factors obey*

$$s_{ij}^{\min} \leqslant s_{ij} \leqslant s_{ij}^{\max},$$

where

$$s_{ij}^{\max} = \min\left\{\begin{array}{cccc} \frac{b-z_{i-1,j-1}}{b-\hat{z}_{k-1,l-1}}, & \frac{b-z_{i-1,j}}{b-\hat{z}_{k-1,l}}, & \frac{b-z_{i,j-1}}{b-\hat{z}_{k,l-1}} & \frac{b-z_{i,j}}{b-\hat{z}_{k,l}}, \\ \frac{a-z_{i-1,j-1}}{a-\hat{z}_{k-1,l-1}}, & \frac{a-z_{i-1,j}}{a-\hat{z}_{k-1,l}}, & \frac{a-z_{i,j-1}}{a-\hat{z}_{k,l-1}}, & \frac{a-z_{i,j}}{a-\hat{z}_{k,l}} \end{array}\right\},$$

$$s_{ij}^{\min} = \max\left\{\begin{array}{cccc} \frac{b-z_{i-1,j-1}}{a-\hat{z}_{k-1,l-1}}, & \frac{b-z_{i-1,j}}{a-\hat{z}_{k-1,l}}, & \frac{b-z_{i,j-1}}{a-\hat{z}_{k,l-1}}, & \frac{b-z_{i,j}}{a-\hat{z}_{k,l}}, \\ \frac{a-z_{i-1,j-1}}{b-\hat{z}_{k-1,l-1}}, & \frac{a-z_{i-1,j}}{b-\hat{z}_{k-1,l}}, & \frac{a-z_{i,j-1}}{b-\hat{z}_{k,l-1}}, & \frac{a-z_{i,j}}{b-\hat{z}_{k,l}} \end{array}\right\},$$

and $(k,l) = \mathbb{J}(i,j)$*,* $i = 1,2,\ldots,N$*,* $j = 1,2,\ldots,M$*, for all nonzero denominators.*

Proof. The proof is similar to the one found in [4]. Consider the function

$$F_{ij}(x,y,z) = e_{ij}x + f_{ij}y + g_{ij}xy + s_{ij}z + k_{ij}$$

defined on the parallelepiped $R_{kl} = [x_{k-1}, x_k] \times [y_{l-1}, y_l] \times [a,b]$, where $(k,l) = \mathbb{J}(i,j)$. If one fixes two of the variables of this function and considers the function defined on a closed subset of $\mathbb{R}$, the graph will be a line segment. Therefore, F_{ij} attains its maximum and minimum values at the vertices of R_{kl}. The eight vertices of R_{kl} may be written as $(x_{k+\mu}, y_{l+\nu}, a)$ or $(x_{k+\mu}, y_{l+\nu}, b)$, $\mu, \nu = -1, 0$. We select the contraction factors s_{ij} such that

$$\left.\begin{array}{l} a \leqslant F_{ij}(x_{k+\mu}, y_{l+\nu}, a) \leqslant b \\ a \leqslant F_{ij}(x_{k+\mu}, y_{l+\nu}, b) \leqslant b \end{array}\right\}$$

$$\Rightarrow \quad \left.\begin{array}{l} a \leqslant e_{ij}x_{k+\mu} + f_{ij}y_{l+\nu} + g_{ij}x_{k+\mu}y_{l+\nu} + s_{ij}a + k_{ij} \leqslant b \\ a \leqslant e_{ij}x_{k+\mu} + f_{ij}y_{l+\nu} + g_{ij}x_{k+\mu}y_{l+\nu} + s_{ij}b + k_{ij} \leqslant b \end{array}\right\}$$

$$\Rightarrow \quad \left.\begin{array}{l} a \leqslant e_{ij}x_{k+\mu} + f_{ij}y_{l+\nu} + g_{ij}x_{k+\mu}y_{l+\nu} + s_{ij}a + k_{ij} \\ \qquad + s_{ij}\hat{z}_{k+\mu,l+\nu} - s_{ij}\hat{z}_{k+\mu,l+\nu} \leqslant b \\ a \leqslant e_{ij}x_{k+\mu} + f_{ij}y_{l+\nu} + g_{ij}x_{k+\mu}y_{l+\nu} + s_{ij}b + k_{ij} \\ \qquad + s_{ij}\hat{z}_{k+\mu,l+\nu} - s_{ij}\hat{z}_{k+\mu,l+\nu} \leqslant b \end{array}\right\}$$

$$\overset{\text{by}(3)}{\Rightarrow} \quad \left.\begin{array}{l} a \leqslant z_{i+\mu,j+\nu} + s_{ij}a - s_{ij}\hat{z}_{k+\mu,l+\nu} \leqslant b \\ a \leqslant z_{i+\mu,j+\nu} + s_{ij}b - s_{ij}\hat{z}_{k+\mu,l+\nu} \leqslant b \end{array}\right\}$$

$$\Rightarrow \quad \left.\begin{array}{l} a - z_{i+\mu,j+\nu} \leqslant s_{ij}(a - \hat{z}_{k+\mu,l+\nu}) \leqslant b - z_{i+\mu,j+\nu} \\ a - z_{i+\mu,j+\nu} \leqslant s_{ij}(b - \hat{z}_{k+\mu,l+\nu}) \leqslant b - z_{i+\mu,j+\nu} \end{array}\right\}$$

$$\Rightarrow \quad \left.\begin{array}{l} \frac{a-z_{i+\mu,j+\nu}}{a-\hat{z}_{k+\mu,l+\nu}} \geqslant s_{ij} \geqslant \frac{b-z_{i+\mu,j+\nu}}{a-\hat{z}_{k+\mu,l+\nu}} \\ \frac{a-z_{i+\mu,j+\nu}}{b-\hat{z}_{k+\mu,l+\nu}} \leqslant s_{ij} \leqslant \frac{b-z_{i+\mu,j+\nu}}{b-\hat{z}_{k+\mu,l+\nu}} \end{array}\right\},$$

for $\mu, \nu = -1, 0$ and nonzero denominators, thus we have the result. □

4. Closed spherical fractal interpolation surfaces

A well-known (and in many areas useful) system of coordinates are the spherical coordinates. This system is ideal for describing positions on a sphere or spheroid. We let

$$0 \leqslant \theta < 2\pi, \quad -\frac{\pi}{2} \leqslant \phi \leqslant \frac{\pi}{2}, \quad r > 0,$$

and define $\boldsymbol{g} = (g_1, g_2, g_3)$ to be the transformation from spherical coordinates to Cartesian coordinates, where

$$x = g_1(\theta, \phi, r) = r\cos\phi\cos\theta,$$
$$y = g_2(\theta, \phi, r) = r\cos\phi\sin\theta,$$
$$z = g_3(\theta, \phi, r) = r\sin\phi.$$

We can construct a closed fractal surface using the next theorem.

Theorem 3. *Consider a set of equidistant interpolation points*

$$\Delta_S = \{(\theta_i, \phi_j, r_{ij}):\ i = 0, 1, \ldots, N;\ j = 0, 1, \ldots, M\}$$

given in spherical coordinates, such that $\theta_0 = 0$, $\theta_N = 2\pi$, $\theta_i - \theta_{i-1} = \delta$, $\phi_0 = -\frac{\pi}{2}$, $\phi_M = \frac{\pi}{2}$, $\phi_j - \phi_{j-1} = \delta$. *Consider, also,* $Q_S \subset \Delta_S = \{(\hat{\theta}_k, \hat{\phi}_l, \hat{r}_{kl}):\ k = 0, 1, \ldots, K;\ l = 0, 1, \ldots, L\}$ *such that* $\hat{\theta}_0 = 0$, $\hat{\theta}_N = 2\pi$, $\hat{\theta}_i - \hat{\theta}_{i-1} = \psi$, $\hat{\phi}_0 = -\frac{\pi}{2}$, $\hat{\phi}_M = \frac{\pi}{2}$, $\hat{\phi}_j - \hat{\phi}_{j-1} = \psi$ *and a labelling map* $\mathbb{J}$ *as defined in Section* 2. *Let G be the graph of the function* $r(\theta, \phi)$ *which arises as the attractor of this RIFS. Then* $\boldsymbol{g}(G)$ *is a continuous closed fractal interpolation surface if and only if the following conditions apply*:

(1) $r_{i,0} = r_{i,M} = R,\ i = 0, 1, \ldots, N.$
(2) $r_{0,j} = r_{N,j},\ j = 0, 1, \ldots, M.$
(3) *The contraction factors are chosen such that* $G \subset [0, 2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times [\epsilon, +\infty)$, *for given* $\epsilon > 0$.

Proof. The proof is straightforward. That G is a continuous surface has been shown in [2]. The first condition ensures that the boundaries of this surface (for $\phi = -\frac{\pi}{2}$ and $\phi = \frac{\pi}{2}$) are line segments, parallel to the $\theta\phi$ plane. This is necessary, because $\boldsymbol{g}$ maps these boundaries to two single points. Every point with $\phi = -\frac{\pi}{2}$ is mapped to the south pole and every point with $\phi = \frac{\pi}{2}$ is mapped to the north pole. Condition (2) ensures (see [2]) that the other two boundaries will be symmetric, with respect to the plane $\theta = \pi$, thus they will fit together after the application of $\boldsymbol{g}$. The last condition ensures that $r > 0$. Considering that $\boldsymbol{g}$ is a continuous function, one can easily get that $\boldsymbol{g}(G)$ is a continuous closed fractal interpolation surface. □

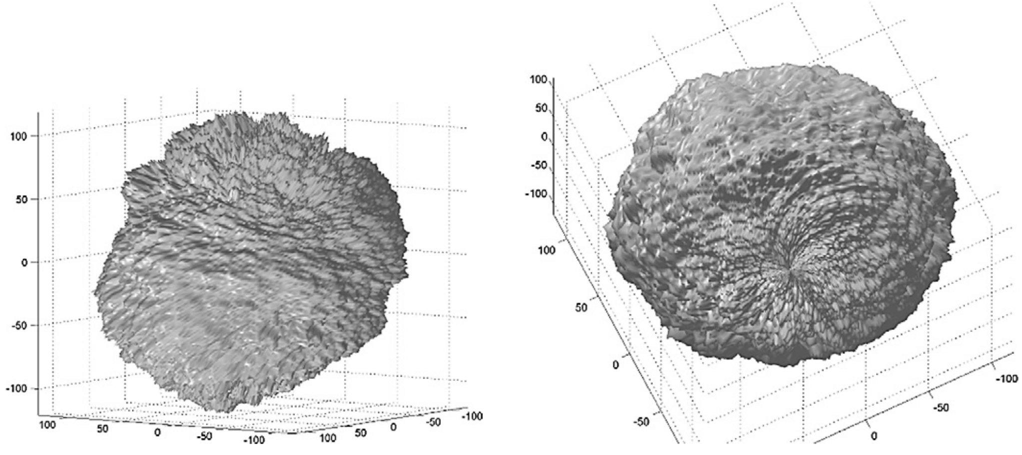


Fig. 2. For this closed surface 5×5 interpolation points were used. The interpolation points and the other parameters of the RIFS satisfy the conditions of Theorem 3.

Corollary 1. *Consider the RIFS defined in Theorem* 3 *satisfying all the specified conditions. The application of the transformation* $\boldsymbol{h} = (h_1, h_2, h_3)$, *where*

$$x = h_1(\theta, \phi, r) = \hat{a} \cdot r \cos\phi \cos\theta,$$
$$y = h_2(\theta, \phi, r) = \hat{b} \cdot r \cos\phi \sin\theta,$$
$$z = h_3(\theta, \phi, r) = \hat{c} \cdot r \sin\phi,$$

with $\hat{a}, \hat{b}, \hat{c} > 0$, *will yield a continuous closed fractal interpolation surface.*

In Figs. 2–5 we give some examples of closed FISs using either $\boldsymbol{g}$ or $\boldsymbol{h}$ as indicated.

5. Dimension of closed spherical FIS

In this section we will prove that the new closed spherical FIS $\hat{G} = \boldsymbol{g}(G)$ has the same Hausdorff dimension ($\dim_{\mathrm{H}}$) as G. We also prove some useful inequalities for the box-counting dimension ($\dim_{\mathrm{B}}$). First we need the following lemma.

Lemma 1. *Let* $f : \mathbb{R}^n \to \mathbb{R}^m$ *be a Lipschitz continuous function and* $G \subset \mathbb{R}^n$; *then*

$$\dim_{\mathrm{H}} f(G) \leqslant \dim_{\mathrm{H}} G, \qquad \dim_{\mathrm{B}} f(G) \leqslant \dim_{\mathrm{B}} G.$$

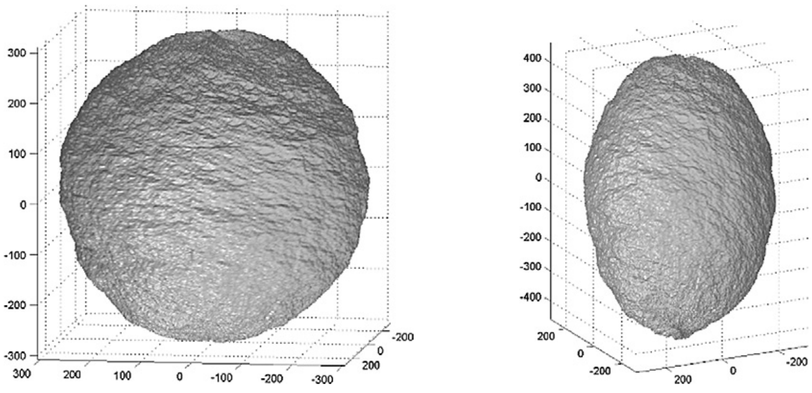
(See, for example, [7, p. 30].)

Lemma 2. *The Hausdorff dimension is countably stable, that is*

$$\dim_{\mathrm{H}} \bigcup_{i=1}^{\infty} E_i = \sup_{1 \leqslant i < \infty} \dim_{\mathrm{H}} E_i,$$

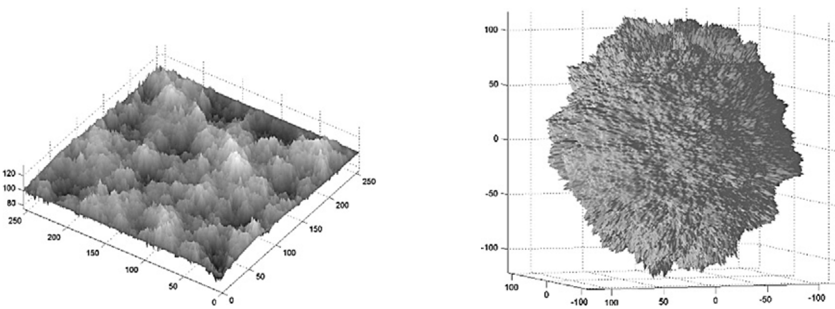
where $E_i \subset \mathbb{R}^n$.

(See, for example, [6, p. 24] or [12, p. 59].)



(a) (b)

Fig. 3. The surface of Fig. 2(a) (in spherical coordinates) is transformed through $\boldsymbol{g}$ (in (a)) and $\boldsymbol{h}$ (in (b)) (in Cartesian coordinates) with $\hat{a} = 1$, $\hat{b} = 1$, $\hat{c} = 1.5$ into a closed fractal interpolation surface.



(a) (b)

Fig. 4. The fractal surface (a) (spherical coordinates) interpolates 9×9 points and it is transformed through $\boldsymbol{g}$ into a closed fractal interpolation surface (b) (Cartesian coordinates).

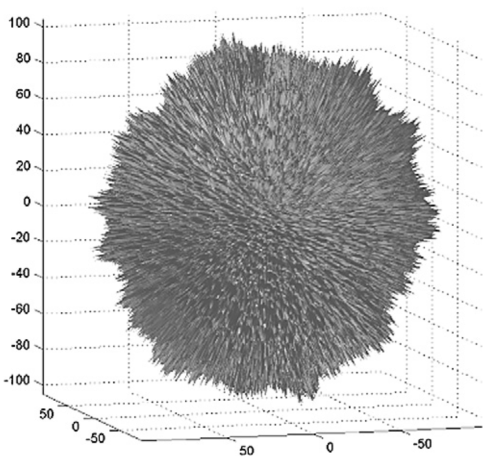


Fig. 5. Another closed fractal interpolation surface (9×9 interpolation points).

Theorem 4. *Let G and $\hat{G} = \boldsymbol{g}(G)$ denote the graph of the FIS and the closed spherical FIS, respectively, as described in Theorem* 3. *Then*

$$\dim_{\mathrm{H}} G = \dim_{\mathrm{H}} \hat{G}.$$

Proof. Suppose that $r(\theta,\phi)$, $(\theta,\phi) \in [0,2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the continuous function with graph G and a, b are its minimum and maximum values. We define $B_n = [0,2\pi] \times [-\frac{\pi}{2} + \frac{1}{n}, \frac{\pi}{2} - \frac{1}{n}] \times [a,b]$ and

$$\boldsymbol{g}_n : B_n \to \mathbb{R}^3$$

the restriction of $\boldsymbol{g}$ on B_n, which is Lipschitz continuous. Now, let $\hat{G}_n = \boldsymbol{g}_n(G)$ and $\hat{G}_n^{(\nu)}$ be the intersection of $\boldsymbol{g}_n(G)$ with the νth octant, $\nu = 1, 2, \ldots, 8$. From the construction of G (condition 3 of Theorem 3), we have that $x + y + z \geqslant a$, for $(x,y,z) \in \hat{G}_n^{(1)}$, thus the convex hull $S_n = \operatorname{conv}(\hat{G}_n^{(1)})$ is a convex and compact set which does not contain the origin. The function $\boldsymbol{g}_n^{-1}$ is well defined on S_n and it has all the partial derivatives continuous (thus bounded) on S_n. Using the Mean Value Theorem we deduce that $\boldsymbol{g}_n^{-1}$ is a Lipschitz continuous function also. Thus (Lemma 1),

$$\dim_{\mathrm{H}} \hat{G}_n^{(1)} = \dim_{\mathrm{H}}\left(G \cap \left(\left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2} - \frac{1}{n}\right) \times [a,b]\right)\right).$$

Proceeding similarly we attain analogous relations for the other octants. Hence,

$$\dim_{\mathrm{H}} \hat{G}_n = \dim_{\mathrm{H}}\bigl(\boldsymbol{g}_n(G)\bigr) = \dim_H(G \cap B_n). \tag{4}$$

Therefore,

$$\begin{aligned}
\dim_{\mathrm{H}} G &= \dim_{\mathrm{H}}\left(\bigcup_{n=1}^{\infty}(G \cap B_n) \cup \left\{r\left(\theta, \pm\frac{\pi}{2}\right),\ \theta \in [0,2\pi]\right\}\right) \\
&= \dim_{\mathrm{H}}\left(\bigcup_{n=1}^{\infty}(G \cap B_n)\right) \\
&= \sup\bigl\{\dim_{\mathrm{H}}(G \cap B_n),\ n \in \mathbb{N}\bigr\} \quad \text{(by Lemma 2)} \\
&= \sup\bigl\{\dim_{\mathrm{H}} \boldsymbol{g}_n(G),\ n \in \mathbb{N}\bigr\} \quad \bigl(\text{by (4)}\bigr) \\
&= \dim_{\mathrm{H}}\left(\bigcup_{n=1}^{\infty} \boldsymbol{g}_n(G)\right) \quad \text{(by Lemma2)} \\
&= \dim_{\mathrm{H}}\left(\bigcup_{n=1}^{\infty} \hat{G}_n \cup \left\{\left(0,0,r\left(0,\frac{\pi}{2}\right)\right), \left(0,0,r\left(0,-\frac{\pi}{2}\right)\right)\right\}\right) \\
&= \dim_{\mathrm{H}} \hat{G}. \qquad \square
\end{aligned}$$

Remark 1. Any other "countably stable" dimension (like the packing dimension, see [12, p. 81]) satisfies, also, Theorem 4.

Remark 2. For the box-counting dimension, which is not countably stable, we have that

$$\dim_{\mathrm{B}}(G \cap B_n) = \dim_{\mathrm{B}} \hat{G}_n, \quad \text{for any } n \in \mathbb{N} \quad \text{and}$$
$$\sup\bigl\{\dim_{\mathrm{B}}(G \cap B_n):\ n \in \mathbb{N}\bigr\} \leqslant \dim_{\mathrm{B}} \hat{G} = \dim_{\mathrm{B}} \boldsymbol{g}(G) \leqslant \dim_{\mathrm{B}} G.$$

6. Approximating closed surfaces with FIS

In the above sections we described in detail the construction of closed spherical FISs that emerge from FIS through a change of coordinates. Here, we address the problem of approximation of a given closed surface (e.g., the surface of planets, comets, rocks).

We consider a natural closed surface $\hat{G}$, which is described by a parametrization $r = r(\theta, \phi)$ in spherical coordinates and we write $r_S = r(\theta, -\frac{\pi}{2})$ and $r_N = r(\theta, \frac{\pi}{2})$, $\forall\theta \in [0, 2\pi]$, for the south and the north pole of the surface. Thus we have a continuous surface defined on $[0, 2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times (0, +\infty)$, which we can approximate using fractal interpolation as described in Section 2.

We construct a grid of $N \times M$ interpolation points as follows:

$$\begin{aligned}
&\theta_0 = 0, \qquad \theta_N = 2\pi, \qquad \theta_i - \theta_{i-1} = \delta, \quad \text{for } i = 1, 2, \ldots, N, \\
&\phi_0 = -\frac{\pi}{2}, \qquad \phi_M = \frac{\pi}{2}, \qquad \phi_j - \phi_{j-1} = \psi, \quad \text{for } j = 1, 2, \ldots, M, \\
&r_{i,j} = r(\theta_i, \phi_j), \quad \text{for } i = 0, 1, \ldots, N-1;\ j = 1, 2, \ldots, M-1, \\
&r_{i,0} = r_S, \quad i = 0, 1, \ldots, N, \\
&r_{i,M} = r_N, \quad i = 0, 1, \ldots, N, \quad \text{and} \\
&r_{N,j} = r_{0,j}, \quad j = 1, 2, \ldots, M-1.
\end{aligned}$$

Some minor modifications of the algorithm described in [3] are needed so that the emerging RIFS will obey the conditions of Theorem 3. In addition, the contraction factors must satisfy the conditions of Theorem 2, so that the surface remains within a given rectangle where $r > 0$. Therefore the attractor G of the RIFS will be the graph of a function that interpolates the above points and $\boldsymbol{g}(G)$ will approximate the original surface $\hat{G}$.

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