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BIVARIATE FRACTAL INTERPOLATION FUNCTIONS ON GRIDS

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Abstract

In this paper, a method of construction of fractal interpolation functions (FIF) on random grids in $\mathbb{R}^2$ is examined.

1. INTRODUCTION

In Ref. 1, M. F. Barnsley based on a theorem of J. E. Hutchinson in Ref. 2, introduced a method of construction of fractal interpolation functions (FIF). For this, he used the theory of Iterated Function Systems, IFS for short. These are continuous functions $f : [a, b] \to \mathbb{R}$, the graph of which interpolates given data $\{(x_i, y_i) : i = 0, \dots, N\} \subset [a, b] \times \mathbb{R}$.

P. R. Massopust in Ref. 3 was the first to put forward the construction of fractal surfaces via IFS. These are continuous functions $f : D \subset \mathbb{R}^2 \to R$, which interpolate given points $\{(x_0, y_0, z_{00}), \dots, (x_N, y_M, z_{NM})\}$ in $\mathbb{R}^3$. He considered the case in which D is a triangular domain and the interpolation points on the boundary of D are co-planar. A slightly more general construction was examined by Geronimo and Hardin in Ref. 4. They considered more general cases of the domain D and boundary data. In particular, they examine the case when D is a polygonal region and the interpolating points are arbitrary. Also in Ref. 5, D. P. Hardin and P. R. Massopust investigate multivariate FIF $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$.

In this paper, we give a method of construction of bivariate fractal interpolation functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$. Firstly, we investigate the case where the interpolation points on each edge of $[a, b] \times [c, d]$ are colinear and secondly we consider the general case. These results also improve and correct a construction quoted in the paper by H. Xie and H. Sun in Ref. 6.

2. CONSTRUCTION OF FRACTAL INTERPOLATION SURFACES

2.1 Special Case

In this section, we construct FIF $f : [a, b] \times [c, d] \to \mathbb{R}$, when the data obeys some special rules.

Let $I = [a, b]$, $J = [c, d]$ and $a = x_0 < x_1 < \cdots < x_N = b$, $c = y_0 < y_1 < \cdots < y_M = d$. Consider the data

$$\{(x_n, y_m, z_{nm}) : n=0, 1, \ldots, N,\ m=0, 1, \ldots, M\}$$

where the interpolation points are such that each of the sets

$$\begin{aligned}&\{(x_0, y_m, z_{0m}) : m = 0, 1, \ldots, M\}\\ &\{(x_N, y_m, z_{Nm}) : m = 0, 1, \ldots, M\}\\ &\{(x_n, y_0, z_{n0}) : n = 0, 1, \ldots, N\}\\ &\{(x_n, y_M, z_{nM}) : n = 0, 1, \ldots, N\}\end{aligned} \qquad (*)$$

is colinear.

We define $w_{nm} : I \times J \times \mathbb{R} \to \mathbb{R}^3$

$$w_{nm}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_n x + b_n \\ c_m y + d_m \\ e_{nm}x + f_{nm}y + g_{nm}xy + a_{nm}z + k_{nm} \end{pmatrix} = \begin{pmatrix} \phi_n(x) \\ \psi_m(y) \\ F_{nm}(x, y, z) \end{pmatrix}$$

where the constants a_n, b_n, c_m, d_m, e_{nm}, f_{nm}, g_{nm}, k_{nm} are defined by the equations

$$w_{nm}\begin{pmatrix} x_0 \\ y_0 \\ z_{00} \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ y_{m-1} \\ z_{n-1,\, m-1} \end{pmatrix}$$

$$w_{nm}\begin{pmatrix} x_N \\ y_0 \\ z_{N0} \end{pmatrix} = \begin{pmatrix} x_n \\ y_{m-1} \\ z_{n,\, m-1} \end{pmatrix}$$

$$w_{nm}\begin{pmatrix} x_0 \\ y_M \\ z_{0M} \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ y_m \\ z_{n-1,\, m} \end{pmatrix}$$

$$w_{nm}\begin{pmatrix} x_N \\ y_M \\ z_{NM} \end{pmatrix} = \begin{pmatrix} x_n \\ y_m \\ z_{n,m} \end{pmatrix}.$$

Analytically, we obtain

$$a_n x_0 + b_n = x_{n-1}, \quad n \in \{1, \ldots, N\}$$

$$a_n x_N + b_n = x_n, \quad n \in \{1, \ldots, N\}$$

$$c_m y_0 + b_m = y_{m-1}, \quad m \in \{1, \ldots, M\}$$

$$c_m y_M + b_m = y_m, \quad m \in \{1, \ldots, M\}$$

from which we yield

$$a_n = \frac{x_n - x_{n-1}}{x_N - x_0}, \quad b_n = \frac{x_{n-1}x_N - x_n x_0}{x_N - x_0}$$

$$c_m = \frac{y_m - y_{m-1}}{y_M - y_0}, \quad d_m = \frac{y_{m-1}y_M - y_m y_0}{y_M - y_0}.$$

Thus

$$\phi_n(x) = \frac{x_n - x_{n-1}}{x_N - x_0}(x - x_0) + x_{n-1}$$

$$\psi_m(y) = \frac{y_m - y_{m-1}}{y_M - y_0}(y - y_0) + y_{m-1}.$$

In the same manner, from the preceding equations, we have

$$e_{nm}x_0 + f_{nm}y_0 + g_{nm}x_0y_0 + a_{nm}z_{00} + k_{nm} = z_{n-1,\, m-1}$$

$$e_{nm}x_N + f_{nm}y_0 + g_{nm}x_Ny_0 + a_{nm}z_{N0} + k_{nm} = z_{n,\, m-1}$$

$$e_{nm}x_0 + f_{nm}y_M + g_{nm}x_0y_M + a_{nm}z_{0M} + k_{nm} = z_{n-1,\, m}$$

$$e_{nm}x_N + f_{nm}y_M + g_{nm}x_Ny_M + a_{nm}z_{NM} + k_{nm} = z_{n,\, m}$$

and thus,

$$g_{nm} = \frac{z_{nm} + z_{n-1,\, m-1} - z_{n-1,\, m} - z_{n,\, m-1} - a_{nm}(z_{NM} + z_{00} - z_{0M} - z_{N0})}{(y_M - y_0)(x_N - x_0)}$$

$$e_{nm} = \frac{z_{n-1,\,m-1} - z_{n,\,m-1} - a_{nm}(z_{00} - z_{N0}) - g_{nm}y_0(x_0 - x_N)}{x_0 - x_N}$$

$$f_{nm} = \frac{z_{n-1,\,m-1} - z_{n-1,\,m} - a_{nm}(z_{00} - z_{0M}) - g_{nm}x_0(y_0 - y_M)}{y_0 - y_M}$$

$$k_{nm} = z_{nm} - e_{nm}x_N - f_{nm}y_M - a_{nm}z_{NM} - g_{nm}x_Ny_M\,.$$

The constants a_{nm} are called vertical scaling factors. Let $0 < |a_{nm}| < 1$. Now define the IFS

$$\{[a,\,b] \times [c,\,d] \times \mathbb{R},\, w_{nm} : n = 1, \ldots,\, N \quad m = 1, \ldots,\, M\}\,.$$

We now establish that the above IFS has a unique attractor G, which is the graph of a continuous function $f : [a,\,b] \times [c,\,d] \to \mathbb{R}$ that interpolates the given data. We adopt the notation and terminology of Refs. 1, 6–8. We have the following.

Proposition 2.1. *Consider the IFS defined above, associated with the set of data* $\{(x_n,\,y_m,\,z_{nm}) : n = 0, \ldots,\, N,\, m = 0, \ldots,\, M\}$. *Assume that the vertical scaling factors* a_{nm} *are given and are such that* $0 < |a_{nm}| < 1$. *Then, there exists a metric* ρ *on* $[a,\,b]\times[c,\,d]\times\mathbb{R}$, *equivalent to the Euclidean metric, such that the IFS is hyperbolic.*

There exists a unique, non-empty, compact set $G \subset \mathbb{R}^3$, *such that*

$$G = \bigcup_{n=1}^{N} \bigcup_{m=1}^{M} w_{nm}(G)\,.$$

Proof. Let $(x,\,y,\,z),\,(x',\,y',\,z') \in [a,\,b]\times[c,\,d]\times\mathbb{R}$. We may assume without loss of generality that a, $c \geq 0$. Let

$$\rho((x,\,y,\,z),\,(x',\,y',\,z')) = |x - x'| + |y - y'| + \theta|z - z'|$$

where θ is

$$\theta = \min\left\{\frac{\min_n\{1 - a_n\}}{\max_{n,\,m}\{2(|e_{nm}| + d|g_{nm}|)\}},\, \frac{\min_n\{1 - c_m\}}{\max_{n,\,m}\{2(|f_{nm}| + b|g_{nm}|)\}}\right\}.$$

The fact that ρ is a metric equivalent to the Euclidean metric, can be easily established.

$$\begin{aligned}
&\rho(w_{nm}(x,\,y,\,z),\, w_{nm}(x',\,y',\,z'))\\
&\quad = |\phi_n(x) - \phi_n(x')| + |\psi_m(y) - \psi_m(y')| + \theta|F_{nm}(x,\,y,\,z) - F_{nm}(x',\,y',\,z')|\\
&\quad = a_n|x - x'| + c_m|y - y'| + \theta|e_{nm}(x - x') + f_{nm}(y - y') + g_{nm}(xy - x'y') + a_{nm}(z - z')|\\
&\quad = a_n|x - x'| + c_m|y - y'| + \theta|e_{nm}(x - x') + f_{nm}(y - y') + g_{nm}[x(y - y') - y'(x - x')] + a_{nm}(z - z')|\\
&\quad \leq a_n|x - x'| + c_m|y - y'| + \theta|e_{nm}||x - x'|\\
&\qquad + \theta|f_{nm}||y - y'| + \theta b|g_{nm}||y - y'| + \theta d|g_{nm}||x - x'| + \theta|a_{nm}||z - z'|\\
&\quad \leq \left(a_n + \frac{1 - a_n}{2(|e_{nm}| + d|g_{nm}|)}\,(|e_{nm}| + d|g_{nm}|)\right)|x - x'|\\
&\qquad + \left(c_m + \frac{1 - c_m}{2(|f_{nm}| + b|g_{nm}|)}\,(|f_{nm}| + b|g_{nm}|)\right)|y - y'| + \theta|a_{nm}||z - z'|\\
&\quad = \frac{1 + a_n}{2}\,|x - x'| + \frac{1 + c_m}{2}\,|y - y'| + \theta|a_{nm}||z - z'|\,.
\end{aligned}$$

Now, set

$$\alpha = \max_n\left\{\frac{1 + a_n}{2}\right\} < 1\,, \quad \gamma = \max_m\left\{\frac{1 + c_m}{2}\right\} < 1\,, \quad \delta = \max_{n,\,m}\{|a_{nm}|\} < 1\,.$$

Then, we conclude

$$\begin{aligned}\rho(w_{nm}(x, y, z), w_{nm}(x', y', z')) &\le \alpha|x - x'| + \gamma|y - y'| + \theta\delta|z - z'| \\ &\le \max\{\alpha, \gamma, \delta\}(|x - x'| + |y - y'| + \theta|z - z'|) \\ &= s \cdot \rho((x, y, z), (x', y', z')) .\end{aligned}$$

Therefore, w_{nm}, $n = 1, \dots, N$, $m = 1, \dots, M$ are contraction mappings. Thus, there is a unique, non-empty compact set $G \subset \mathbb{R}^3$, such that $G = \bigcup_{n=1}^{N} \bigcup_{m=1}^{M} w_{nm}(G)$. □

Proposition 2.2. *Let* $\{[a, b] \times [c, d] \times \mathbb{R}, w_{nm} : n = 1, \dots, N, m = 1, \dots, M\}$ *denote the IFS defined above and* G *its attractor. Assume that the vertical scaling factors* a_{nm} *obey* $0 < |a_{nm}| < 1$, $n = 1, \dots, N$, $m = 1, \dots, M$, *so that the IFS is hyperbolic. Then there exists a continuous function* $f : [a, b] \times [c, d] \to \mathbb{R}$ *that interpolates the given data* $\{(x_n, y_m, z_{nm}) : n = 1, \dots, N, m = 1, \dots, M\}$ *and its graph* $\tilde{G} = \{(x, y, f(x, y)) : (x, y) \in [a, b] \times [c, d]\} = G$.

Proof. Let $\mathcal{F}$ be the set of continuous functions defined on $[a, b] \times [c, d]$, such that

$$f(x_0, (1 - \lambda)y_0 + \lambda y_M) = (1 - \lambda)z_{00} + \lambda z_{0M}$$

$$f(x_N, (1 - \lambda)y_0 + \lambda y_M) = (1 - \lambda)z_{N0} + \lambda z_{NM}$$

$$f((1 - \lambda)x_0 + \lambda x_N, y_0) = (1 - \lambda)z_{00} + \lambda z_{N0}$$

$$f((1 - \lambda)x_0 + \lambda x_N, y_M) = (1 - \lambda)z_{0M} + \lambda z_{NM}$$

for $\lambda \in [0, 1]$.

Then, $\mathcal{F}$ is a closed subset of $(C([a, b] \times [c, d]), \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ denotes the sup-norm. Hence, $\mathcal{F}$ is a complete metric space.

We define $T : \mathcal{F} \to \mathcal{F}$ such that,

$$\begin{aligned}Tf(x, y) = e_{nm}\phi_n^{-1}(x) &+ f_{nm}\psi_m^{-1}(y) \\ &+ g_{nm}\phi_n^{-1}(x)\psi_m^{-1}(y) \\ &+ a_{nm}f(\phi_n^{-1}(x), \psi_m^{-1}(y)) + k_{nm}\end{aligned}$$

$(x, y) \in I_n \times J_m = [x_{n-1}, x_n] \times [y_{m-1}, y_m]$. We will verify that T is well-defined.

If $(x, y) \in (x_{n-1}, x_n) \times (y_{m-1}, y_m)$, there is nothing to prove.

If (x, y) is on the boundary $\partial(I_n \times J_m)$ of $I_n \times J_m$, then at least one of the following holds true:

(i) $(x, y) \in \partial(I_n \times J_m) \cap \partial(I_{n+1} \times J_m)$,
(ii) $(x, y) \in \partial(I_n \times J_m) \cap \partial(I_n \times J_{m+1})$,
(iii) $(x, y) \in \partial(I_n \times J_m) \cap \partial(I_{n-1} \times J_m)$.
(iv) $(x, y) \in \partial(I_n \times J_m) \cap \partial(I_n \times J_{m-1})$.

Consider case (i). (The others are dealt with similarly.) Then $(x, y) = (x_n, (1 - \lambda)y_{m-1} + \lambda y_m)$ for some $\lambda \in [0, 1]$. It suffices to show that $Tf(x, y)$ has the same value if (x, y) is either considered to be a point of $I_n \times J_m$, or of $I_{n+1} \times J_m$.

If we consider (x, y) as a point of $I_n \times J_m$, we have that $\phi_n^{-1}(x_n) = x_N$ and $\psi_m^{-1}((1 - \lambda)y_{m-1} + \lambda y_m) = (1 - \lambda)y_0 + \lambda y_M$. Hence,

$$\begin{aligned}&e_{nm}\phi_n^{-1}(x) + f_{nm}\psi_m^{-1}(y) + g_{nm}\phi_n^{-1}(x)\psi_m^{-1}(y) + a_{nm}f(\phi_n^{-1}(x), \psi_m^{-1}(y)) + k_{nm} \\ &\quad = e_{nm}x_N + f_{nm}((1 - \lambda)y_0 + \lambda y_M) + g_{nm}x_N((1 - \lambda)y_0 + \lambda y_M) \\ &\qquad + a_{nm}f(x_N, (1 - \lambda)y_0 + \lambda y_M) + k_{nm} \quad (f \in \mathcal{F}) \\ &\quad = e_{nm}x_N + f_{nm}((1 - \lambda)y_0 + \lambda y_M) + g_{nm}x_N((1 - \lambda)y_0 + \lambda y_M) \\ &\qquad + a_{nm}((1 - \lambda)z_{N0} + \lambda z_{NM}) + k_{nm} \\ &\quad = [e_{nm}x_N + f_{nm}y_0 + g_{nm}x_N y_0 + a_{nm}z_{N0} + k_{nm}] \\ &\qquad - \lambda[f_{nm}y_0 + g_{nm}x_N y_0 + a_{nm}z_{N0}] + \lambda[f_{nm}y_M + g_{nm}x_N y_M + a_{nm}z_{NM}] \\ &\quad = z_{n,m-1} - \lambda[z_{n,m-1} - e_{nm}x_N - k_{nm}] + \lambda[z_{nm} - e_{nm}x_N - k_{nm}] \\ &\quad = z_{n,m-1} - \lambda z_{n,m-1} + \lambda z_{nm} \\ &\quad = (1 - \lambda)z_{n,m-1} + \lambda z_{nm} .\end{aligned}$$

Next, we consider (x, y) as a point of $I_{n+1} \times J_m$. We have $\phi_{n+1}^{-1}(x_n) = x_0$ and $\psi_m^{-1}((1-\lambda)y_{m-1} + \lambda y_m) = (1-\lambda)y_0 + \lambda y_M$. Hence, working in an analogous way, we obtain

$$\begin{aligned}
&e_{n+1,\,m}\phi_{n+1}^{-1}(x) + f_{n+1,\,m}\psi_m^{-1}(y) + g_{n+1,\,m}\phi_{n+1}^{-1}(x)\psi_m^{-1}(y) \\
&\quad + a_{n+1,\,m}f(\phi_{n+1}^{-1}(x),\, \psi_m^{-1}(y)) + k_{n+1,\,m} \\
&= [e_{n+1,\,m}x_0 + f_{n+1,\,m}y_0 + g_{n+1,\,m}x_0y_0 + a_{n+1,\,m}z_{00} + k_{n+1,\,m}] \\
&\quad - \lambda[f_{n+1,\,m}y_0 + g_{n+1,\,m}x_0y_0 + a_{n+1,\,m}z_{00}] \\
&\quad + \lambda[f_{n+1,\,m}y_M + g_{n+1,\,m}x_0y_M + a_{n+1,\,m}z_{0M}] \\
&= (1-\lambda)z_{n,\,m-1} + \lambda z_{nm}\,.
\end{aligned}$$

Hence, Tf is well-defined on $[a, b] \times [c, d]$. In order to show that $Tf \in \mathcal{F}$, we need to show that the value of Tf on each edge of the data, obeys the rule of the definition of $\mathcal{F}$. To see this, let us consider the case when $(x_N,\, (1-\lambda)y_{m-1} + \lambda y_m) \in I_N \times J_m$. Then,

$$Tf(x_N,\, (1-\lambda)y_{m-1}+\lambda y_m) = (1-\lambda)z_{N,\,m-1}+\lambda z_{NM}\,.$$

Hence, the graph of Tf on $\{x_N\} \times J_m$ is the line segment that joins the points $(x_N,\, y_{m-1},\, z_{N,\,m-1})$, $(x_N,\, y_m,\, z_{Nm})$. Since the points $(x_N,\, y_m,\, z_{Nm})$, $m = 0, \ldots, M$ are colinear, the graph of Tf on $\{x_N\} \times [c = y_0,\, y_M = d]$ is the line segment that joins the points $(x_N,\, y_0,\, z_{N0})$, $(x_N,\, y_M,\, z_{NM})$. The other cases can be proved similarly. Thus, $Tf \in \mathcal{F}$. It is easy to verify that T is a contraction mapping on $\mathcal{F}$ with respect to the $\|\cdot\|_\infty$ norm, with contractivity factor $\delta = \max_{n,\,m}\{|a_{nm}|\}$. Thus, there exists a continuous function $f \in \mathcal{F}$ such that $Tf = f$. Then, f is a polygonal function on $\{x_n\} \times [c, d]$, $n = 1, \ldots, N$ and $[a, b] \times \{y_m\}$, $m = 1, \ldots, M$. Indeed, let us consider the first case. Let $(x, y) \in \{x_n\} \times [c, d]$. Then, there is $m \in \{1, \ldots, M\}$ such that $(x, y) \in (x_n,\, (1-\lambda)y_{m-1} + \lambda y_m)$, for some $\lambda \in [0, 1]$. Then, since f is the fixed point of T, we have

$$\begin{aligned}
&(Tf)(x_n,\, (1-\lambda)y_{m-1} + \lambda y_m) \\
&\quad = (1-\lambda)z_{n,\,m-1} + \lambda z_{nm} \\
&\quad = f(x_n,\, (1-\lambda)y_{m-1} + \lambda y_m)\,.
\end{aligned}$$

Similarly, we prove the second claim.

In the special case where $\lambda = 1$, we have that $f(x_n,\, y_m) = z_{nm}$. Hence, f is a polygonal function on the grid, which interpolates the given data.

If $\tilde{G}$ is the graph of f, we have that $\tilde{G} = \bigcup_{n=1}^{N}\bigcup_{m=1}^{M} w_{nm}(\tilde{G})$, i.e. $\tilde{G}$ is the attractor of the hyperbolic IFS $\{[a, b] \times [c, d] \times \mathbb{R},\, w_{nm} : n = 1, \ldots, N,\, m = 1, \ldots, M\}$. As the attractor G is unique (Proposition 2.1), we conclude that $\tilde{G} = G$. The proof of the theorem is complete. □

2.2 General Construction

Let $\{(x_n,\, y_m,\, z_{nm}) : n = 0, 1, \ldots, N, m = 0, 1, \ldots, M\}$ be a given set of data. Let the points $(x_{-1},\, y_{-1})$, $(x_{N+1},\, y_{-1})$, $(x_{-1},\, y_{M+1})$, $(x_{N+1},\, y_{M+1})$ be such that $x_{-1} < x_0$, $y_{-1} < y_0$, $x_N < x_{N+1}$, $y_M < y_{M+1}$. Now let us consider the data

$$\begin{aligned}
\{(x_n,\, y_m,\, \hat{z}_{nm}) : n &= -1,\, 0, \ldots, N+1 \\
m &= -1,\, 0, \ldots, M+1\}
\end{aligned}$$

where $\hat{z}_{nm} = z_{nm}$ for $n = 0, 1, \ldots, N$, $m = 0, 1, \ldots, M$ and for the remaining cases $\hat{z}_{nm}$ obey condition $(*)$ in Sec. 2.1. We construct a FIF

$$\hat{f} : [x_{-1},\, x_{N+1}] \times [y_{-1},\, y_{M+1}] \to \mathbb{R}$$

according to Proposition 2.2. The function f which interpolates the given data is the restriction of $\hat{f}$ on $[x_0,\, x_N] \times [y_0,\, y_M]$.

2.3 Examples

We now use the following data, to construct a bivariate fractal surface, by using our general construction method. The data is given in the following table. The method used for constructing the figures presented here is based on the Deterministic Iteration Algorithm (see Ref. 7 for details).

y \ x	**0**	**100**	**200**	**300**
0	1	4	6	2
100	2	1	3	6
200	5	0	4	3
300	3	6	3	4

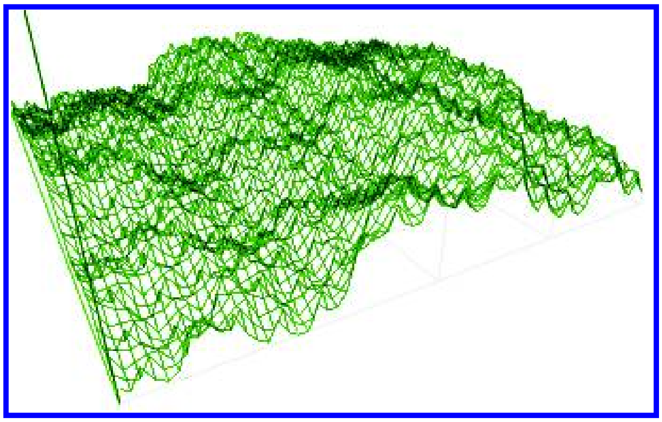

Fig. 1 The graph of a bivariate FIS.

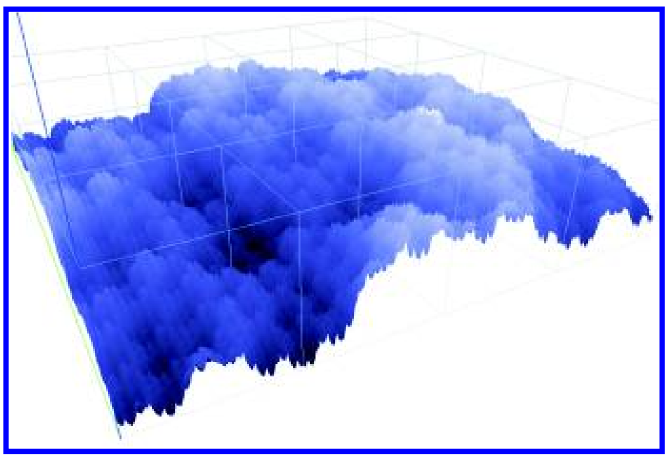

Fig. 2 A solid model analogue of the previous figure.

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