Math. Nachr. 123 (1985) 131—135

The n-Dimensional Hausdorff Measure of the n-Skeleton of a Convex W-Compact Set (Body)

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(Received December 28, 1983)

1. Introduction

When C is a convex weakly compact set in a normed linear space E and n is a nonnegative integer, the n-skeleton of C, denoted by $\text{skel}_n C$, consists of those points of C that do not lie in the relative interior of any $(n + 1)$-dimensional convex subset of C. In this paper, we study the n-dimensional HAUSDORFF measure of the n-skeleton of a convex w-compact set, where n is a positive integer.

A convex body is a bounded closed convex set having nonempty interior. In the case that C is a convex body in a reflexive BANACH space we may regard C as a weakly compact convex set with nonempty interior. In this case J. LINDENSTRAUSS and R. R. PHELPS [6] have proved that the set of the extreme points of C is uncountable, that is the 0-dimensional HAUSDORFF measure of $\text{skel}_0 C$ is not σ-finite. Using this result, we prove that $\text{skel}_n C$ has no σ-finite n-dimensional HAUSDORFF measure for any positive integer n.

2. The Results

The following theorems will be proved.

Theorem 1. *Let C be a w-compact convex set of infinite dimension in a normed linear space E. If $H^n(.)$ denotes the n-dimensional* HAUSDORFF *measure then* $H^n(\text{skel}_n C) = +\infty$ *for every* $n = 1, 2, \ldots$

Theorem 2. *If C is a convex body in a reflexive* BANACH *space then the n-dimensional* HAUSDORFF *measure of* $\text{skel}_n C$ *is not σ-finite for any* $n = 1, 2, \ldots$

At first we quote and prove two lemmata that will be used in the proof of Theorem 1.

Lemma 1. *If Σ is a compact convex set in the* EUCLIDean *space E^k, containing a unit cube of dimension n, $1 \leqq n \leqq k - 1$ then* $H^n(\text{skel}_n \Sigma) \geqq k + 1 - n$.

Proof. By Theorem 1 in [1] we have that

$$(1) \qquad H^n(\text{skel}_n \Sigma) = \frac{\binom{k}{n}}{ka(k-n)\,a(n)} \int n_0(E_{k-n} \cap \Sigma)\, d\mu^k_{k-n}(E_{k-n})$$

where $a(r)$ is the content of the r-dimensional unit ball, the integral is taken over those $(k - n)$-dimensional flats that intersect the interior of Σ and $n_0(E_{k-n} \cap \Sigma)$ is the number of the extreme points of the intersection of Σ with a $(n-k)$-flat E_{k-n}. Then

$$n_0(E_{k-n} \cap \Sigma) \geqq k + 1 - n. \tag{2}$$

Since Σ contains a unit cube of dimension n, the μ^k_{k-n}-measure of $(k-n)$-flats, that intersect Σ, is greater than or equal of the μ^k_{k-n}-measure of $(k-n)$-flats that intersect the n-dimensional unit cube. But the measure of the $(k-n)$-flats in E^k that meet a unit cube of dimension n is $k\,a(k-n)\,a(n)\Big/\binom{k}{k-n}$. Hence, using (1) and (2), we have that

$$H^n(\mathrm{skel}_n\,\Sigma) \geqq k + 1 - n$$

and this conclude the proof of the lemma.

Lemma 2. *Let C be a w-compact set in a normed linear space E and π be a linear projection from E onto a subspace F. Then*

$$\mathrm{Skel}_n\,\pi(C) \subseteqq \pi(\mathrm{skel}_n\,C) \quad \textit{for} \quad n + 1 < \dim F.$$

Proof. Let $y \in \mathrm{skel}_n\,\pi(C)$. We consider the following cases.

Case 1. Suppose $y \in \mathrm{skel}_0\,\pi(C)$. Then y is an extreme point of $\pi(C)$. We have that $F = \pi^{-1}(y) \cap C$ is w-compact and convex. If ϱ in F is the mid-point $(c_1 + c_2)/2$ of two points c_1, c_2 of C then $\pi(\varrho) = \big(\pi(c_1) + \pi(c_2)\big)/2$. Since y is an extreme point of $\pi(C)$, we have that $\pi(c_1) = \pi(c_2) = y$, so $c_1, c_2 \in F$. This implies that F is an extreme face of C. If x is an extreme point of F, then x must also be an extreme point of C. This implies that $x \in F \cap \mathrm{ext}\,C$. Therefore $y = \pi(x) \in \pi(\mathrm{ext}\,C)$. Hence $\mathrm{skel}_0\,\pi(C) \subseteqq \pi(\mathrm{skel}_0\,C)$.

Case 2. Let $y \in \mathrm{skel}_i\,\pi(C) - \bigcup_{j=1}^{i-1} \mathrm{skel}_j\,\pi(C)$, $1 \leqq i \leqq n$. Then there exists a face A of $\pi(C)$ of dimension i such that $y \in \mathrm{relint}\,A$. Let $F = \pi^{-1}(A) \cap C$ and $\varrho \in F$, and $c_1, c_2 \in C$ as in case 1. Then $\pi(\varrho) = \big(\pi(c_1) + \pi(c_2)\big)/2 \in A$ and since A is an extreme face of $\pi(C)$, we have $\pi(c_1), \pi(c_2) \in A$. So $c_1, c_2 \in F$, that is F is an extreme face of C of co-dimension $(k-i)$.

Let $x \in \mathrm{ext}\,\big(\pi^{-1}(y) \cap C\big) \subseteqq F$. Since $\pi^{-1}(y)$ has co-dimension k the co-dimension of $\pi^{-1}(y) \cap C$ relative to the face F is i. Then the faces of F, whose relative interior contains x have dimension at most i. Hence $x \in \mathrm{skel}_i\,F \subseteqq \mathrm{skel}_i\,C$ as F is an extreme face of C.

Therefore

$$\mathrm{skel}_i\,\pi(C) - \bigcup_{j=0}^{i-1} \mathrm{skel}_j\,\pi(C) \subseteqq \pi(\mathrm{skel}_i\,C), \qquad 1 \leqq i \leqq n$$

and

$$\mathrm{skel}_n\,\pi(C) = \mathrm{skel}_0\,\pi(C) \cup \bigcup_{i=1}^{n} \left[\mathrm{skel}_i\,\pi(C) - \bigcup_{j=0}^{i-1} \mathrm{skel}_j\,\pi(C)\right] \subseteqq \pi(\mathrm{skel}_n\,C).$$

Proof of Theorem 1. Let $n \geqq 1$ and $k \geqq n + 1$, n being fixed but arbitrary. We suppose that the zero vector belongs to C. Let $\{e_1, e_2, \ldots, e_n\}$ be a set of linearly independent unit vectors in E, and K_n be the unit cube of dimension n.

After C has been scaled and translated in a suitable way, we may suppose that K_n is a subset of C. Let $\{e_{n+1}, \ldots, e_k\}$ be a set of linearly independent unit vectors in E such that the set $\{e_1, \ldots, e_k\}$ spans the subspace E^k of E of dimension k.

Let π_k be the identity map on E^k. Then π_k has a linear extension π defined from E into E^k such that $\|\pi\| \leqq 1$. (see [3] p. 105). Therefore $\pi(K_n) = \pi_k(K_n) = K_n \subseteqq \pi(C)$. Hence $\pi(C)$ is a compact convex set in E^k, that contains the n-dimensional unit cube K_n.

Since π is non expansive, we have that

$$H^n(\mathrm{skel}_n\, C) \geqq H^n\big(\pi(\mathrm{skel}_n\, C)\big).$$

By lemma 2

$$\mathrm{skel}_n\, \pi(C) \subseteqq \pi(\mathrm{skel}_n\, C).$$

By lemma 1

$$H^n\big(\mathrm{skel}_n\, \pi(C)\big) \geqq k + 1 - n$$

so $H^n(\mathrm{skel}_n\, C) \geqq H^n\big(\pi(\mathrm{skel}_n\, C)\big) \geqq k + 1 - n$. Hence $H^n(\mathrm{skel}_n\, C) \geqq k + 1 - n$ as this holds for every $k \geqq n + 1$. Hence $H^n(\mathrm{skel}_n\, C) = +\infty$ for every $n = 1, 2, \ldots$

This proves the theorem.

In the proof of Theorem 2 the following definitions are used:

Definition 1. The map $f\colon X \to Y$ is said to be a *homeomorphism of class* 0, 1 between the topological spaces X and Y if and only if

i) f is $1 - 1$, continuous and
ii) for every closed subset F of X, the set $f(F)$ is a G_σ subset of Y.

Definition 2. If W is a subset of $I^k \times I$, where $I = [0, 1]$ and h is a map from $I^k \times I$ into I, we say that $h(., y)$, $y \in I$ is a *selector* of W if and only if for each $x_0 \in I^k$, $h(x_0, y) \in W_{x_0} = \{(x_0, z) \in I^k \times I : (x_0, z) \in W\}$.

Proof of Theorem 2. Let C be a convex body in the reflexive Banach space E In any reflexive Banach space E, there exists a separable infinite dimensional closed subspace F and a linear projection Π of norm 1 from E onto F [Prop. 1, 7]. Then $\Pi(C)$ is a convex body in the separable reflexive Banach space F. By lemma 2, if the n-dimensional Hausdorff measure of $\mathrm{skel}_n\, \Pi(C)$ is not σ-finite, then the n-dimensional Hausdorff measure of $\mathrm{skel}_n\, C$ is not σ-finite. Hence we may suppose that the space E is separable. The convex body C is compact, metrizable (in the weak topology).

By Theorem 1 in [5] $\mathrm{skel}_k\, C$ is an absolute G_σ-set in the closed set C. Hence $\mathrm{skel}_k\, C$ is a Borel set in E. By § 36, III, Vol. I in [4] there exists a mapping $f\colon K \to E$ from a closed subset K of the set of the irrationals in $[0, 1]$ onto E, such that f is a homeomorphism of class 0, 1. Let $\pi\colon E \to E^k$, be a projection of E on the Euclidean space E^k of dimension k. Then we may suppose that $I^k = [0, 1]^k \subseteqq \mathrm{int}\, \pi(C)$.

Let $x \in I^k$, then $\pi^{-1}(x)$ is a hyperplane of co-dimension k and $C \cap \pi^{-1}(x)$ is a convex body of infinite dimension. Then by Corollary 1.2 in [6] the set $\mathrm{ext}\,\big(C \cap \pi^{-1}(x)\big)$ is uncountable. Since $\pi^{-1}(x)$ is of co-dimension k we have that

$$\mathrm{ext}\,\big(C \cap \pi^{-1}(x)\big) \subseteqq \mathrm{skel}_k\, C$$

so for every $x \in I^k$ the set $\mathrm{skel}_k\, [C \cap \pi^{-1}(x)]$ is uncountable.

Let $E^{k+1} = E^k \times E^1$ and consider K to be a subset of the interval $\{(0, y), 0 \in E^k, 0 < y < 1\}$. We define a map $\varphi\colon K \to E^k \times E^1$ such that

$$\varphi(y) = \big(\pi(f(y)), y\big).$$

Then

(i) φ is a $1-1$ map.

(ii) φ is a continuous as π and f are.

(iii) φ^{-1} is continuous on $\varphi(K)$ (as projection map).

The set $\operatorname{skel}_k C$ is a BOREL set so $f^{-1}(\operatorname{skel}_k C)$ is a BOREL subset of K as f is continuous and $\varphi(f^{-1}(\operatorname{skel}_k C)) = W'$ is a BOREL subset if E^{k+1} as φ^{-1} is continuous. Then as $\operatorname{skel}_k [C \cap \pi^{-1}(x)]$ is uncountable, $W_x = \{(x, y) \in E^{k+1}: (x, y) \in W,\ x \in E^k\}$ is an uncountable set for each $x \in I^k$. Let $W = W' \cap (I^k \times I)$. By Theorem 7 in [2] there exists a map $h: I^k \times I \to I$ such that:

(i) h is an $\mathcal{L}(I^k \times I)$ measurable map, where $\mathcal{L}(I^n)$ denotes the family of the LEBESGUE measurable subsets of I^n, n being an integer;

(ii) for each $x \in I^k$, $h(x, \cdot)$ is a BOREL isomorphism of I into $W_x = \{(x, y) \in E^{k+1}: (x, y) \in W, x \in E^k\}$ and

(iii) for each y, $h(\cdot, y)$ is an $\mathcal{L}(I^k)$ measurable selector of W.

Let $\{A_y : y \in W_0\}$ be the uncountable family of these selectors. Then $A_y \cap A_{y'} = \emptyset$ for $y \neq y'$ and $H^k(A_y) > 0$. Then $D_y = f(\varphi^{-1}(A_y))$ is an H^k-measurable set in T as f and φ^{-1} are continuous. Hence $\operatorname{skel}_k C$ contains an uncountable family $\{D_y\}_{y \in W_0}$ with $H^k(D_y) > 0$ and $D_y \cap D_{y'} = \emptyset$ for $y \neq y'$. Therefore $\operatorname{skel}_k C$ has no σ-finite H^k-measure ([8], p. 123, Theorem 58).

Finally we give an example of a convex closed bounded set with empty interior in a reflexive BANACH space such that $\operatorname{skel}_n C$ has σ-finite n-dimensional HAUSDORFF measure for $n = 0, 1, \ldots$ This shows that the assumption of the non-emptiness of the interior of the set C in Theorem 2 can not be removed.

Example. Let l_2 denote the space of all sequences $x = \{x_1, x_2, \ldots\}$. of scalar such that $\sum_{n=1} |x_n|^2 < +\infty$ and $\{e_i\}_{i=0}^{\infty}$ be the set

$$e_0 = \{0, 0, 0, \ldots\}$$
$$e_1 = \{1, 0, 0, \ldots\}$$
$$e_2 = \{0, 1/2, 0, \ldots\} \quad \text{and so on.}$$

Let $C = \overline{\operatorname{con}}\left(\bigcup_{n=0}^{\infty} e_n\right)$. Then C is a convex closed set in l_2 with empty interior. The set of the extreme points of C is $\bigcup_{n=0}^{\infty} \{e_n\}$ and its n-skeleton is

$$\operatorname{skel}_n C = \bigcup_{k=n}^{\infty} \bigcup_{0 \le i_1 < i_2 \cdots < i_n \le k-1} \operatorname{con}\{e_k, e_{i_1}, \ldots, e_{i_n}\}.$$

We have $H^n(\operatorname{con}(e_k, e_{i_1}, \ldots, e_{i_n})) < +\infty$. Hence the n-dimensional skeleton has σ-finite n-dimensional HAUSDORFF measure for every $n \geqq 0$.

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