

## STRICT CONVEXITY OF SETS IN ANALYTIC TERMS

LEONI DALLA and TELEMACHOS HATZIAFRATIS<sup>✉</sup>

(Received 11 July 2004; revised 21 March 2005)

Communicated by A. Rubinov

### Abstract

We compare the geometric concept of strict convexity of open subsets of  $\mathbb{R}^n$  with the analytic concept of 2-strict convexity, which is based on the defining functions of the set, and we do this by introducing the class of  $2N$ -strictly convex sets. We also describe an exhaustion process of convex sets by a sequence of 2-strictly convex sets.

2000 *Mathematics subject classification*: primary 52A20.

*Keywords and phrases*: strictly convex set,  $2N$ -strictly convex set, defining function, exhaustion of a convex set.

### 1. Introduction

Let  $\mathcal{X}(\mathbb{R}^n)$  be the set of convex compact subsets of  $\mathbb{R}^n$  with non-empty interior. Then  $\mathcal{X}(\mathbb{R}^n)$ , equipped with the Hausdorff metric, is a complete metric space. A set in  $\mathcal{X}(\mathbb{R}^n)$  is called *strictly convex* if its boundary does not contain any line segments. Klee [1] proved that the subset of  $\mathcal{X}(\mathbb{R}^n)$ , which contains the sets which are not strictly convex, is a set of first category in  $\mathcal{X}(\mathbb{R}^n)$ . Zamfirescu [4, 3] improved this result by proving that the above set is  $\sigma$ -porous. In view of these results we can say that the set of strictly convex sets is not only dense in  $\mathcal{X}(\mathbb{R}^n)$ , but is a large set. On the other hand, there is another concept of strict convexity—this is what we call *2-strict convexity*—defined in terms of a defining function of  $D$ . More precisely, if  $D \subset \mathbb{R}^n$  is a bounded open set with  $C^2$  boundary and  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$  is a  $C^2$  function so that

$$D = \{x \in \mathbb{R}^n : \rho(x) < 0\}, \quad \partial D = \{x \in \mathbb{R}^n : \rho(x) = 0\},$$

and  $d\rho(x) \neq 0$  for each  $x \in \partial D$ , then  $D$  is said to be 2-strictly convex if for every  $x \in \partial D$  and every  $y = (y_1, \dots, y_n) \in \mathbb{R}^n \setminus \{0\}$ ,

$$\sum_{j=1}^n \frac{\partial \rho(x)}{\partial x_j} y_j = 0 \quad \text{implies that} \quad \sum_{1 \leq j, k \leq n} \frac{\partial^2 \rho(x)}{\partial x_j \partial x_k} y_j y_k > 0.$$

This concept of 2-strict convexity plays an important role in complex analysis. For example, 2-strict convexity affects the solvability of the inhomogeneous Cauchy-Riemann equations in the domain, with Lipschitz estimates (see Range [2]). Thus, in a sense, it is natural to ask what is the relation between the geometric and analytic strict convexity. It is easy to show that if a set is 2-strictly convex, then it is also strictly convex in the geometric sense. The converse does not hold however, and in order to deal with this question we generalize 2-strict convexity and we obtain the concept of  $2N$ -strictly convexity. Then, using this, we state and prove Theorem 2.4. We will also describe a process of exhausting an arbitrary convex set by 2-strictly convex sets with smooth boundary. The main results are Theorems 2.4 and 3.5 and their corollaries, and the various lemmas that we present, are needed in their proof. Although some of the ideas involved in these lemmas are essentially known, we include them here for completeness.

## 2. Strictly convex sets

Let us first recall the definition of defining functions. Let  $D \subset \mathbb{R}^n$  be a bounded open set with  $C^1$  boundary. A  $C^1$  function  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be a *defining function* of  $D$  if  $D = \{x \in \mathbb{R}^n : \rho(x) < 0\}$ ,  $\partial D = \{x \in \mathbb{R}^n : \rho(x) = 0\}$ , and  $d\rho(x) \neq 0$  for each  $x \in \partial D$ . If the boundary of  $D$  is assumed to be  $C^\infty$ , then we can choose a defining function  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$  to be  $C^\infty$  (see [2]). Also if the boundary of  $D$  is assumed to be real analytic, then we can choose a  $C^\infty$  defining function  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$  which is moreover real analytic in a neighborhood of  $\partial D$ .

We will also use the following notation about higher order differentials. For  $m \in \mathbb{N}$ ,  $x \in \mathbb{R}^n$ , and  $y = (y_1, \dots, y_n) \in \mathbb{R}^n$

$$d_m \rho(x)(y) = \sum_{1 \leq j_1, \dots, j_m \leq n} \frac{\partial^m \rho(x)}{\partial x_{j_1} \dots \partial x_{j_m}} y_{j_1} \dots y_{j_m},$$

defined in the case  $\rho$  is of class  $C^m$ . In particular,  $d_1 \rho(x) = d\rho(x)$ .

**LEMMA 2.1.** *Let  $D \subset \mathbb{R}^n$  be a bounded open set with  $C^1$  boundary and  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$  a  $C^1$  defining function of it. Then  $D$  is convex if and only if  $d_1 \rho(x)(x - y) > 0$  for every  $x \in \partial D$  and  $y \in D$ .*

PROOF. Assume that  $D$  is convex. Suppose that for some  $x \in \partial D$  and  $y \in D$ ,

$$d_1\rho(x)(x - y) < 0.$$

By the convexity of  $D$ ,  $r(t) = tx + (1 - t)y \in D$ , for  $t \in [0, 1]$ . Define

$$h(t) = \rho(r(t)), \quad \text{for } t \in [0, 1].$$

Then  $h$  is a  $C^1$  function,  $h(1) = \rho(x) = 0$  and  $h(t) < 0$  for  $t \in [0, 1)$ . Also

$$h'(1) = d_1\rho(x)(x - y) < 0.$$

Therefore there is an  $\varepsilon$  with  $0 < \varepsilon < 1$  such that  $h'(t) < 0$  for  $t \in [1 - \varepsilon, 1]$ . It follows that  $h$  is strictly decreasing in the interval  $[1 - \varepsilon, 1]$ . So  $h(1 - \varepsilon) > h(1) = 0$ , which contradicts the fact that  $h(t) < 0$  for  $t \in [0, 1)$ .

This proves that

$$(1) \quad d_1\rho(x)(x - y) \geq 0 \quad \text{for every } x \in \partial D \text{ and } y \in D.$$

Now suppose that for some  $x \in \partial D$  and  $y \in D$ ,  $d_1\rho(x)(x - y) = 0$ . Since  $D$  is assumed open, there is a  $\delta > 0$  so that the ball

$$B(y, \delta) = \{y + su : s \in (-\delta, \delta) \text{ and } u \in \mathbb{R}^n \text{ with } |u| = 1\} \subset D.$$

Let  $u \in \mathbb{R}^n$  with  $|u| = 1$ . Then  $y + su \in D$ , for  $s \in (-\delta, \delta)$ . By (1),

$$d_1\rho(x)(x - (y + su)) \geq 0.$$

Since  $d_1\rho(x)(x - y) = 0$ , it follows that  $sd_1\rho(x)(u) \leq 0$ , for  $s \in (-\delta, \delta)$ . Hence  $d_1\rho(x)(u) = 0$ , for every  $u \in \mathbb{R}^n$  with  $|u| = 1$ . This contradicts the assumption that  $d\rho(x) \neq 0$  and completes the proof that  $d_1\rho(x)(x - y) > 0$  for every  $x \in \partial D$  and  $y \in D$ .

Conversely, assume that  $d_1\rho(x)(x - y) > 0$  for every  $x \in \partial D$  and  $y \in D$ . Let  $y, z \in D$ ,  $y \neq z$ , and suppose that for some  $x \in [y, z]$ ,  $x \in \partial D$ . Let us write

$$x = y + t(z - y) = z + s(y - z), \quad \text{for some } t, s \in (0, 1).$$

Then

$$d_1\rho(x)(x - y) > 0 \quad \text{and} \quad d_1\rho(x)(x - z) > 0,$$

which implies that

$$d_1\rho(x)(z - y) > 0 \quad \text{and} \quad d_1\rho(x)(y - z) > 0,$$

which is impossible.

Thus for  $x \in [y, z]$ , either  $x \in D$  or  $x \notin \bar{D}$ . If  $x \notin \bar{D}$  then  $\rho(x) > 0$ . Since  $\rho(y) < 0$ , there exists a  $\tau \in (0, 1)$  so that  $\rho(y + \tau(x - y)) = 0$ . Then  $y + \tau(x - y) \in \partial D \cap [y, z]$ , which has already been excluded. This proves that  $D$  is convex.  $\square$

LEMMA 2.2. *Let  $D \subset \mathbb{R}^n$  be a bounded open and convex set with  $C^\infty$  boundary,  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$  a  $C^\infty$  defining function of  $D$ ,  $x \in \partial D$ , and  $y \in \mathbb{R}^n$ . If  $d_m \rho(x)(y) = 0$  for  $m = 1, 2, \dots, 2k$ , then  $d_{2k+1} \rho(x)(y) = 0$ .*

PROOF. Since  $d_m \rho(x)(s y) = s^m d_m \rho(x)(y)$ , for every  $m$ , we may assume that  $|y| = 1$ . If  $d_{2k+1} \rho(x)(y) > 0$ , then there is an  $\varepsilon > 0$  such that  $d_{2k+1} \rho(z)(y) > 0$  for  $z \in B(x, \varepsilon)$ . By Taylor's theorem, for  $0 < |t| < \varepsilon$ ,

$$\begin{aligned} \rho(x + ty) &= \rho(x) + \sum_{m=1}^{2k} \frac{1}{m!} d_m \rho(x)(ty) + \frac{1}{(2k+1)!} d_{2k+1} \rho(z_t)(ty) \\ &= \frac{1}{(2k+1)!} t^{2k+1} d_{2k+1} \rho(z_t)(y), \end{aligned}$$

for some  $z_t \in [x, x + ty] \subset B(x, \varepsilon)$ . It follows that for  $t \in (-\varepsilon, 0)$ ,  $\rho(x + ty) < 0$ , and therefore  $x + ty \in D$ . Then, by Lemma 2.1,

$$d_1 \rho(x)(x - (x + ty)) = d_1 \rho(x)(-ty) > 0.$$

This is a contradiction since, by assumption,  $d_1 \rho(x)(y) = 0$ . This shows that  $d_{2k+1} \rho(x)(y) > 0$  cannot hold.

Similarly we prove that the inequality  $d_{2k+1} \rho(x)(y) < 0$  cannot hold either, and the equation  $d_{2k+1} \rho(x)(y) = 0$  follows.  $\square$

LEMMA 2.3. *Let  $D \subset \mathbb{R}^n$  be a bounded open and convex set with  $C^\infty$  boundary,  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$  a  $C^\infty$  defining function of  $D$ ,  $x \in \partial D$  and  $y \in \mathbb{R}^n$ . If  $d_m \rho(x)(y) = 0$  for  $m = 1, 2, \dots, 2k - 1$ , and  $d_{2k} \rho(x)(y) \neq 0$ , then  $d_{2k} \rho(x)(y) > 0$ .*

PROOF. If  $\ell$  is the straight line  $\ell = \{r(t) = x + ty : t \in \mathbb{R}\}$ , then the intersection  $\ell \cap \bar{D}$  is a line segment, that is,  $\ell \cap \bar{D} = [a, b]$  for some  $a, b \in \mathbb{R}^n$ , and  $x \in [a, b]$ . We consider the following cases.

(i)  $a = b$ . Then the function  $h(t) = \rho(r(t))$  for  $t \in \mathbb{R}$  has the property  $h(t) > 0$  for  $t \neq 0$  and  $h(0) = 0$ . Thus  $h$  has a minimum at  $t = 0$ . Since

$$h^{(m)}(0) = \begin{cases} d_m \rho(x)(y) = 0, & \text{for } m = 1, 2, \dots, 2k - 1, \\ d_{2k} \rho(x)(y) \neq 0, & \text{for } m = 2k, \end{cases}$$

it follows that  $h^{(2k)}(0) = d_{2k} \rho(x)(y) > 0$ .

(ii)  $a \neq b$  and  $[a, b] \subset \partial D$ . In this case  $\rho(x + ty) = 0$  for every  $t \in [\tau, \sigma]$ , where  $\tau, \sigma \in \mathbb{R}$  with  $\tau < \sigma$  and  $0 \in [\tau, \sigma]$ . Therefore,

$$d_{2k} \rho(x)(y) = \left. \frac{d^{2k}}{dt^{2k}} [\rho(x + ty)] \right|_{t=0} = 0.$$

Thus this case cannot occur.

(iii)  $a \neq b$  and  $[a, b] \not\subset \partial D$ . Since  $[a, b] \subset \bar{D} = D \cup \partial D$ , there is a  $t_0 \neq 0$  such that  $x + t_0 y \in D$ . By Lemma 2.1,  $d_1 \rho(x)(x - (x + t_0 y)) = d_1 \rho(x)(-t_0 y) > 0$ . However, this implies that  $d_1 \rho(t_0 y) \neq 0$  and therefore this case cannot occur either.

This completes the proof of the lemma.  $\square$

DEFINITION. Let  $D \subset \mathbb{R}^n$  be a bounded open set with  $C^\infty$  boundary and  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$  a  $C^\infty$  defining function of it. For  $x \in \partial D$ , let  $T(x) = \{y \in \mathbb{R}^n : d_1 \rho(x)(y) = 0\}$  and

$$S_{2m}(x) = \left\{ y \in \mathbb{R}^n \setminus \{0\} \left| \begin{array}{l} d_j \rho(x)(y) = 0, \text{ for } j = 1, 2, \dots, 2m-1, \\ d_j \rho(x)(y) > 0, \text{ for } j = 2m. \end{array} \right. \right\}.$$

The set  $D$  is called *2k-strictly convex at the point x* if for every  $y \in T(x) \setminus \{0\}$ , there exists a positive integer  $m(y) \leq k$  such that  $y \in S_{2m(y)}(x)$ , and  $k = \max\{m(y) : y \in T(x)\}$ . The set  $D$  is called *2N-strictly convex* if for each  $x \in \partial D$ , there is  $k(x) \leq N$  such that  $D$  is  $2k(x)$ -strictly convex at  $x$ , and  $N = \max\{k(x) : x \in \partial D\}$ . If  $D$  is 2-strictly convex, then  $N = 1$ , and therefore  $k(x) = 1$  for every  $x \in \partial D$ . Hence for  $x \in \partial D$ ,  $m(y) = 1$  for every  $y \in T(x) - \{0\}$ , that is,

$$d_2 \rho(x)(y) > 0 \quad \text{when} \quad d_1 \rho(x)(y) = 0 \quad \text{and} \quad y \neq 0.$$

In some sense, the 2-strictly convex sets are the most strictly convex sets.

REMARK 1. For each fixed  $x \in \partial D$ , the sets  $T(x)$  and  $S_{2m}(x)$  do not depend on the defining function  $\rho$ , that is, they depend only on the set  $D$ . This can be justified as follows. Let  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$  be another  $C^\infty$  defining function of  $D$ . Then there is a  $C^\infty$  function  $h$ , defined in a neighborhood  $W$  of  $\partial D$ , so that  $h > 0$  and  $\lambda = h\rho$  in  $W$  (see [2, page 51]). Thus  $d_1 \lambda(x)(y) = h(x)d_1 \rho(x)(y)$  if  $\rho(x) = 0$ , and the independence of  $T(x)$  of the defining function, follows.

To show that the sets  $S_{2m}(x)$  are also independent of the defining function, it suffices to prove that

$$(2) \quad \begin{array}{ll} d_k \lambda(x)(y) = h(x)d_k \rho(x)(y), & \text{if } x \in \partial D \quad \text{and} \\ d_j \rho(x)(y) = 0 & \text{for } j = 1, 2, \dots, k-1. \end{array}$$

Let us prove this for  $k = 2$ . A straightforward computation shows that

$$\begin{aligned} \sum_{1 \leq i_1, i_2 \leq n} \frac{\partial^2 \lambda}{\partial x_{i_1} \partial x_{i_2}}(x) y_{i_1} y_{i_2} &= h(x) \sum_{1 \leq i_1, i_2 \leq n} \frac{\partial^2 \rho}{\partial x_{i_1} \partial x_{i_2}}(x) y_{i_1} y_{i_2} \\ &+ 2 \left( \sum_{i_1=1}^n \frac{\partial h}{\partial x_{i_1}}(x) y_{i_1} \right) \left( \sum_{i_2=1}^n \frac{\partial \rho}{\partial x_{i_2}}(x) y_{i_2} \right) \\ &+ \rho(x) \sum_{1 \leq i_1, i_2 \leq n} \frac{\partial^2 h}{\partial x_{i_1} \partial x_{i_2}}(x) y_{i_1} y_{i_2}, \end{aligned}$$

and therefore  $d_2\lambda(x)(y) = h(x)d_2\rho(x)(y)$ , if  $\rho(x) = 0$  and  $d_1\rho(x)(y) = 0$ .

The proof of (2) in the general case is a computation, which is simplified if we consider the functions

$$l(t) = \lambda(x + ty) \quad \text{and} \quad r(t) = \rho(x + ty), \quad t \in \mathbb{R},$$

and observe that

$$d_j\lambda(x)(y) = \left. \frac{d^j l}{dt^j} \right|_{t=0} \quad \text{and} \quad d_j\rho(x)(y) = \left. \frac{d^j r}{dt^j} \right|_{t=0}.$$

Thus (2) follows from the formula

$$\frac{d^k(\chi r)}{dt^k} = \sum_{j=0}^k \binom{k}{j} \frac{d^j \chi}{dt^j} \frac{d^{k-j} r}{dt^{k-j}},$$

where  $\chi(t) = h(x + ty)$ .

Notice also that (2) implies that for each  $k$ ,

$$\begin{aligned} \{x \in \partial D : d_j\rho(x)(y) = 0 \text{ for } j = 1, \dots, k-1\} \\ = \{x \in \partial D : d_j\lambda(x)(y) = 0 \text{ for } j = 1, \dots, k-1\}. \end{aligned}$$

**THEOREM 2.4.** *Let  $D \subset \mathbb{R}^n$  be a bounded open set with real analytic boundary. The following are equivalent:*

- (i)  $D$  is connected and  $2N$ -strictly convex, for some  $N \in \mathbb{N}$ .
- (ii)  $D$  is convex and its boundary does not contain line segments.

**PROOF.** (i) implies (ii). Let  $\Omega = \{(x, y) \in D \times D : (1-t)x + ty \in D \text{ for every } t \in [0, 1]\}$ . Then  $\Omega$  is an open subset of  $D \times D$ . We claim that  $\Omega$  is also closed in  $D \times D$ , for otherwise there would exist a sequence  $(x_\nu, y_\nu) \in \Omega$ ,  $(x_\nu, y_\nu) \rightarrow (x, y) \in D \times D$  and  $(x, y) \notin \Omega$ . Then for some  $\tau \in (0, 1)$ ,  $z = (1-\tau)x + \tau y \notin D$ . Since  $x_\nu \rightarrow x$ ,  $y_\nu \rightarrow y$ , and  $(1-t)x_\nu + ty_\nu \in D$ , it follows that  $(1-t)x + ty \in \overline{D}$  for  $t \in [0, 1]$ . Hence  $h(t) = \rho((1-t)x + ty) \leq 0$  for  $t \in [0, 1]$ , and  $h(\tau) = \rho(z) = 0$ , since  $z \in \partial D$ . Throughout this proof,  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$  is a  $C^\infty$  defining function of  $D$  which is real analytic in a neighborhood of  $\partial D$ . Thus  $h(t)$  takes its maximum at  $t = \tau$ , and therefore  $h'(\tau) = d_1\rho(z)(y-x) = 0$ . By assumption, there is an  $m \leq N$  such that

$$d_j\rho(z)(y-x) = 0, \quad j = 1, 2, \dots, 2m-1, \quad \text{and} \quad d_{2m}\rho(z)(y-x) > 0.$$

In other words,

$$h^{(j)}(\tau) = 0 \text{ for } j = 1, 2, \dots, 2m-1, \quad \text{and} \quad h^{(2m)}(\tau) > 0,$$

and this is a contradiction since  $h(t)$  takes its maximum at  $t = \tau$ . This proves that  $\Omega$  is closed in  $D \times D$ .

Since  $D$  is assumed to be connected, it follows that  $\Omega = D \times D$ , that is,  $D$  is convex.

Now we will show that  $\partial D$  does not contain line segments. Assume, to reach a contradiction, that  $[x, y] \subset \partial D$  and  $x \neq y$ . Then

$$h(t) = \rho((1-t)x + ty), \quad \text{for } t \in [0, 1],$$

and therefore  $h^{(m)}(0) = d_m \rho(x)(y-x) = 0$  for every  $m \in \mathbb{N}$ . However, this contradicts the assumption that  $D$  is  $2N$ -strictly convex.

(ii) implies (i). Let  $x \in \partial D$  and  $y \in T(x) \setminus \{0\}$ . If  $d_{2m} \rho(x)(y) = 0$  for every  $m \in \mathbb{N}$ , then by Lemma 2.2,  $d_{2m+1} \rho(x)(y) = 0$  for every  $m \in \mathbb{N}$ . Since  $\rho$  is assumed to be real analytic in a neighborhood of  $\partial D$ ,

$$\rho(x + ty) = \rho(x) + \sum_{m=1}^{\infty} d_m \rho(x)(ty) = 0, \quad \text{for } t \in (-\varepsilon, \varepsilon),$$

where  $\varepsilon > 0$ . However, this implies that  $x + ty \in \partial D$  for  $t \in (-\varepsilon, \varepsilon)$ , that is,  $\partial D$  contains line segments, and this is a contradiction.

It follows that there exists  $l(y)$  so that  $d_{2l(y)} \rho(x)(y) \neq 0$ . Let  $m(y)$  be the smallest integer with this property, that is,  $d_{2m(y)} \rho(x)(y) \neq 0$ . If  $m(y) = 1$  then, by Lemma 2.3,  $d_2 \rho(x)(y) > 0$ . If  $m(y) \geq 2$ , then  $d_{2j} \rho(x)(y) = 0$  for  $j = 1, \dots, m(y) - 1$ , and by Lemma 2.2,  $d_{2j+1} \rho(x)(y) = 0$  for  $j = 1, \dots, m(y) - 1$ . Hence by Lemma 2.3,  $d_{2m(y)} \rho(x)(y) > 0$ . This shows that for every  $y \in T(x) \setminus \{0\}$  there is  $m(y)$  such that  $y \in \mathcal{S}_{2m(y)}(x)$ .

Let  $k(x) = \sup\{m(y) : y \in T(x) \setminus \{0\}\}$ . We will prove that  $k(x) < \infty$ . To reach a contradiction assume that  $k(x) = \infty$ . Since  $m(y/|y|) = m(y)$  for  $y \in T(x) \setminus \{0\}$ , there exists a sequence  $y_\nu \in T(x) \setminus \{0\}$  with  $|y_\nu| = 1$  and  $m(y_\nu) \rightarrow \infty$ . By the compactness of the set  $\{y : y \in T(x) \setminus \{0\} \text{ and } |y| = 1\}$ , we may assume that  $y_\nu \rightarrow y \in T(x) \setminus \{0\}$  with  $|y| = 1$ . Let  $m(y)$  be such that  $y \in \mathcal{S}_{2m(y)}(x)$ . Then  $d_{2m(y)} \rho(x)(y) > 0$ , and therefore  $d_{2m(y)} \rho(x)(y_\nu) > 0$  for  $\nu \geq \nu_0$ , where  $\nu_0 \in \mathbb{N}$ . By the choice of  $m(y_\nu)$ ,  $m(y_\nu) \leq m(y)$  for  $\nu \geq \nu_0$ , which contradicts  $m(y_\nu) \rightarrow \infty$ .

So far we proved that for each  $x \in \partial D$  there is  $k(x)$  so that  $D$  is  $2k(x)$ -strictly convex at  $x$ . Let  $N = \sup\{k(x) : x \in \partial D\}$ . It remains to show that  $N < \infty$ . To reach a contradiction, assume that  $N = \infty$ . By the compactness of  $\partial D$ , there is a sequence  $x_\nu \in \partial D$  such that  $x_\nu \rightarrow x \in \partial D$  and  $k(x_\nu) \rightarrow \infty$ . Let  $y_\nu \in T(x_\nu)$  with  $|y_\nu| = 1$  and  $k(x_\nu) = m_{x_\nu}(y_\nu)$ , where  $m_{x_\nu}(y_\nu)$  is the  $m(y_\nu)$ -integer associated to  $y_\nu \in T(x_\nu)$ . Passing to a subsequence, we may assume that  $y_\nu \rightarrow y$  with  $|y| = 1$ . Since  $y_\nu \in T(x_\nu)$ ,  $d_1 \rho(x_\nu)(y_\nu) = 0$ , and therefore, by continuity,  $d_1 \rho(x)(y) = 0$ , that is,  $y \in T(x) \setminus \{0\}$ . Then  $d_{2m_x(y)} \rho(x)(y) > 0$ , and therefore  $d_{2m_x(y)} \rho(x_\nu)(y_\nu) > 0$ , for  $\nu \geq \nu_0$ , where  $\nu_0 \in \mathbb{N}$ . By the choice of  $m_{x_\nu}(y_\nu)$ ,  $m_{x_\nu}(y_\nu) \leq m_x(y)$  for  $\nu \geq \nu_0$ , and consequently  $k(x_\nu) = m_{x_\nu}(y_\nu) \leq m_x(y)$ , which contradicts  $k(x_\nu) \rightarrow \infty$ .

This proves that  $N \in \mathbb{N}$  and that  $D$  is  $2N$ -strictly convex.  $\square$

**COROLLARY 2.5.** *Let  $D \subset \mathbb{R}^n$  be a bounded open set with real analytic boundary. Then the following are equivalent:*

- (i)  $D$  is connected and  $2N$ -strictly convex for some  $N \in \mathbb{N}$ .
- (ii) For every  $x \in \partial D$  and  $y \in \overline{D} \setminus \{x\}$ ,  $d_1\rho(x)(x - y) > 0$ .

**PROOF.** (i) implies (ii). By Theorem 2.4,  $D$  is convex and its boundary does not contain line segments. If  $y \in D$ ,  $d_1\rho(x)(x - y) > 0$  by Lemma 2.1. If  $y \in \partial D$ , then  $x + t(y - x) \in D$  for every  $t \in (0, 1)$ . Then, by Lemma 2.1,

$$d_1\rho(x)(x - t(y - x)) = td_1\rho(x)(x - y) > 0,$$

and (ii) follows.

(ii) implies (i). By Lemma 2.1,  $D$  is convex. We claim that  $\partial D$  does not contain line segments. If  $[x, y] \subset \partial D$  with  $x \neq y$ , then  $\rho((1 - t)x + ty) = 0$  for  $t \in [0, 1]$ , and this would imply that  $d_1(x)(x - y) = 0$ . Hence  $\partial D$  does not contain line segments, and (i) follows from Theorem 2.4.  $\square$

**REMARK 2.** By examining the proof of Theorem 2.4 we see that if  $\partial D$  is of class  $C^{2N}$  and  $D$  is  $2N$ -strictly convex then  $D$  is convex and its boundary does not contain line segments, that is, one direction of the theorem does not require real analyticity of the boundary ( $C^{2N}$  is enough). However, for the other direction real analyticity is necessary as seen by the example of the set  $\{(x, y) \in \mathbb{R}^2 : y > \phi(x)\}$ , where

$$\phi(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

examining its boundary locally at the point  $(0, 0)$ .

### 3. Exhaustion of convex sets by 2-strictly convex sets

The notation  $A \subset\subset B$  that we will be using, is defined as follows:  $A \subset\subset B$  if and only if there is a compact set  $E$  so that  $A \subset E \subset B$ .

**LEMMA 3.1.** *Let  $D \subset \mathbb{R}^n$  be an open bounded and convex set. Then there is a continuous convex function  $\lambda : D \rightarrow \mathbb{R}$  such that for every  $a \in \mathbb{R}^n$ ,*

$$\{x \in D : \lambda(x) < a\} \subset\subset D.$$

**PROOF.** We may assume without loss of generality that  $0 \in D$ . Let us consider Minkowski's functional

$$\lambda_D(x) = \inf \left\{ t > 0 : \frac{1}{t}x \in D \right\},$$



defined for  $x \in \mathbb{R}^n$ . Then  $\lambda_D : \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuous convex function with the properties  $D = \{x \in \mathbb{R}^n : \lambda_D(x) < 1\}$ ,  $\partial D = \{x \in \mathbb{R}^n : \lambda_D(x) = 1\}$ , and  $\mathbb{R}^n \setminus \overline{D} = \{x \in \mathbb{R}^n : \lambda_D(x) > 1\}$ . Define

$$\lambda(x) = \frac{1}{1 - \lambda_D(x)}, \quad \text{for } x \in D.$$

It is straightforward to check that the function  $\lambda : D \rightarrow \mathbb{R}$  has the required properties.  $\square$

LEMMA 3.2. *Let  $f : K \times G \rightarrow \mathbb{R}$  be a continuous function, where  $G \subset \mathbb{R}^n$  is a closed rectangle and  $K$  is a compact set. Then for every  $\varepsilon > 0$  there exist points  $\xi_j \in G$  and positive numbers  $c_j$ ,  $j = 1, 2, \dots, M$ , such that*

$$\left| \sum_{j=1}^M c_j f(x, \xi_j) - \int_G f(x, y) dy \right| < \varepsilon \quad \text{for } x \in K.$$

*More generally this holds in the case where  $G \subset \mathbb{R}^n$  is a compact set whose boundary has measure zero.*

PROOF. This follows from the uniform continuity of  $f$  and the definition of the integral.  $\square$

LEMMA 3.3. *Let  $D \subset \mathbb{R}^n$  be an open set and  $f : D \rightarrow \mathbb{R}$  a continuous function. Let us also consider  $\phi \in C^\infty(\mathbb{R}^n)$ ,  $\phi \geq 0$ ,  $\phi(x) = 0$  for  $|x| \geq 1$ , and  $\int \phi(x) dx = 1$ . For sufficiently small  $\varepsilon > 0$ , set  $D_\varepsilon = \{x \in D : \text{dist}(x, \partial D) > \varepsilon\}$  and define the function*

$$f_\varepsilon(x) = \int_{y \in \mathbb{R}^n} f(y) \frac{1}{\varepsilon^n} \phi\left(\frac{x-y}{\varepsilon}\right) dy, \quad x \in D_\varepsilon.$$

*Then  $f_\varepsilon$  is well-defined and  $C^\infty$  in  $D_\varepsilon$ , and  $f_\varepsilon$  converges to  $f$  uniformly on compact subsets of  $D$  as  $\varepsilon \rightarrow 0^+$ .*

*Moreover, if the function  $f$  is convex in  $D$ , then  $f_\varepsilon$  is convex in  $D_\varepsilon$  too.*

PROOF. Firstly the existence of functions like  $\phi$  is quite standard, but we can also give a specific example as follows: For  $t \in \mathbb{R}$ , set

$$\psi(t) = \begin{cases} \exp[-1/(t+1)^2] \exp[-1/(t-1)^2] & \text{if } |t| < 1, \\ 0 & \text{if } |t| \geq 1, \end{cases}$$

and define  $\tilde{\psi}(x) = \psi(|x|^2)$  and  $\phi = \tilde{\psi} / \int \tilde{\psi} dx$ .

Now, since  $\text{supp}(\phi) \subset B(0, 1)$ ,

$$f_\varepsilon(x) = \int_{y \in B(x, \varepsilon)} f(y) \frac{1}{\varepsilon^n} \phi\left(\frac{x-y}{\varepsilon}\right) dy, \quad x \in D_\varepsilon.$$

and therefore  $f_\varepsilon$  is well-defined for  $x \in D_\varepsilon$ .

Furthermore, if  $a \in D_\varepsilon$ , there is a sufficiently small  $r > 0$ , such that  $\overline{B(a, r)} \subset D$  and for  $x$  close to  $a$ ,

$$f_\varepsilon(x) = \int_{y \in B(a, r)} f(y) \frac{1}{\varepsilon^n} \phi\left(\frac{x-y}{\varepsilon}\right) dy.$$

It follows that  $f_\varepsilon$  is  $C^\infty$  in  $D_\varepsilon$ .

Next, changing the variable in the integral which defines  $f_\varepsilon$ , we obtain

$$f_\varepsilon(x) = \int_{y \in B(0, 1)} f(x - \varepsilon y) \phi(y) dy, \quad \text{for } x \in D_\varepsilon.$$

Also if  $K$  is a compact subset of  $D$ , then the set  $K_\varepsilon = \{x \in \mathbb{R}^n : \text{dist}(x, K) \leq \varepsilon\}$  is a compact subset of  $D$ , for  $\varepsilon > 0$  and sufficiently small.

By observing that the function  $f(u)\phi(y)$  for  $(u, y) \in K_\varepsilon \times \overline{B(0, 1)}$  is uniformly continuous, and writing

$$f_\varepsilon(x) - f(x) = \int_{y \in B(0, 1)} [f(x - \varepsilon y) - f(x)] \phi(y) dy,$$

which follows from the assumption  $\int \phi dx = 1$ , we obtain that  $f_\varepsilon$  converges to  $f$ , uniformly on  $K$ , as  $\varepsilon \rightarrow 0^+$ .

Finally, it follows from Lemma 3.2 and the fact that the uniform limit of convex functions is also convex, that each  $f_\varepsilon$  is convex, if  $f$  is a convex function. Indeed,  $f_\varepsilon$  is a – uniform on compact subsets of  $D_\varepsilon$  – limit of functions of the form

$$x \longrightarrow \sum_{j=1}^M f(x - \varepsilon y_j) c_j, \quad \text{with } c_j > 0,$$

which are convex functions. □

A  $C^2$ -function  $\mu : W \rightarrow \mathbb{R}$ , defined in an open set  $W \subset \mathbb{R}^n$ , is 2-strictly convex if for every  $x \in W$ ,

$$d_2 \mu(x)(y) = \sum_{1 \leq j, k \leq n} \frac{\partial^2 \mu(x)}{\partial x_j \partial x_k} y_j y_k > 0 \quad \text{for } y \in \mathbb{R}^n \setminus \{0\}.$$

In the following we will use the term *strictly convex* function to mean 2-strictly convex function.

LEMMA 3.4. *Let  $\mu$  be a strictly convex  $C^2$ -function in a neighborhood of a compact set  $K \subset \mathbb{R}^n$ . Then there exists an  $\varepsilon > 0$  with the following property: If  $g$  is a  $C^2$ -function in a neighborhood of  $K$  and  $|\partial^2 g / \partial x_j \partial x_k| < \varepsilon$  on  $K$  for every  $j, k$ , then the function  $\mu + g$  is strictly convex in a neighborhood of  $K$ .*

PROOF. Define

$$\varepsilon = \frac{1}{n^2} \min \left\{ \sum_{1 \leq j, k \leq n} \frac{\partial^2 \mu(x)}{\partial x_j \partial x_k} y_j y_k : x \in K \text{ and } y \in \mathbb{R}^n \text{ with } |y| = 1 \right\}.$$

Then  $\varepsilon > 0$  and it is straightforward to check that it has the required property.  $\square$

THEOREM 3.5. *Let  $D \subset \mathbb{R}^n$  be a bounded open and convex set,  $K \subset D$  a compact convex subset, and  $U \subset D$  an open neighborhood of  $K$ . Then there exists a  $C^\infty$  strictly convex function  $\mu : D \rightarrow \mathbb{R}$  such that for every  $a \in \mathbb{R}$ ,  $\{x \in D : \mu(x) < a\} \subset\subset D$ , and moreover  $\mu < 0$  on  $K$  and  $\mu > 0$  on  $D \setminus U$ .*

PROOF. By Lemma 3.1, there is a continuous convex function  $\lambda : D \rightarrow \mathbb{R}$  such that for every  $a \in \mathbb{R}$ ,  $\{x \in D : \lambda(x) < a\} \subset\subset D$ . Adding a negative constant to the function  $\lambda$ , if necessary, we may assume that  $\lambda < 0$  on  $K$ .

Now consider the set  $K' = \{x \in D : \lambda(x) \leq 0\}$ . Since  $K$  is assumed to be convex, for every  $x \in K' \cap (D \setminus U)$ , there is an affine function  $u_x$  (that is, a function of the form  $u_x(t) = c_0 + c_1 t_1 + \cdots + c_n t_n$ , with  $c_i$  constants) so that  $u_x(x) > 0$  while  $u_x < 0$  on  $K$ . By the compactness of the set  $K' \cap (D \setminus U)$ , we may choose affine functions  $u_1, \dots, u_M$  in such a way that  $\max(u_1, \dots, u_M) > 0$  on  $K' \cap (D \setminus U)$  and  $\max(u_1, \dots, u_M) < 0$  on  $K$ .

Setting  $u = \max(\lambda, u_1, \dots, u_M)$ , we have  $u < 0$  on  $K$ ,  $u > 0$  on  $D \setminus U$ , and of course  $u$  is continuous and convex on  $D$ . Also,  $\{x \in D : u(x) < a\} \subset\subset D$  for every  $a \in \mathbb{R}$ .

In order to construct a  $C^\infty$  and strictly convex function with these properties, we consider the sets  $D_j = \{x \in D : u(x) < j\}$  for  $j = 0, 1, 2, \dots$ , and we construct a sequence of functions  $\mu_j$ ,  $j = 1, 2, \dots$ , such that

- (1)  $\mu_j$  is  $C^\infty$  and strictly convex in a neighborhood of  $\overline{D}_j$ ,
- (2)  $\mu_j < 0$  on  $K$ ,
- (3)  $\mu_j \geq u$  on  $\overline{D}_j$ , and
- (4)  $\mu_j = \mu_{j-1}$  on  $D_{j-2}$ , for  $j \geq 2$ .

Then, setting  $\mu = \mu_j$  on  $D_{j-2}$ , we obtain the desired function.

First we construct  $\mu_1$ . Let us choose  $\varepsilon > 0$  so that  $u + \varepsilon < 0$  on  $K$ . By Lemma 3.3, there is a  $C^\infty$  and convex function  $\tilde{\mu}_1$  in a neighborhood of  $\overline{D}_1$ , so that  $|\tilde{\mu}_1 - u| < \varepsilon/2$  on  $\overline{D}_1$ . Then, for  $\delta > 0$ , the function  $\tilde{\mu}_1(x) + \delta|x|^2$  is strictly convex in a neighborhood

of  $\overline{D}_1$ , and if  $\delta$  is sufficiently small, the function  $\mu_1(x) = \tilde{\mu}_1(x) + \delta|x|^2 + \varepsilon/2$  satisfies the required conditions.

Next let us assume that the functions  $\mu_1, \dots, \mu_{j-1}$  have been constructed for  $j \geq 2$ . Since  $\mu_{j-1}$  is  $C^\infty$  in a neighborhood of  $\overline{D}_{j-1}$ , there is a  $C^\infty$  function  $\widehat{\mu}_{j-1} : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\widehat{\mu}_{j-1} = \mu_{j-1}$  in a neighborhood of  $\overline{D}_{j-1}$ . By Lemma 3.3, there is a  $C^\infty$  and convex function  $\tilde{\mu}_j$  in a neighborhood of  $\overline{D}_j$ , so that  $|\tilde{\mu}_j - u| < 1/2$  on  $\overline{D}_j$ . Let  $\delta > 0$  be sufficiently small so that the strictly convex  $C^\infty$

$$\widehat{\mu}_j(x) = \tilde{\mu}_j(x) + \delta|x|^2,$$

defined in a neighborhood of  $\overline{D}_j$ , satisfies  $|\widehat{\mu}_j - u| < 1/2$  on  $\overline{D}_j$ . Then (and in combination with the definition of the sets  $D_j$ )

$$\widehat{\mu}_j < j - 3/2 \text{ on } \overline{D}_{j-2} \quad \text{and} \quad \widehat{\mu}_j > j - 3/2 \text{ on } \overline{D}_j \setminus D_{j-1}.$$

Now let us take a  $C^\infty$  and convex function  $\chi : \mathbb{R} \rightarrow \mathbb{R}$  with  $\chi(t) = 0$  for  $t \leq j - 3/2$  and  $\chi' > 0$  for  $t > j - 3/2$ . For example, we may choose

$$\chi(t) = \begin{cases} \int_{s=0}^t \exp[-1/(s-j+3/2)^2] ds & \text{if } t > j - 3/2, \\ 0 & \text{if } t \leq j - 3/2. \end{cases}$$

Then  $\chi \circ \widehat{\mu}_j = 0$  on  $\overline{D}_{j-2}$  and  $\chi \circ \widehat{\mu}_j > 0$  on  $\overline{D}_j \setminus D_{j-1}$ . Moreover  $\chi \circ \widehat{\mu}_j$  is convex in a neighborhood of  $\overline{D}_j$  and strictly convex in a neighborhood of  $\overline{D}_j \setminus D_{j-1}$ . Here we used the facts that  $\widehat{\mu}_j$  is strictly convex in a neighborhood of  $\overline{D}_j$ ,  $\widehat{\mu}_j > j - 3/2$  on  $\overline{D}_j \setminus D_{j-1}$  and that  $\chi'(t) > 0$  for  $t > j - 3/2$ , to conclude that  $\chi \circ \widehat{\mu}_j$  is strictly convex in a neighborhood of  $\overline{D}_j \setminus D_{j-1}$ .

Now we claim that for a sufficiently large  $C > 0$ , the function  $\mu_j = \widehat{\mu}_{j-1} + C\chi \circ \widehat{\mu}_j$  satisfies the conditions (1)–(4). First let us observe that (4) and (2) are satisfied for every  $C > 0$ . Indeed, since  $\chi \circ \widehat{\mu}_j = 0$  and  $\widehat{\mu}_{j-1} = \mu_{j-1}$  on  $\overline{D}_{j-2}$ , we obtain (4), and (2) follows, from  $K \subset D_0 \subset D_{j-2}$ , because  $\mu_{j-1}$  satisfies (2). That (3) is satisfied for  $C > 0$ , sufficiently large, follows from the facts that  $\widehat{\mu}_{j-1} (= \mu_{j-1}) \geq u$  on  $\overline{D}_{j-1}$ ,  $\chi \circ \widehat{\mu}_j \geq 0$  on  $\overline{D}_j$ , and  $\chi \circ \widehat{\mu}_j > 0$  on  $\overline{D}_j \setminus D_{j-1}$ .

It remains to show that (1) also holds, if  $C > 0$  and sufficiently large. Since the function  $\widehat{\mu}_{j-1} (= \mu_{j-1})$  is strictly convex and  $\chi \circ \widehat{\mu}_j$  is convex in a neighborhood of  $\overline{D}_{j-1}$ , it follows that  $\mu_j$  is strictly convex in a neighborhood of  $\overline{D}_{j-1}$  for  $C > 0$ . Moreover, since  $\chi \circ \widehat{\mu}_j$  is strictly convex in a neighborhood of  $\overline{D}_j \setminus D_{j-1}$ , it follows from Lemma 3.4, that

$$\mu_j = C \left( \chi \circ \widehat{\mu}_j + \frac{1}{C} \widehat{\mu}_{j-1} \right)$$

is also strictly convex in a neighborhood of  $\overline{D}_j \setminus D_{j-1}$ , if  $C > 0$  is sufficiently large, so that  $1/C$  is small enough. Here we also used the fact that the second order derivatives of the function  $\widehat{\mu}_{j-1}$  are all bounded on a compact set.

This completes the proof of the theorem.  $\square$

**COROLLARY 3.6.** *Let  $D \subset \mathbb{R}^n$  be an open convex set. Then  $D$  can be exhausted by a sequence of 2-strictly convex sets with  $C^\infty$  boundary, that is, there exists a sequence  $D_j$  of 2-strictly convex sets with  $C^\infty$  boundary such that  $D_j \subset\subset D_{j+1} \subset\subset D$  and  $\bigcup D_j = D$ .*

**PROOF.** Assume that  $D$  is bounded. According to Theorem 3.5, there exists a  $C^\infty$  and strictly convex function  $\mu : D \rightarrow \mathbb{R}$  such that for  $a \in \mathbb{R}$ ,

$$\{x \in D : \mu(x) < a\} \subset\subset D.$$

However, the function  $\mu$ , being strictly convex, has at most one critical point in  $D$ . Therefore, the sets  $D_j = \{x \in D : \mu(x) < j\}$  have  $C^\infty$  boundary, for  $j \in \mathbb{N}$  and sufficiently large, and, since  $\mu$  is a strictly convex function, they are 2-strictly convex. Now it is clear that these sets have the required properties.

Finally if  $D$  is unbounded we may write  $D = \bigcup_{j=1}^{\infty} G_j$ , where  $G_j \subset\subset G_{j+1} \subset\subset D$  are open convex sets, and choose the 2-strictly convex sets  $D_j$  with  $C^\infty$  boundary such that  $G_j \subset\subset D_j \subset\subset G_{j+1}$ .  $\square$

## References

- [1] V. Klee. 'Some new results on smoothness and rotundity in normed linear spaces', *Math. Ann.* **139** (1959), 51–63.
- [2] R. M. Range. *Holomorphic functions and integral representations in several complex variables*, Graduate Texts in Mathematics 108 (Springer, New York, 1986).
- [3] T. Zamfirescu. 'Nearly all convex bodies are smooth and strictly convex', *Monatsh. Math.* **103** (1987), 57–62.
- [4] ———, 'Baire categories in convexity', *Atti Sem. Mat. Fis. Univ. Modena* **39** (1991), 139–164.

The University of Athens  
 Department of Mathematics  
 Panepistemiopolis 15784  
 Athens  
 Greece

e-mail: ldalla@math.uoa.gr, thatziaf@math.uoa.gr

