FEW POINTS TO GENERATE A RANDOM POLYTOPE

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Abstract. A random polytope, K_n, is the convex hull of n points chosen randomly, independently, and uniformly from a convex body $K \subseteq R^d$. It is shown here that, with high probability, K_n can be obtained by taking the convex hull of $m = o(n)$ points chosen independently and uniformly from a small neighbourhood of the boundary of K.

§1. *Introduction and results.* A random polytope K_n, inscribed in a convex body $K \subseteq R^d$ is usually defined [**2, 8**] (see also [**4**] for extensive references) as the convex hull of points $x_1, \ldots, x_n$ drawn randomly, independently and uniformly from K. With high probability most points chosen are interior to K_n and are not needed when forming the convex hull. The aim of this paper is to give this observation a more precise, quantitative form. Before proceeding, some definitions are needed.

Given $x \in K$, the Macbeath region, or M-region for short, with coefficient $\lambda > 0$ is defined as

$$M(x, \lambda) = M_K(x, \lambda) = x + \lambda[(K - x) \cap (x - K)].$$

On the convex body, we define the function $u(x)$, given by

$$u(x) = \text{vol } M(x, 1).$$

Set $K(u \geq t) = \{x \in K : u(x) \geq t\}$.

Macbeath [**7**] proved the convexity of the set $K(u \geq t)$. It is shown in [**2**] that the expectation of vol $(K \backslash K_n)$ is of the same order as vol $K(u \leq 1/n)$. This means, rough' speaking, that K_n and $K(u \geq 1/n)$ are "close".

We are interested in the case when $K_n \supseteq K(u \geq t)$, so it is natural to define

$$p(n, t) = \text{Prob } (K_n \supseteq K(u \geq t)).$$

This function is increasing both in n and t. Moreover, $p(n, 0) = 0$ and $p(n, t)$ tends to one as $n \to \infty$, for any fixed $t > 0$. Our main result shows that $p(n, t)$ gets very close to 1 when $t = \text{const } (\log n)/n$.

In what follows $c, c_1, \ldots, c(d), c_1(d), \ldots$ denote constants that depend only on d.

THEOREM 1. *For every* $\beta > 0$

$$p\left(n, \beta \frac{\log n}{n}\right) \geq 1 - c(d) n^{1 - (\beta/(d 2^{d+1}))} (\beta \log n)^{d-2}.$$

This shows that $K_n \supseteq K(u \geqslant \beta(\log n)/n)$ with high probability (if β is large enough). Assume now vol $K=1$. This does not change anything except the normalization factor. Write

$$\{y_1, \ldots, y_m\} = \{x_1, \ldots, x_n\} \cap K\left(u \leqslant \beta \frac{\log n}{n}\right).$$

The points $y_1, \ldots, y_m$ form a random sample of size m from $K(u \leqslant \beta(\log n)/n)$ and $K_n = \operatorname{conv}\{y_1, \ldots, y_m\}$ with probability $p(n, \beta(\log n)/n)$; thus the number of points that generate K_n is less than n.

The number m of points needed to form K_n, is a random variable following the binomial distribution with parameters n and $p = \operatorname{vol} K(u \leqslant \beta(\log n)/n)$. It is a consequence of the affine isoperimetric inequality (see [**2**]) that

$$\operatorname{vol} K(u \leqslant \varepsilon) \leqslant c_1(d)\, \varepsilon^{2/(d+1)},$$

for any convex body $K \subseteq R^d$ (with vol $K=1$) and for every $\varepsilon > 0$. Then

$$p \leqslant c_1(d) \left(\beta \frac{\log n}{n}\right)^{2/(d+1)}.$$

Now to generate K_n with few points (and high probability) the following two-step random procedure can be applied. Fix n large, determine p and choose $m \in (0, \ldots, n\}$ according to binomial distribution $\binom{n}{m} p^m (1-p)^{n-m}$ (notice that m is concentrated around its expectation np, so it is much less than n). Select m points $y_1, \ldots, y_m$ randomly, independently and uniformly from $K(u \leqslant \beta(\log n)/n))$. Then $\operatorname{conv}\{y_1, \ldots, y_m\}$ is a random polytope K_n with probability $p(n, \beta(\log n)/n)$ which is large by Theorem 1.

The expectation of the Hausdorff distance between K and K_n is of order $((\log n)/n)^{2/(d+1)}$ when K is smooth enough (see [**1**]) while the Hausdorff distance between K and $K(u \geqslant t)$ is of order $t^{2/(d+1)}$ (more precisely information is available when $d=2$ (see [**3**])). This shows that the order of magnitude of $t = \beta(\log n)/n$ in Theorem 1 cannot be improved.

Our next theorem proves this for all convex bodies, not only for the smooth ones.

THEOREM 2. *For every $\beta > 0$ and large enough n*

$$p\left(n, \beta \frac{\log n}{n}\right) \leq 1 - n^{-2(3d)^d \beta}.$$

The exponent here can be replaced by

$$-\frac{d^d \beta}{[(d-1)^2+1]2^d}$$

in the case of polytopes.

The case of a polytope is the content of Lemma 2 which also improves an old result of Levi [**6**] about the maximum volume of a symmetric subset of a convex body (see also [**5**] for further information).

§2. *Auxiliary lemmas.*

LEMMA 1. *Assume $z \in K$ and $\eta > 0$. Then*
(i) $M(z, \frac{1}{2}) \subseteq K(u \geqslant u(z)/(d2^d))$ *and*
(ii) *if* $K(u \geqslant \eta) \setminus K_n \neq \emptyset$ *then* $vol\,(K(u \geqslant \eta/(d2^d))\eta \setminus K_n) \geqslant 1/(2^{d+1})\eta$.

Proof. (i) Let $x \in M(z, \frac{1}{2})$ with $x = \alpha y + (1-\alpha)z$ for some $0 \leqslant \alpha \leqslant \frac{1}{2}$ and $y \in \mathrm{bd}\, K$ with $2z - y \in K$. Let H_y be a supporting hyperplane of K at y and denote by H_x and H_z the hyperplanes parallel to H_y that pass through x and z respectively.

As $x = \alpha y + (1-\alpha)z$ and $y \in M(x, 1)$ we can easily prove that the pyramid $B_x = \mathrm{conv}\,(\{y\} \cup (M(z, 1) \cup H_x))$ is a subset of $M(x, 1)$. So

$$u(x) = \mathrm{vol}\, M(x, 1) \geqslant 2\, \mathrm{vol}\,(B_x). \tag{1}$$

On the other hand the pyramid $B_z = \mathrm{conv}\,(\{y\} \cup (M(z, 1) \cap H_z))$ is a subset of $M(z, 1)$ with

$$\mathrm{vol}\,(B_z) \geqslant \frac{1}{2d} u(z). \tag{2}$$

Comparing the volumes of the above pyramids which have common vertex at y and parallel bases, using (1) and (2) we obtain

$$u(x) \geqslant 2\, \mathrm{vol}\,(B_x) = 2 \left\| \frac{y-x}{y-z} \right\|^d \mathrm{vol}\,(B_z) \geqslant 2(1-\alpha)^d\, \mathrm{vol}\,(B_z) \geqslant \frac{1}{d2^d} u(z).$$

Thus $x \in K(u \geqslant u(z)/(d2^d))$. As $M(z, \frac{1}{2})$ is centrally symmetric, this completes the proof of (i).

(ii) Let $z \in K(u \geqslant \eta) \setminus K_n$. Then there exists a half-space H with $z \in \mathrm{bd}\, H$ and

$$M(z, \tfrac{1}{2}) \cap H \subseteq K \setminus K_n. \tag{3}$$

By (i) and (3) we have that

$$M(z, \tfrac{1}{2}) \cap H \subseteq K\left(u \geqslant \frac{1}{d2^d}\, \eta\right) \setminus K_n,$$

and

$$\mathrm{vol}\left(M\left(u \geqslant \frac{1}{d2^d}\, \eta\right) \setminus K_n\right) \geqslant \frac{1}{2}\, \mathrm{vol}\left(M\left(z, \frac{1}{2}\right)\right)$$

$$= \frac{1}{2^{d+1}}\, \mathrm{vol}\, M(z, 1) = \frac{1}{2^{d+1}} u(z) \geqslant \frac{1}{2^{d+1}}\, \eta.$$

LEMMA 2. *Let B be a convex compact set lying in a hyperplane H of R^d, with y_0 as its centre of gravity. Assume $P = \mathrm{conv}\,(B \cup \{x_0\})$ where $x_0 \notin H$. Then*

(i) *the set $M_P(\frac{1}{2}(x_0 + y_0), 1)$ contains the pyramid C whose vertex is at x_0, its basis is parallel to B and is passing through a point of the line segment $[x_0, y_0]$ at a distance $(1/d)\|x_0 - y_0\|$ from x_0; and*

(ii) $$\mathrm{vol}\, P \leqslant \frac{1}{(d-1)^2 + 1}\, d^d u(\tfrac{1}{2}(x_0 + y_0)).$$

Proof. For the sake of simplicity we assume $x_0=(0,\ldots,0,1)$ and the basis B to be on the hyperplane $x_d=-1$ with centre of gravity $y_0=(0,\ldots,0,-1)$.

(i) The statement is trivial when $d=2$. So assume $d\geqslant 3$ and let C be the pyramid with vertex at x_0 and basis parallel to B passing through $(0,\ldots,0,1-(2/d))$. We have to prove that $C\subseteq M(0,1)=P\cap(-P)$. Suppose it does not hold. As $C\subseteq P$ there exists a point $y\in C$ of the form $y=(-\alpha,1-(2/d))$ with $\alpha\in R^{d-1}$ and such that $-y\notin P$.

Denote by H_x, the hyperplane parallel to the basis B and passing through $x\in[x_0,y_0]$. As x_0 belongs to the cone C and $-y\notin P$, the point

$$z=\left(\frac{1}{d-1}\alpha,\ 1-\frac{2}{d}\right)=\frac{d-2}{d-1}x_0+\frac{1}{d-1}(-y)\notin P$$

and belongs to $H_{(1-(2/d))x_0}$. The point $y_0=(0,\ldots,0,-1)$ is the centre of gravity of B so the point $(0,1-(2/d))$ is the centre of gravity of the $(d-1)$-dimensional convex set $H_{(1-(2/d))x_0}\cap P$ which contains $y=(-\alpha,1-(2/d))$.

From a well known result it follows that the point

$$z=\left(\frac{1}{d-1}\alpha,\ 1-\left(\frac{2}{d}\right)\right)$$

belongs to P. This contradiction proves part (i).

(ii) For the symmetric body $M(0,1)$ we have that

$$\operatorname{vol}(M(0,1)\cap H_y)\geqslant\operatorname{vol}(M(0,1)\cap H_{(1-(2/d))x_0})=\operatorname{vol}(M(0,1)\cap H_{(-1+(2/d))x_0}),$$

for any point $y\in[(-1+(2/d))x_0,(1-(2/d))x_0]$. Hence the volume of the part of $M(0,1)$ lying between $H_{(-1+(2/d))x_0}$ and $H_{(1-(2/d))x_0}$ is at least $d(d-2)\operatorname{vol}C$.

As $\operatorname{vol}C=(2/d^2)\operatorname{vol}(H_{(1-(2/d))x_0}\cap P)$ we conclude from (i) that

$$u(0)\geqslant(d(d-2)+2)\operatorname{vol}C=((d-1)^2+1)\frac{1}{d^d}\operatorname{vol}P$$

The validity of (ii) now follows since the ratio $u(\frac{1}{2}(x_0+y_0))/\operatorname{vol}P$ is invariant under affine transformations.

Let $K\subseteq R^d$ be a convex body with g its centre of gravity and let $F_g(K)=\operatorname{vol}M(g,1)/\operatorname{vol}K$. It is known that for all $K\subseteq R^2$, $F_g(K)\geqslant\frac{2}{3}$ (see **[5]** for references). When we are in R^d with $d>2$ then $F_g(K)>2/(1+d^d)$ (**[6]**). With the help of Lemma 2, a better bound will be established.

COROLLARY 1.

$$F_g(K)>\left(\frac{2}{1+d}\right)^d.$$

Proof. Let K be embedded in R^{d+1}, lying on $x_{d+1}=0$ with $g=0$. We construct a cone P with vertex at $x_0=(0,1)$ and basis on $x_{d+1}=-1$ in such a way that $P\cap(x_{d+1}=0)=K$. Then the intersection of the Macbeath region $M(0,1)$ of P with $x_{d+1}=0$ is $M(g,1)$ of K. By Lemma 2 (i) and the Brünn–Minkowski inequality applied on the symmetric set $M(0,1)$ it follows that

$$\operatorname{vol}M(g,1)>\operatorname{vol}P\cap\left\{x\in R^{d+1}: x_{d+1}=1-\frac{1}{d+1}\right\}=\left(\frac{2}{d+1}\right)^d\operatorname{vol}K.$$

§3. *Proof of Theorem* 1. The method of Bárány–Larman [2] is used to establish an upper bound for the expectation of $\mathrm{vol}\,(K(u \geq \delta)\backslash K_n)$ for $\delta > 0$ with $1 < [\delta n] < n$. So

$$E(\mathrm{vol}\,(K(u \geq \delta)\backslash K_n) = \int_{K(u\geq\delta)} \mathrm{Prob}\,(x \notin K_n) dx$$

$$\leq \int_{K(u\geq\delta)} 2 \sum_{i=0}^{d-1} \binom{n}{i} \left(\frac{u(x)}{2}\right)^i \left(1 - \frac{u(x)}{2}\right)^{n-i} dx$$

$$\leq 2 \sum_{i=0}^{d-1} \sum_{\lambda=[\delta n]}^{n} \int_{(\lambda-1)/n \leq u(x) \leq \lambda/n} \binom{n}{i} \left(\frac{u(x)}{2}\right)^i \left(1 - \frac{u(x)}{2}\right)^{n-i} dx$$

$$\leq 2 \sum_{i=0}^{d-1} \sum_{\lambda=[\delta n]}^{n} \binom{n}{i} \left(\frac{\lambda}{2n}\right)^i_i \left(1 - \frac{\lambda-1}{2n}\right)^{n-1} \mathrm{vol}\left(K\left(u \leq \frac{\lambda}{n}\right)\right)$$

$$\leq 2 \sum_{\lambda=[\delta n]}^{n} \sum_{i=0}^{d-1} \frac{\lambda^i}{i!} e^{-(\lambda-1)/2} \leq 2e^{3/2} \sum_{\lambda=[\delta n]}^{n} \lambda^{d-1} e^{-\lambda/2}$$

$$\leq 2e^{3/2} \int_{[\delta n]}^{n} \lambda^{d-1} e^{-\lambda/2} d\lambda \leq c_0(d) e^{-\delta n/2} (\delta n)^{d-1} \tag{4}$$

for n large enough. In the proof of the above the following inequalities were used:

$$\binom{n}{i}\left(\frac{\lambda}{2n}\right)^i \leq \frac{\lambda^i}{2^i i!}, \qquad \left(1 - \frac{\lambda-1}{2n}\right)^{-i} \leq 2^i, \qquad \left(1 - \frac{\lambda-1}{2n}\right)^n \leq e^{-(\lambda-1)/2};$$

and

$$\int_\gamma^\infty t^k e^{-t} dt = e^{-\gamma}\left(\gamma^k + \sum_{i=0}^{k-1} k(k-1)\ldots(k-i)\gamma^{k-1-i}\right) \leq (k+1)e^{-\gamma}\gamma^k \quad \text{for} \quad \gamma > k.$$

Setting $\delta n = (\beta/(d2^d)) \log n$ in (4), we conclude that

$$E\left(\mathrm{vol}\left(K\left(u \geq \frac{\beta}{d2^d} \frac{\log n}{n}\right)\backslash K_n\right)\right) \leq c_1(d) n^{-\beta(d2^{d+1})} (\beta \log n)^{d-1} \tag{5}$$

Using now the Markov inequality, Lemma 1 (ii) with $\eta = (\beta \log n)/n$ and (5) we find

$$\mathrm{Prob}\left(K\left(u \geq \frac{\log n}{n}\right)\backslash K_n \neq \varnothing\right) \leq \mathrm{Prob}\left(\mathrm{vol}\left(K\left(u \geq \frac{\beta}{d2^d}\frac{\log n}{n}\right)\backslash K_n\right) \geq \frac{\beta}{2^{d+1}} \frac{\log n}{n}\right)$$

$$\leq E\left(\mathrm{vol}\left(K\left(u \geq \frac{\beta}{d2^d}\frac{\log n}{n}\right)\backslash K_n\right)\right) \Big/ \frac{\beta}{2^{d+1}} \frac{\log n}{n}$$

$$\leq c(d) n^{1-(\beta/(d2^{d+1}))} (\beta \log n)^{d-2}. \tag{6}$$

§4. *Proof of Theorem* 2. Let $x \in K$ and denote by $v(x) = \min\{\mathrm{vol}\,(K \cap H) : x \in H, H \text{ a half space}\}$. Then $C_K(x)$ is a minimal cap if $C_K(x) = K \cap H$, $x \in H$ and its volume is $v(x)$. For sufficiently small value of $v(x)$ we have that $v(x) \leqslant (3d)^d u(x)$ (Lemma 2, [2]). Then

$$\begin{aligned} 1 - p\left(n, \beta \frac{\log n}{n}\right) &= \mathrm{Prob}\left(K\left(u \geqslant \beta \frac{\log n}{n}\right) \setminus K_n \neq \varnothing\right) \\ &\geqslant \mathrm{Prob}\left(\text{there exists } x : u(x) = \beta \frac{\log n}{n} \text{ and } x \notin K_n\right) \\ &\geqslant \mathrm{Prob}\left(\text{there exists } x : u(x) = \beta \frac{\log n}{n} \text{ and } C_K(x) \cap K_n = \varnothing\right) \\ &\geqslant (1 - v(x))^n \\ &\geqslant (1 - (3d)^d u(x))^n \geqslant e^{-2(3d)^d \beta \log n} \\ &= n^{-2(3d)^d \beta}. \end{aligned}$$

Now we establish a better bound in the case when K is a polytope. Let x_0 be a vertex of K and Q be the minimal cone with vertex at x_0 containing K. The minimal caps for K and Q are the same for suitable points of K near x_0. Hence we may suppose that there exists a hyperplane H_{x_0} supporting K at x_0 such that

$$C_K(x) = H(x_0, x) \cap K = H(x_0, x) \cap Q = C_Q(x)$$

for $x \in \mathrm{int}\, K$, near x_0, where $H(x_0, x)$ is the slab between the parallel hyperplanes H_{x_0} and H_x with $x \in H_x$. We may also suppose that the only vertex of K contained in the slab $H(x_0, 2x - x_0)$ is x_0. Hence the set $H(x_0, 2x - x_0) \cap Q$ is just like the set P in Lemma 2 with centre of gravity of its basis at $2x - x_0$ and $C_K(y) = H(x_0, y) \cap Q$ for $y \in [x_0, 2x - x_0]$. This implies as is Lemma 2 (ii) that

$$v(x) \leqslant \frac{1}{(d-1)^2 + 1} \left(\frac{d}{2}\right)^d u(x)$$

Using now the same argument as in the general case, we obtain

$$1 - p\left(n, \beta \frac{\log n}{n}\right) \geqslant n^{[2\beta/((d-1)^2+1)](d/2)^d}.$$

Corollary 2. *For a convex body K in R^d with a simple vertex (one that lies in exactly d facets), the following inequality holds*:

$$p\left(n, \beta \frac{\log n}{n}\right) \leqslant 1 - n^{(2\beta/d!)(d/2)^d}.$$

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