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Volumes of a Random Polytope in a Convex Set

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ABSTRACT. The classical result of Blaschke on the expected areas of triangles in a plane convex body is extended to the expected areas of the convex hull of n points in a plane convex body.

Let K be a convex body in Euclidean d-space R^d. If n points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are randomly and independently selected from K, the convex hull $K(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ of these points can be interpreted as a random polytope with at most n vertices. The expected value of the volume of this polytope is defined by

$$m(n, K) = (\mathrm{Vol}\, K)^{-n} \int_{\mathbf{x}_1 \in K} \cdots \int_{\mathbf{x} \in K} \mathrm{Vol}(K(\mathbf{x}_1, \ldots, \mathbf{x}_n))\, d\mathbf{x}_1, \ldots, d\mathbf{x}_n.$$

In **[1]**, H. Groemer has shown that for fixed volume $m(n, K)$ attains its minimum value when, and only when, K is an ellipsoid. In **[2]**, I. Barany and D. G. Larman show that for n large

$$(1) \qquad (1 - c_1 n^{-2/d+1})\, \mathrm{Vol}\, K \le m(n, k) \le \left(1 - c_2 \frac{(\log n)^{d-1}}{n}\right) \mathrm{Vol}\, K,$$

where c_1 depends on d and c_2 depends on K. Further, for n large, polytopes behave like the r.h.s. of (1) and ellipsoids like the l.h.s. of (1).

Consequently it is reasonable to conjecture that for fixed volume the maximum of $m(n, K)$ will be obtained at a polytope and perhaps even at a simplex. We shall prove that for $d = 2$ the maximum is attained at a triangle and, in general, that $m(n, P)$ is a maximum taken over all d-polytopes with at most $d + 2$ vertices when P is a d-simplex. It is worth remarking that Blaschke **[5]** showed that the maximum of $m(3, K)$ in two dimensions is attained exactly when K is a triangle. A comprehensive survey is contained in a recent article by R. Schneider **[6]**.

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THEOREM 1. *Let K be a convex body in R^2. Then, if T^2 denotes a triangle of the same area as K,*

$$m(n, K) \leq m(n, T^2)$$

with strict inequality if K is any polygon other than a triangle.

REMARK. It seems reasonable to conjecture that equality holds in Theorem 1 only if K is a triangle. Also the methods of Theorem 1 can be applied to show that $m(n, C) \leq m(n, T^3)$ for all convex cylinders C in R^3 of the same volume as the simplex T^3.

THEOREM 2. *Let P be a d-polytope in R^d with at most $d+2$ vertices and let T^d be a d-simplex of the same volume as P. Then*

$$m(n, P) \leq m(n, T^d)$$

with equality if and only if P is a d-simplex.

REMARK. The methods used in Theorem 2 can be applied to many other polytopes. The first case in which some modification of the method cannot obviously be used seems to be the dodecahedron in R^3.

LEMMA 1. *If n, p are positive integers greater than 2, then there exists a convex polygon K_0, of area 1, with at most p vertices and*

$$m(n, K) \leq m(n, K_0)$$

for any other convex polygon K, of area 1, with at most p vertices.

PROOF. By a result of F. John [3], we may suppose that each convex polygon K considered contains a disk of radius $\frac{1}{3}\sqrt{2/\sqrt{3}}$ and is contained in a concentric disk of radius $\frac{2}{3}\sqrt{2/\sqrt{3}}$. So the convex bodies being considered form a compact metric space in the Hausdorff metric. As $m(n, K)$ is a continuous function of K in the Hausdorff metric, Lemma 1 follows.

REMARK. The existence of maximal bodies from a given class in other situations considered in Theorem 2 may be proved in a similar way.

We next need a lemma proved by H. Groemer [1]. If $\mathbf{q} = (q_1, \dots, q_{d-1})$ is a point of R^{d-1} and if $z \in \mathbb{R}$, the point $(q_1, \dots, q_{d-1}, z)$ of R^d will be denoted by $(\mathbf{q}, z)$.

LEMMA 2. *Let H be the hyperplane which consists of all points $x_d = 0$ and let $\mathbf{c}_1, \dots, \mathbf{c}_n$ be n given point in H, where $n > d$. If $z_1, \dots, z_n$ are real numbers, put $Z = (z_1, \dots, z_n)$ and*

$$V(Z) = \mathrm{Vol}(\mathrm{conv}((\mathbf{c}_1, z_1), \dots, (\mathbf{c}_n, z_n))). \tag{2}$$

Then, if Z' and Z'' are two points of R^n,

$$V(\tfrac{1}{2}Z' + \tfrac{1}{2}Z'') \leq \tfrac{1}{2}V(Z') + \tfrac{1}{2}V(Z'').$$

For completeness we give the proof of Lemma 2.

PROOF. Let

$$K = \operatorname{conv}((\mathbf{c}_1', \tfrac{1}{2}z_1' + \tfrac{1}{2}z_1''), \dots, (\mathbf{c}_n, \tfrac{1}{2}z_n' + \tfrac{1}{2}z_n'')),$$
$$K' = \operatorname{conv}((\mathbf{c}_1, z_1'), \dots, (\mathbf{c}_n, z_n')),$$
$$K'' = \operatorname{conv}((\mathbf{c}_1, z_1''), \dots, (\mathbf{c}_n, z_n'')).$$

The sets K, K', K'' have the same orthogonal projection $C = \operatorname{conv}(\mathbf{c}_1, \dots, \mathbf{c}_n)$ on H. If $\mathbf{x} \in C$, let $(\mathbf{x}, \overline{x}')$ and $(\mathbf{x}, \underline{x}')$ denote the upper and lower points in K', and similarly let $(\mathbf{x}, \overline{x}'')$ and $(\mathbf{x}, \underline{x}'')$ denote the upper and lower points in K''. Let K^* denote the convex body whose upper and lower points are $(\mathbf{x}, \frac{1}{2}(\overline{x}'+\overline{x}''))$ and $(\mathbf{x}, \frac{1}{2}(\underline{x}'+\underline{x}''))$, respectively, $\mathbf{x} \in C$. Then

$$\operatorname{Vol} K^* = \int (\tfrac{1}{2}(\overline{x}' - \underline{x}') + \tfrac{1}{2}(\overline{x}'' - \underline{x}''))\, d\mathbf{x} = \tfrac{1}{2}V(Z') + \tfrac{1}{2}V(Z'').$$

Also, as $(\mathbf{c}_i, \frac{1}{2}z_i' + \frac{1}{2}z_i'') \in K$, $i = 1, \dots, n$,

$$K \subset K^*,$$

and hence $\operatorname{Vol} K = V(\frac{1}{2}Z' + \frac{1}{2}Z'') \leq \frac{1}{2}V(Z') + \frac{1}{2}V(Z'')$, as required.

PROOF OF THEOREM 1. We shall assume that all bodies considered have area 1. By Lemma 1, let Q be a polygon with at most p vertices with $m(n, K) \leq m(n, Q)$ for any other polygon K, of area 1, with at most p vertices.

We shall show that Q is a triangle. Suppose not. Then Q has vertices $P_1P_2P_3\cdots P_r$, $r \geq 4$, with $P_1P_2, P_2P_3, \dots, P_rP_1$ consecutive edges. We may suppose that there is a Cartesian coordinate system (x_1, x_2) with P_2P_r in the positive direction of the x_2-axis and $P_1 \equiv (0, 0)$. The line P_3P_2 extended will cut the x_2-axis at $(0, -\beta)$ and the line $P_{r-1}P_r$ extended will cut the x_2-axis at $(0, \alpha)$. We suppose, without loss of generality, that $0 < \alpha \leq \beta$.

Let $P_1^+ = (0, \alpha)$ and $P_1^- = (0, -\alpha)$. Consider now the two polygons

$$Q^+ = \operatorname{conv}(P_1^+, P_2, \dots, P_{r-1}), \qquad Q^- = \operatorname{conv}(P_1^-, P_2, \dots, P_r).$$

Notice that Q^+ has only $r-1$ vertices and that Q^- has r (or possibly $r-1$ if $\alpha = \beta$) vertices. By the definition of Q,

$$m(n, Q^+) \leq m(n, Q), \qquad m(n, Q^-) \leq m(n, Q). \tag{3}$$

Now Q, Q^+, and Q^- have the same orthogonal projection $(0, \gamma)$ onto the x_1-axis. Let (P_2, P_r) extended meet the x_1-axis in $(0, \delta)$, where $0 < \delta < \gamma$. Let $c_1, \dots, c_n$ be any choice of n numbers in the interval $[0, \gamma]$. Let the vertical line through $(c_i, 0)$ meet Q, Q^-, Q^+ in the intervals $[(c_i, \alpha_i - l_i), (c_i, \alpha_1 + l_i)]$, $[(c_i, \alpha_i^- - l_i), (c_i, \alpha_i^- + l_i)]$, $[(c_i, \alpha_i^+ - l_i), (c_i, \alpha_i^+ + l_i)]$, respectively, $i = 1, \dots, n$. Of course, if $\delta \leq c_i \leq \gamma$, then all three intervals are equal. In any case, all three intervals will have the same length. Let

t_i, $i = 1, \ldots, n$, satisfy $|t_i| \le l_i$ and consider the following points, where $\mathbf{t} = (t_i, \ldots, t_n)$:

$$Z(\mathbf{t}) = ((\alpha_1 + t_1), \ldots, (\alpha_n + t_n)),$$
$$Z^+(\mathbf{t}) = ((\alpha_1^+ + t_1), \ldots, (\alpha_n^+ + t_n)),$$
$$Z^-(\mathbf{t}) = ((\alpha_1^- + t_1), \ldots, (\alpha_n^- + t_n)),$$

Then $Z(\mathbf{t}) = \frac{1}{2}Z^+(\mathbf{t}) + \frac{1}{2}Z^-(\mathbf{t})$ and so, by Lemma 2,

$$V(Z(\mathbf{t})) \le \tfrac{1}{2}V(Z^+(\mathbf{t})) + \tfrac{1}{2}V(Z^-(\mathbf{t})). \tag{4}$$

Consequently, integrating (4) over $t_1, \ldots, t_n$ and then over $c_1, \ldots, c_n$, we conclude from (4) that

$$m(n, Q) \le \tfrac{1}{2}m(n, Q^+) + \tfrac{1}{2}m(n, Q^-). \tag{5}$$

However, suppose P_s is a vertex of Q whose orthogonal projection onto the x_1-axis is $(0, \gamma)$ and we choose points $(c_1, t_1), \ldots, (c_n, t_n)$ on (P_1, P_s) with $c_1 < \delta$, $c_n > \delta$. Then, if $\mathbf{t}^* = (t_1^*, \ldots, t_n^*)$, $V(Z(\mathbf{t})) = 0$, $V(Z^+(\mathbf{t}^*)) > 0$, and $V(Z^-(\mathbf{t}^*)) > 0$. So, by continuity, we have an improvement on (5) to

$$m(n, Q) < \tfrac{1}{2}m(n, Q^+) + \tfrac{1}{2}m(n, Q^-). \tag{6}$$

Thus (6) contradicts the maximality of $m(n, Q)$. Consequently Q is a triangle. Finally, since any convex body in R^2 can be approximated arbitrarily closely from within and without by polygons, we conclude that Theorem 1 holds.

PROOF OF THEOREM 2. We first note that every d-polytope P with $d + 2$ vertices is either (i) pyramidal about one of its vertices or (ii) (affinely equivalent to) the convex hull of an r-simplex T^r and a $(d-r)$-simplex T^{d-r}, $1 \le r \le d-1$, in orthogonal r- and $(d-r)$-dimensional subspaces, respectively, such that $\mathbf{0} \in \operatorname{relint} T^r \cap \operatorname{relint} T^{d-r}$ (see Grünbaum **[4, p. 97]**).

So let P be a d-polytope of unit volume with at most $d + 2$ vertices at which $\max m(n, P')$ is achieved, where the maximum is taken over all d-polytopes P' of unit volume with at most $d+2$ vertices. We suppose that P has form (ii) and hence show that there exists another polytope P^* of form (i) at which the maximum is also achieved.

(7)

We suppose that T^r is embedded in the r-space determined by the first r coordinates and that T^{d-r} is embedded in the $(d-r)$-subspace determined by the last $d-r$ coordinates. Let $\mathbf{e}_d$ denote the dth unit vector, which in our case lies in $\operatorname{aff} T^{d-r}$. Consider $T^r + t\mathbf{e}_d$, t real. There exists $\alpha > 0$ such that $T^r + t\mathbf{e}_d$ meets the relative interior of T^{d-r} for $|t| < \alpha$ but (say) $T^r + \alpha\mathbf{e}_d$ meets the relative boundary of T^{d-r}.

We shall show that

$$P^+ = \operatorname{conv}(T^r + \alpha \mathbf{e}_d, T^{d-r})$$

satisfies $m(n, P^+) = m(n, P)$ which will establish (5). Let

$$P^- = \operatorname{conv}(T^r - \alpha \mathbf{e}_d, T^{d-r})$$

and let Q denote the orthogonal projection of P, P^+, and P^- onto the $(d-1)$-subspace $x_d = 0$. Let $\mathbf{q} = (q_1, \dots, q_{d-1}, 0) \in Q$. Then the line $\ell_q = \{(x_1, \dots, x_d): x_1 = q_1, \dots, x_{d-1} = q_{d-1}\}$ meets P in a line segment determined by $q_d^- \le x_d \le q_d^+$, say, and $q^- = (q_1, \dots, q_{d-1}, q_d^-)$ and $q^+ = (q_1, \dots, q_{d-1}, q_d^+)$ are boundary points of P. Now every boundary point $\mathbf{z}$ of P has the unique representation $\mathbf{z} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$, where $\mathbf{x} \in \operatorname{relbd} T^r$ and $y \in \operatorname{relbd} T^{d-r}$. Consequently $\mathbf{q}^- = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$, for unique $\mathbf{x} \in \operatorname{relbd} T^r$ and $\mathbf{y} \in \operatorname{relbd} T^{d-r}$, and $\mathbf{q}^+ = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}'$, for unique $\mathbf{y}' \in \operatorname{relbd} T^{d-r}$, $\mathbf{y}' - \mathbf{y}$ in the direction of $\mathbf{e}_d$.

Consider P^+ and the same point $\mathbf{q} \in Q$. The line ℓ_q meets P^+ in a line segment determined by $q_d^{-+} \le x_d \le q_d^{++}$, and $\mathbf{q}^{-+} = (q_1, \dots, q_{d-1}, q_d^{-+})$, and $\mathbf{q}^{++} = (q_1, \dots, q_{d-1}, q^{++})$ are boundary points of P^+. Then

$$\mathbf{q}^{-+} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y} + \lambda\alpha\mathbf{e}_d, \qquad \mathbf{q}^{++} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}' + \lambda\alpha\mathbf{e}_d$$

and hence

$$q_d^{-+} = q_d^- + \lambda\alpha, \qquad q_d^{++} = q_d^+ + \lambda\alpha. \tag{8}$$

Similarly, for the corresponding q^{--}, q^{+-} of P^-,

$$q_d^{--} = q_d^- - \lambda\alpha, \qquad q_d^{+-} = q_d^+ - \lambda\alpha. \tag{9}$$

So, using (8) and (9), the line ℓ_q cuts P^-, P^+, and P in line segments of the same length, with P^- and P^+ also having unit volume. Thus if $\mathbf{q}_1, \dots, \mathbf{q}_n$ is any choice of points in Q, if $Z = (z_1, \dots, z_n)$, where $z_i = q_{id}^- + t_i$, $0 \le t_i \le q_{id}^+ - q_{id}^-$, $i = 1, \dots, n$, if $Z^+ = (z_1^+, \dots, z_n^+)$, where $z_i^+ = q_{id}^- + t_i + \lambda\alpha$, $i = 1, \dots, n$, and if $Z^- = (z_1^-, \dots, z_n^-)$, where $z_i^- = q_{id}^- + t_i - \lambda\alpha$, $i = 1, \dots, n$, then $Z = \frac{1}{2}Z' + \frac{1}{2}Z''$ and so, by Lemma 2,

$$V(Z) \le \tfrac{1}{2}V(Z') + \tfrac{1}{2}V(Z'').$$

Hence, as in Theorem 1,

$$m(n, P) \le \tfrac{1}{2}m(n, P^+) + \tfrac{1}{2}m(n, P^-)$$

and hence $m(n, P^+) = m(n, P^-) = m(n, P)$, which establishes (7).

We repeat this process within the base of the polytope P which is pyramidal about one of its vertices and for which $m(n, P)$ is maximal until the base becomes a (two-dimensional) square. A further application reduces the square to a triangle and hence P to a d-simplex as required. We may produce strict inequality by arguing as in (6) of the proof of Theorem 1.

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