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Asymptotic Behavior of Jensen and Jensen Type Functional Equations

John Michael Rassias
National and Capodistrian University of Athens
Pedagogical Department E.E., Section of Mathematics
4, Agamemnonos Str, Aghia Paraskevi, Athens 15342,Greece
jrassias@primedu.uoa.gr

Matina John Rassias
University of Strathclyde
Statistics and Modelling Science, Livingstone Tower
26 Richmond Str, Glasgow, UK, G1 1XH
matina@stams.strath.ac.uk

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Abstract

In 1941 D.H. Hyers solved the well-known Ulam stability problem for linear mappings. In 1951 D.G. Bourgin was the second author to treat the Ulam problem for additive mappings. In 1982-2004 we established the Hyers-Ulam stability for the Ulam problem of linear and nonlinear mappings. In 1983 F. Skof was the first author to solve the Ulam problem for additive mappings on a restricted domain. In 1998 S.-M. Jung and in 2002-2004 the authors of this paper investigated the Hyers-Ulam stability of additive and quadratic mappings on restricted domains. In this paper we improve our bounds and thus our results obtained ,in 2003 for Jensen and Jensen type mappings and establish new theorems about the Ulam stability of additive mappings of the first (and the second) form on restricted domains. Finally, we apply our recent results to the asymptotic behavior of functional equations of these types.

1 Introduction

In 1940 and in 1964 S. M. Ulam [26] proposed the *general Ulam stability problem*:

"When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true ?"

In 1941 D.H. Hyers [13] solved this problem for linear mappings. In 1951 D.G. Bourgin [3] was the second author to treat the Ulam problem for additive mappings. In 1978, according to P.M. Gruber [12], this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1978 Th.M. Rassias [24] employed Hyers' ideas to new linear mappings. In 1980 and in 1987, I. Fenyö [7, 8] established the stability of the Ulam problem for quadratic and other mappings. In 1987 Z. Gajda and R. Ger [10] showed that one can get analogous stability results for subadditive multifunctions. Other interesting stability results have been achieved also by the following authors : J. Aczél [1], C. Borelli and G.L. Forti [2, 9], P.W. Cholewa [4], St. Czerwik [5], and H. Drljevic [6], and Pl. Kannappan [15]. In 1982-2004 J.M. Rassias ([16-21],[23]) and in 2003 the authors [22] solved the above Ulam problem for different mappings. In 1999 P. Gavruta [11] answered a question of ours [18] concerning the stability of the Cauchy equation. In 1983 F. Skof [25] was the first author to solve the Ulam problem for additive mappings on a restricted domain. In 1998 S.-M. Jung [14] and in 2002-2003 the authors [21,22] investigated the Hyers-Ulam stability for additive and quadratic mappings on restricted domains. In this paper we improve our bounds and thus our results obtained, in 2003 for Jensen and Jensen type mappings and establish new theorems about the Ulam stability of additive mappings of the first (and the second) form on restricted domains. Finally, we apply our recent results to the asymptotic behavior of functional equations of these types.

Throughout this paper, let X be a real normed space and Y be a real Banach space in the case of functional inequalities, as well as let X and Y be real linear spaces for functional equations.

Definition 1.1. A mapping $A : X \to Y$ is called *additive of the first form* if A satisfies the functional equation

$$A(x_1 + x_2) + A(x_1 - x_2) = 2A(x_1) \tag{1.1}$$

for all $x_1, x_2 \in X$. We note that (1.1) is equivalent to *the Jensen equation*

$$A\left(\frac{x+y}{2}\right)=\frac{1}{2}\left[A(x)+A(y)\right], \tag{1.1a}$$

or

$$2A\left(\frac{x+y}{2}\right)=A(x)+A(y) \tag{1.1b}$$

for $x = x_1 + x_2$, $y = x_1 - x_2$. A mapping $A : X \to Y$ is called a *Jensen mapping* if A satisfies functional equation (1.1a) (or (1.1b)).

Definition 1.2. A mapping $A : X \to Y$ is called *additive of the second form* if A satisfies the functional equation

$$A(x_1 + x_2) - A(x_1 - x_2) = 2A(x_2) \tag{1.2}$$

for all $x_1, x_2 \in X$. We note that (1.2) is equivalent to *the Jensen type equation*

$$A\left(\frac{x-y}{2}\right)=\frac{1}{2}\left[A(x)-A(y)\right], \tag{1.2a}$$

or

$$2A\left(\frac{x-y}{2}\right)=A(x)-A(y) \tag{1.2b}$$

for $x = x_1 + x_2$, $y = x_1 - x_2$. A mapping $A : X \to Y$ is called a *Jensen type mapping* if A satisfies functional equation (1.2a) (or (1.2b)).

Definition 1.3. A mapping $f : X \to Y$ is called *approximately odd* if f satisfies the functional inequality

$$\|f(x)+f(-x)\|\le\theta \tag{1.3}$$

for some fixed $\theta \ge 0$ and for all $x \in X$.

In this section we state Theorem 1.1 which was proved by the first author of this paper [19], in 1994.

Theorem 1.1. *If a mapping $f : X \to Y$ satisfies the inequalities*

$$\|f(x_1+x_2)+f(x_1-x_2)-2f(x_1)\|\le\delta\ ,$$

$$\|f(0)\|\le\delta_0$$

for some fixed δ, $\delta_0 \geq 0$ *and all* $x_1, x_2 \in X$, *then there exists a unique additive mapping* $A : X \to Y$ *of the first form which satisfies the inequality*

$$\|f(x) - A(x)\| \leq \delta + \delta_0$$

for all $x \in X$. *If, moreover, f is measurable or* $f(tx)$ *is continuous in t for each fixed* $x \in X$ *then* $A(tx) = tA(x)$ *for all* $x \in X$ *and* $t \in \mathbb{R}$.

This last assertion holds according to the work of the first author [16], in 1982.

Therefore it is obvious that the following stability Theorem 1.2 holds for additive mappings of the first form, which is sharper than the above Theorem 1.1.

Theorem 1.2. *If a mapping* $f : X \to Y$ *satisfies the inequalities*

$$\|f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1)\| \leq \delta \ , \tag{1.4}$$

$$\|f(0)\| \leq \delta_0 \tag{1.5}$$

for some fixed $\delta \geq 0$, $\delta_0 \geq 0$ *and all* $x_1, x_2 \in X$, *then there exists a unique additive mapping* $A : X \to Y$ *of the first form, which satisfies the inequality*

$$\|f(x) - A(x)\| \leq \delta + \|f(0)\| \ (\leq \delta + \delta_0) \tag{1.6}$$

for all $x \in X$. *If, moreover, f is measurable or* $f(tx)$ *is continuous in t for each fixed* $x \in X$ *then* $A(tx) = tA(x)$ *for all* $x \in X$ *and* $t \in \mathbb{R}$.

2. Stability of the additive equation (1.1) of the first form on a restricted domain

We establish the following new stability Theorem 2.1 for additive mappings of the first form on a restricted domain, which is sharper than the analogous Theorem 2 of the authors [22], in 2003.

Theorem 2.1. *Let* $d > 0$ *and* $\delta \geq 0$ *be fixed. If a mapping* $f : X \to Y$ *satisfies the inequalities* (1.4) *for all* $x_1, x_2 \in X$, *with* $\|x_1\| + \|x_2\| \geq d$, *and* (1.5), *then there exists a unique additive mapping* $A : X \to Y$ *of the first form such that*

$$\|f(x) - A(x)\| \leq \frac{5}{2}\delta + \|f(0)\| \ (\leq \frac{5}{2}\delta + \delta_0) \tag{2.1}$$

for all $x \in X$. *If, moreover, f is measurable or* $f(tx)$ *is continuous in t for each fixed* $x \in X$, *then* $A(tx) = tA(x)$ *for all* $x \in X$ *and* $t \in \mathbb{R}$.

Proof. Assume $\|x_1\|+\|x_2\|< d$. If $x_1 = x_2 = 0$, then we choose a $t \in X$ with $\| t \| = d$. Otherwise, let us choose

$$t=\left(1+\frac{d}{\|x_1\|}\right)x_1 \ , \ if \ \ \|x_1\|\ge\|x_2\| \ ; \ t=\left(1+\frac{d}{\|x_2\|}\right)x_2 \ , \ if \ \ \|x_1\|\le\|x_2\| .$$

Clearly, we see

$$\|x_1-t\|+\|x_2+t\|\ge 2\|t\|-(\|x_1\|+\|x_2\|)\ge d \ , \|x_1-x_2\|+\|2t\|\ge d \ ,$$
$$\|x_1+t\|+\|-x_2+t\|\ge 2\|t\|-(\|x_1\|+\|x_2\|)\ge d \ , \ \|x_1\|+\|t\|\ge d \tag{2.2}$$

These inequalities (2.2) come from the corresponding substitutions attached between the right-hand sided parentheses of the following functional identity.

Therefore from (1.4), (2.2), the triangle inequality, and the *functional identity*

$$2[f(x_1+x_2)+f(x_1-x_2)-2f(x_1)]$$
$$=[f(x_1+x_2)+f(x_1-x_2-2t)-2f(x_1-t)] \text{ (with } x_1-t \text{ on } x_1\text{, and } x_2+t \text{ on } x_2\text{)}$$
$$-[f(x_1-x_2-2t)+f(x_1-x_2+2t)-2f(x_1-x_2)] \text{ (with } x_1-x_2 \text{ on } x_1\text{, and } 2t \text{ on } x_2\text{)}$$
$$+[f(x_1-x_2+2t)+f(x_1+x_2)-2f(x_1+t)] \text{ (with } x_1+t \text{ on } x_1\text{, and } -x_2+t \text{ on } x_2\text{)}$$
$$+2[f(x_1+t)+f(x_1-t)-2f(x_1)] \text{ (with } x_1 \text{ on } x_1\text{, and } t \text{ on } x_2\text{),}$$

we get

$$\|f(x_1+x_2)+f(x_1-x_2)-2f(x_1)\|\le\frac{5}{2}\delta . \tag{2.3}$$

Applying now Theorem 1.2 and the above inequality (2.3) , one gets that there exists a unique additive mapping $A : X\to Y$ of the first form that satisfies the additive equation (1.1) and the inequality (2.1), such that $A(x)=\lim_{n\to\infty} 2^{-n} f(2^n x)$. Our last assertion is trivial according to the Theorem 1.2.

We note that, if we define $S_2=\{(x_1,x_2)\in X^2 : \|x_i\| \langle\, d, i=1,2\}$ for some $d > 0$, then

$$\{(x_1,x_2)\in X^2 : \|x_1\|+\|x_2\|\ge 2d\}\subset X^2 \setminus S_2 .$$

Corollary 2.1. *If we assume that a mapping* $f : X \to Y$ *satisfies the inequalities* (1.4)-(1.5) *for some fixed* $\delta, \delta_0 \geq 0$ *and for all* $(x_1, x_2) \in X^2 \backslash S_2$, *then there exists a unique additive mapping* $A : X \to Y$ *of the first form, satisfying* (2.1) *for all* $x \in X$. *If, moreover,* f *is measurable or* $f(tx)$ *is continuous in* t *for each fixed* $x \in X$, *then* $A(tx)=tA(x)$ *for all* $x \in X$ *and all* $t \in \mathbb{R}$.

Corollary 2.2. *A mapping* $f : X \to Y$ *is additive of the first form, if and only if the asymptotic condition*

$$\left\| f(x_1 + x_2) + f(x_1 - x_2) - 2f(x_1) \right\| \to 0, \text{ as } \|x_1\| + \|x_2\| \to \infty \tag{2.4}$$

holds.

However, in 1983 F. Skof [25] proved an asymptotic property for *the additive mappings* $A : X \to Y$, such that

$$A(x_1 + x_2) = A(x_1) + A(x_2) \tag{2.5}$$

holds for all $x_1, x_2 \in X$.

3. Stability of the additive equation (1.2) of the second form

We establish the following new stability Theorem 3.1 for additive mappings of the second form, which is sharper than the analogous Theorem 3 of the authors [22], in 2003.

Theorem 3.1. *If a mapping* $f : X \to Y$ *satisfies the inequality*

$$\left\| f(x_1 + x_2) - f(x_1 - x_2) - 2f(x_2) \right\| \leq \delta \tag{3.1}$$

for some $\delta \geq 0$ *and for all* $x_1, x_2 \in X$, *then there exists a unique additive mapping* $A : X \to Y$ *of the second form, such that* $\|f(0)\| \leq \frac{\delta}{2}$, *which satisfies the inequality*

$$\left\| f(x) - A(x) \right\| \leq \delta + \| f(0) \| \left(\leq \frac{3}{2}\delta\right) \tag{3.2}$$

for all $x \in X$. *If, moreover,* f *is measurable or* $f(tx)$ *is continuous in* t *for each fixed* $x \in X$, *then* $A(tx) = tA(x)$ *for all* $x \in X$ *and* $t \in \mathbb{R}$.

Proof. Replacing $x_1 = x_2 = 0$ in (3.1), we find

$$\|f(0)\| \leq \frac{\delta}{2}. \tag{3.3}$$

Thus, substituting $x_1 = x_2 = x$ in (3.1), one gets

$$\left\| f(x) - 2^{-1} f(2x) \right\| \leq (\delta + \| f(0) \|)(1 - 2^{-1}) , \tag{3.4}$$

for some $\delta \geq 0$,and all $x \in X$. Therefore from (3.4) and the triangle inequality, we obtain

$$\left\| f(x) - 2^{-n} f(2^n x) \right\| \leq (\delta + \| f(0) \|)(1 - 2^{-n}) , \tag{3.5}$$

for some $\delta \geq 0$, any $n \in N$, and all $x \in X$.

We prove as in [22] that

$$A(x) = 2^{-n} A(2^n x) \tag{3.6}$$

holds for any $n \in N$, and all $x \in X$.

By (3.5), for $n \geq m > 0$, and $h = 2^m x$, we have

$$\left\| 2^{-n} f(2^n x) - 2^{-m} f(2^m x) \right\| \quad < (\delta + \| f(0) \|) \cdot 2^{-m} \to 0 \text{ , as } m \to \infty . \tag{3.7}$$

Therefore we may apply a direct method to the definition of A, such that the formula

$$A(x) = \lim_{n \to \infty} 2^{-n} f(2^n x) \tag{3.8}$$

holds for all $x \in X$ [16-19]. From this formula (3.8) and the inequality (3.1), it follows that $A : X \to Y$ is an additive mapping of the second form. According to the above inequality (3.5) and the formula (3.8), one gets that the inequality (3.2) holds.

Assume now that there is another additive mapping $A' : X \to Y$ of the second form which satisfies the equation (1.2), the formula (3.6) and the inequality (3.2). Therefore, as in [22],one gets

$$A(x) = A'(x) \tag{3.9}$$

for all $x \in X$, completing the proof of the first part of our Theorem 3.1.

The proof of the last assertion in our Theorem 3.1 is obvious according to the work [16], in 1982.

4. Stability of the additive equation (1.2) of the second form on a restricted domain

We establish the following new stability Theorem 4.1 for additive mappings of the second form on a restricted domain, which is sharper than the analogous Theorem 4 of the authors [22], in 2003.We note that from (1.3) and $\left\| f(-2x) + f(2x) \right\| \leq \theta$ (from (1.3) with

$2x$ on x) and (3.4) as well as $\|f(-2x)-2f(-x)\| \le \delta + \| f(0) \|$ (from (3.4) with $-x$ on x, after we have multiplied (3.4) by 2) and the triangle inequality one gets

$$2\|f(-x)+f(x)\| \le \|-[f(-2x)-2f(-x)]\| + \|-[f(2x)-2f(x)]\| + \|f(-2x)+f(2x)\|$$

or $\| f(-x)+f(x) \| \le (\delta + \| f(0) \|) + \dfrac{\theta}{2} \overset{(1.3)}{=} \theta$.

Therefore $\theta = 2(\delta + \|f(0)\|)$ and thus the inequality

$$\| f(-x)+f(x) \| \le 2(\delta + \| f(0) \|), \tag{4.1a}$$

holds for some fixed $\delta \ge 0$, and all $x \in X$.

Theorem 4.1. *Let* $d>0$ *and* $\delta \ge 0$, *be fixed. If an approximately odd mapping* $f : X \to Y$ *satisfies inequality* (3.1) *for all* $x_1, x_2 \in X$ *with* $\|x_1\| + \|x_2\| \ge d$, *and the inequality* (4.1a) *for all* $x \in X$ *with* $\|x\| \ge d$, *then there exists a unique additive mapping* $A : X \to Y$ *of the second form such that* $\|f(0)\| \le \dfrac{\delta}{2}$ *and*

$$\|f(x)-A(x)\| \le 9\delta + 5 \| f(0) \| \left(\le \frac{23}{2}\delta\right) \tag{4.1}$$

for all $x \in X$. *If, moreover, f is measurable or* $f(tx)$ *is continuous in t for each fixed* $x \in X$, *then* $A(tx) = tA(x)$ *for all* $x \in X$ *and* $t \in \mathbb{R}$.

Proof. Assume $\|x_1\| + \|x_2\| < d$. If $x_1 = x_2 = 0$, then we choose a $t \in X$ with $\| t \| = d$. Otherwise, let us choose

$$t = \left(1 + \frac{d}{\|x_1\|}\right)x_1 \ , \ if \ \ \|x_1\| \ge \|x_2\| \ ; \ t = \left(1 + \frac{d}{\|x_2\|}\right)x_2 \ , \ if \ \ \|x_1\| \le \|x_2\|.$$

We note that:

$$\| t \| = \|x_1\| + d > d, \text{ if } \|x_1\| \ge \|x_2\|; \ \| t \| = \|x_2\| + d > d, \text{ if } \|x_1\| \le \|x_2\|.$$

Clearly, we see

$$\|x_1 - t\| + \|x_2 + t\| \ge 2\| t \| - (\|x_1\| + \|x_2\|) \ge d, \|x_1 - t\| + \|x_2 - t\| \ge 2\| t \| - (\|x_1\| + \|x_2\|) \ge d,$$

$$\|x_1 - 2t\| + \|x_2\| \ge 2\| t \| - (\|x_1\| + \|x_2\|) \ge d, \ \ \| t \| + \|x_2\| \ge d \tag{4.2}$$

and $\|t - x_2\| \ge \| t \| - \| x_2 \| = (\| x_2 \| + d) - \| x_2 \| = d$, because $\| t \| = \| x_2 \| + d$.

Therefore from (4.1a) , (3.1), (4.2), and the following *functional identity*

$$f(x_1+x_2)-f(x_1-x_2)-2f(x_2)$$

$$=\left[f(x_1+x_2)-f(x_1-x_2-2t)-2f(x_2+t)\right] \text{ (with } x_1-t \text{ on } x_1\text{, and } x_2+t \text{ on } x_2\text{)}$$

$$+\left[f(x_1+x_2-2t)-f(x_1-x_2)-2f(x_2-t)\right] \text{ (with } x_1-t \text{ on } x_1\text{, and } x_2\text{-}t \text{ on } x_2\text{)}$$

$$-\left[f(x_1+x_2-2t)-f(x_1-x_2-2t)-2f(x_2)\right] \text{ (with } x_1-2t \text{ on } x_1\text{, and } x_2 \text{ on } x_2\text{)}$$

$$+2\left[f(t+x_2)-f(t-x_2)-2f(x_2)\right] \text{ (with } t \text{ on } x_1\text{, and } x_2 \text{ on } x_2\text{)}$$

$$+2\left[f(t-x_2)+f(-(t-x_2))\right] \text{ (with } t-x_2 \text{ on } x\text{),}$$

we get

$$\left\| f(x_1+x_2)-f(x_1-x_2)-2f(x_2)\right\| \le 9\delta+4\|f(0)\|(\le 11\delta). \tag{4.3}$$

Applying the above Theorem 3.1 and the inequality (4.3), we prove that

$$\left\| f(x)-A(x)\right\| \le (9\delta+4\|f(0)\|)+\|f(0)\|$$

$$=9\delta+5\,\|\,f(0)\,\|\,(\le \frac{23}{2}\delta).$$

Therefore there exists a unique additive mapping $A : X\to Y$ of the second form that satisfies the equation (1.2) and the inequality (4.1), completing the proof of this Theorem . It is obvious that our inequalities (4.1) and (4.3) are sharper than the corresponding inequalities of the authors [22], where the right-hand sides were equal to

$$\frac{33}{2}\delta \ge 9\delta+5\,\|\,f(0)\,\|\,(\le \frac{23}{2}\delta) \text{ and } 11\delta \ge 9\delta+4\,\|\,f(0)\,\|\,(\le 11\delta),$$

respectively, because of the condition $\|f(0)\| \le \delta/2$ for a fixed $\delta \ge 0$..

We note that if we define $S_1=\{x\in X : \|x\|< d\}$ and $S_2=\{(x_1,x_2)\in X^2 : \|x_i\|\langle\, d, i=1,2\}$ for some fixed $d > 0$, then

$$\{x\in X : \|x\|\ge 2d\}\subset X\setminus S_1 \quad \text{and} \quad \{(x_1,x_2)\in X^2 : \|x_1\|+\|x_2\|\ge 2d\}\subset X^2\setminus S_2.$$

Corollary 4.1. *If we assume that a mapping* $f : X\to Y$ *satisfies the inequality* (4.1) *for some fixed* $\delta \ge 0$ *and for all* $(x_1, x_2) \in X^2\setminus S_2$ *and* (4.1a) *for all* $x\in X\setminus S_1$, *then there exists a unique additive mapping* $A : X\to Y$ *of the second form, satisfying* (4.1) *for all* $x \in X$. *If, moreover, f is measurable or* $f(tx)$ *is continuous in t for each fixed* $x \in X$, *then* $A(tx) = tA(x)$ *for all* $x \in X$ *and* $t \in \mathbb{R}$.

Corollary 4.2. *A mapping $f : X \to Y$ is additive of the second form, if and only if the asymptotic conditions*

$$\| f(-x)+f(x)\| \to 0 \text{ and } \|f(x_1+x_2)-f(x_1-x_2)-2f(x_2)\| \to 0, \quad (4.4)$$

as $\| x \| \to \infty$ and $\|x_1\|+\|x_2\| \to \infty$ *hold, respectively.*

5. Stability of the Jensen equation (1.1b)

We establish the following new stability Theorem 5.1 for Jensen mappings .

Theorem 5.1. *If a mapping $f : X \to Y$ satisfies the Jensen inequality*

$$\left\| 2f\left(\frac{x_1+x_2}{2}\right)-f(x_1)-f(x_2) \right\| \le \delta , \quad (5.1)$$

for some fixed $\delta \ge 0$, and all $x_1, x_2 \in X$, then there exists a unique Jensen mapping $A : X \to Y$, satisfying $A(x)=\lim_{n\to\infty} 2^{-n} f(2^n x)$ and the inequality

$$\|f(x)-A(x)\| \le \delta + \| f(0) \| \quad (5.2)$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in$.

Proof. Substituting $x_1 = 2x$ and $x_2 = 0$ in (5.1), one gets

$$\|f(x)-2^{-1} f(2x)\| \le (\delta + \| f(0) \|)(1-2^{-1}) , \quad (5.3)$$

for some $\delta \ge 0$,and all $x \in X$. Therefore from (5.3) and the triangle inequality, we obtain

$$\|f(x)-2^{-n} f(2^n x)\| \le (\delta + \| f(0) \|)(1-2^{-n}) , \quad (5.4)$$

for some $\delta \ge 0$, any $n \in N$, and all $x \in X$. The rest of the proof is omitted as it is similar to the proof of the Theorem 3.1.

6. Stability of the Jensen equation (1.1b) on a restricted domain

We establish the following new stability Theorem 6.1 for Jensen mappings on a restricted domain.

Theorem 6.1. *Let $d > 0$ and $\delta \ge 0$ be fixed. If a mapping $f : X \to Y$ satisfies the Jensen inequality* (5.1) *for all $x_1, x_2 \in X$, with $\|x_1\|+\|x_2\| \ge d$, and the additional inequalities*

$$\|f(-x)+f(x)\| \leq \delta + 2\|f(0)\|, \tag{6.1a}$$

$$\|f(2x)-2f(x)\| \leq \delta + \|f(0)\| \tag{6.1b}$$

for all $x \in X$ *with* $\|x\| \geq d$, *then there exists a unique Jensen mapping* $A : X \to Y$ *,such that the inequality*

$$\|f(x)-A(x)\| \leq \frac{7}{2}\delta + 3\|f(0)\| \tag{6.1}$$

holds for all $x \in X$. *If, moreover,* f *is measurable or* $f(tx)$ *is continuous in* t *for each fixed* $x \in X$, *then* $A(tx) = tA(x)$ *for all* $x \in X$ *and* $t \in \mathbb{R}$.

Proof. It is clear that the *"approximate odd"* inequality (6.1a) holds, if we replace $x_1 = x, x_2 = -x$ in (5.1) and then apply the triangle inequality. We get (6.1b) from (5.3). From (1.1b), the triangle inequality, and the *functional identity*

$$\begin{aligned}
&2f\left(\frac{x_1+x_2}{2}\right)-f(x_1)-f(x_2)\\
&=2f\left(\frac{x_1+x_2}{2}\right)-f(x_1-t)-f(x_2+t) \quad (\textit{with } x_1-t \textit{ on } x_1 \textit{ and } x_2+t \textit{ on } x_2)\\
&\quad+\frac{1}{2}[2f(x_2+t)-f(2x_2)-f(2t)] \quad (\textit{with } 2x_2 \textit{ on } x_1 \textit{ and } 2t \textit{ on } x_2)\\
&\quad+\frac{1}{2}[2f(x_1-t)-f(2x_1)-f(-2t)] \quad (\textit{with } 2x_1 \textit{ on } x_1 \textit{ and } -2t \textit{ on } x_2)\\
&\quad+\frac{1}{2}[f(2x_1)-2f(x_1)] + \frac{1}{2}[f(2x_2)-2f(x_2)] + \frac{1}{2}[f(-2t)+f(2t)]
\end{aligned}$$

we get

$$\left\|2f\left(\frac{x_1+x_2}{2}\right)-f(x_1)-f(x_2)\right\| \leq \frac{7}{2}\delta + 2\|f(0)\| \tag{6.2}$$

Applying now Theorem 5.1 and the above inequality (6.2) , one gets that there exists a unique Jensen mapping $A : X \to Y$ that satisfies the Jensen equation (1.1b) and the inequality (6.1), such that $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ with $A(-x) = -A(x)$ (from (6.1a)).

We note that, if we define $S_1 = \{x \in X : \|x\| < d\}$ and $S_2 = \{(x_1, x_2) \in X^2 : \|x_i\| < d, i = 1, 2\}$ for some $d > 0$, then

$\{x \in X : \|x\| \geq 2d\} \subset X \setminus S_1$ and $\{(x_1, x_2) \in X^2 : \|x_1\| + \|x_2\| \geq 2d\} \subset X^2 \setminus S_2$.

Corollary 6.1. *If we assume that a mapping* $f : X\to Y$ *satisfies the inequality* (5.1) *for some fixed* $\delta \geq 0$ *and for all* $(x_1, x_2) \in X^2\backslash S_2$ *and* (6.1a)-(6.1b) *for all* $x\in X\backslash S_1$, *then there exists a unique Jensen mapping* $A : X\to Y$, *satisfying* (6.1) *for all* $x \in X$. *If, moreover,* f *is measurable or* $f(tx)$ *is continuous in* t *for each fixed* $x \in X$, *then* $A(tx)=tA(x)$ *for all* $x \in X$ *and all* $t \in$ □ .

Corollary 6.2. *A mapping* $f : X\to Y$ *is a Jensen mapping, if and only if the asymptotic conditions* $\|f(-x)+f(x)\| \to 0$ *and* $\|f(2x)-2f(x)\| \to 0$, *as* $\|x\| \to \infty$ *and* $\|2f((x_1+x_2)/2)-f(x_1)-f(x_2)\| \to 0$, *as* $\|x_1\|+\|x_2\| \to \infty$ *hold.*

7. Stability of the Jensen type equation (1.2b)

We establish the following new stability Theorem 7.1 for Jensen type mappings .

Theorem 7.1. *If a mapping* $f : X\to Y$ *satisfies the Jensen type inequality*

$$\left\|2f\left(\frac{x_1-x_2}{2}\right)-f(x_1)+f(x_2)\right\| \leq \delta , \tag{7.1}$$

for some fixed $\delta \geq 0$, *and all* $x_1, x_2 \in X$, *then there exists a unique Jensen type mapping* $A : X\to Y$, *satisfying* $A(x)=\lim_{n\to\infty} 2^{-n} f(2^n x)$ *and the inequality*

$$\|f(x)-A(x)\| \leq \delta+\| f(0)\| \tag{7.2}$$

for all $x \in X$. *If, moreover,* f *is measurable or* $f(tx)$ *is continuous in* t *for each fixed* $x \in X$ *then* $A(tx) = tA(x)$ *for all* $x \in X$ *and* $t \in$ □ .

Proof. Substituting $x_1 = 2x$ and $x_2 = 0$ in (7.1), one gets

$$\left\|f(x)-2^{-1}f(2x)\right\| \leq (\delta+\| f(0)\|)(1-2^{-1}) , \tag{7.3}$$

for some $\delta \geq 0$,and all $x \in X$. The rest of the proof is omitted as similar to the proof of the Theorem 3.1 and Theorem 5.1.

8. Stability of the Jensen type equation (1.2b) on a restricted domain

We establish the following new stability Theorem 8.1 for Jensen type mappings on a restricted domain.

Theorem 8.1. *Let $d > 0$ and $\delta \geq 0$ be fixed. If a mapping $f: X \to Y$ satisfies the Jensen type inequality (7.1) for all x_1, $x_2 \in X$, with $\|x_1\| + \|x_2\| \geq d$, and the additional inequalities*

$$\|f(-x) + f(x)\| \leq \delta, \tag{8.1a}$$

$$\|f(2x) - 2f(x)\| \leq \delta + \|f(0)\| \tag{8.1b}$$

for all $x \in X$ with $\|x\| \geq d$, then there exists a unique Jensen type mapping $A: X \to Y$, such that the inequalitiy

$$\|f(x) - A(x)\| \leq 5\delta + 2\,\|f(0)\| \tag{8.1}$$

holds for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof . It is clear that the *"approximate odd"* inequality (8.1a) holds for all $x \in X$, if we replace $x_1 = -x, x_2 = x$ in (7.1) and then apply the triangle inequality. From (7.3) we get (8.1b). From (1.2b), the triangle inequality, and the *functional identity*

$$\begin{aligned}
&2f\left(\frac{x_1 - x_2}{2}\right) - f(x_1) + f(x_2) \\
&= 2f\left(\frac{x_1 - x_2}{2}\right) - f(x_1 - t) + f(x_2 - t) \quad (\textit{with } x_1 - t \textit{ on } x_1 \textit{ and } x_2 - t \textit{ on } x_2) \\
&\quad + \frac{1}{2}[2f(-x_2 + t) - f(-2x_2) + f(-2t)] \quad (\textit{with } -2x_2 \textit{ on } x_1 \textit{ and } -2t \textit{ on } x_2) \\
&\quad + \frac{1}{2}[2f(x_1 - t) - f(2x_1) + f(2t)] \quad (\textit{with } 2x_1 \textit{ on } x_1 \textit{ and } 2t \textit{ on } x_2) \\
&\quad + \frac{1}{2}[f(2x_1) - 2f(x_1)] - \frac{1}{2}[f(2x_2) - 2f(x_2)] - \frac{1}{2}[f(-2t) + f(2t)] \\
&\quad - [f(-x_2 + t) + f(x_2 - t)] + \frac{1}{2}[f(-2x_2) + f(2x_2)]
\end{aligned}$$

we get

$$\left\| 2f(\frac{x_1 - x_2}{2}) - f(x_1) + f(x_2) \right\| \le 5\delta + \|f(0)\|. \qquad (8.2)$$

Applying now Theorem 7.1 and the above inequality (8.2), one gets that there exists a unique Jensen type mapping $A : X \to Y$ that satisfies the Jensen type equation (1.2b) and the inequality (8.1), such that $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ with $A(-x) = -A(x)$ (from (8.1a)).

We note that, if we define $S_1 = \{x \in X : \|x\| < d\}$ and $S_2 = \{(x_1, x_2) \in X^2 : \|x_i\| \langle\, d, i = 1, 2\}$ for some $d > 0$, then

$\{x \in X : \|x\| \ge 2d\} \subset X \setminus S_1$ and $\{(x_1, x_2) \in X^2 : \|x_1\| + \|x_2\| \ge 2d\} \subset X^2 \setminus S_2$.

Corollary 8.1. *If we assume that a mapping $f : X \to Y$ satisfies the inequality* (7.1) *for some fixed $\delta \ge 0$ and for all* $(x_1, x_2) \in X^2 \setminus S_2$ *and* (8.1a)-(8.1b) *for all $x \in X \setminus S_1$, then there exists a unique Jensen type mapping $A : X \to Y$, satisfying* (8.1) *for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx)=tA(x)$ for all $x \in X$ and all $t \in$ □ .*

Corollary 8.2. *A mapping $f : X \to Y$ is a Jensen type mapping, if and only if the asymptotic conditions $\|f(-x) + f(x)\| \to 0$ and $\|f(2x) - 2f(x)\| \to 0$, as $\|x\| \to \infty$ and*

$$\left\| 2f\left(\frac{x_1 - x_2}{2}\right) - f(x_1) + f(x_2) \right\| \to 0, \text{ as } \|x_1\| + \|x_2\| \to \infty \text{ , hold.}$$

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