イロト イポト イヨト イヨト

3

AUEB

Robust control and applications in economic theory In honour of Professor Emeritus Grigoris Kalogeropoulos on the occasion of his retirement

A. N. Yannacopoulos Department of Statistics – AUEB

24 May 2013

A. N. Yannacopoulos - Department of Statistics

(ロト (四) (注) (注) (注) (三)

AUEB

In collaboration with

- ► W. A. Brock (Professor Emeritus, Economics Department, Wisconsin)
- ► A. Xepapadeas (Professor, Department of European and International Economic Studies, AUEB)

A. N. Yannacopoulos – Department of Statistics

Introduction

- ▶ Robust control is a very important part of stochastic control.
- ▶ In some sense it is the most realistic version of control theory:
- ► We wish to control a system but we do not know the exact law of evolution of the state process.
- ▶ What we have is a family of laws (scenarios), and we want to control the worst possible scenario.
- ▶ The best policy for the worst scenario is our robust control.

イロト 不得 トイラト イラト 二日

<ロト < 団ト < 団ト < 団ト = 団ト = 団 =

AUEB

- This theme has become extremely useful in economics and finance
- ▶ T. J. Sargent, Nobel Prize in Economics 2011 has devoted most of his research in this field.
- Furthermore robust control has interesting connections with game theory and in particular with stochastic differential games.

Control



◆□ > ◆□ > ◆豆 > ◆豆 > ̄豆 = のへで

AUEB

A. N. Yannacopoulos – Department of Statistics

Robust control



◆□▶ ◆□▶ ◆三▶ ◆三▶ ・三 ・ ○へ⊙

AUEB

A. N. Yannacopoulos – Department of Statistics

A linear model

► Consider a spatially extended economic system, located on a discrete set D for simplicity, with state variable x_n and with state equation:

$$dx_n = (\sum_m a_{nm}x_m + \sum_m b_{nm}u_m)dt + \sum_m c_{nm}dw_m, \ n \in \mathcal{D}$$

- Stochastic fluctuations are understood in the sense of the lto theory of stochastic integration.
- In compact form this can be expressed as

dx = (Ax + Bu) dt + Cdw

where A, B, C : $\ell^2 \rightarrow \ell^2$ are linear operators, related to the doubly infinite matrices with elements a_{nm}, b_{nm}, c_{nm} , respectively.

► This can be understood as an infinite dimensional Ornstein-Uhlenbeck equation on the Hilbert space l².

A. N. Yannacopoulos - Department of Statistics



- ► The operator A gives us the interconnection of the various economic units with each other.
- ► The operator B gives us how a control which is applied at site m affects the state of the system at site n.
- ► The operator C is the covariance operator, and tells us how uncertainty at site *m* affect the state of the system at site *n*.
- ► Of course in the finite dimensional case the model makes perfect sense and all the above operators become matrices.
- Our model can be a model for e.g. a spatially extended fishery: x is biomass at various compartments, u is harvesting rate.

▲ロト ▲母 ト ▲ヨ ト ▲ヨ ト つのべ

- Assume now that there is some uncertainty concerning the "true" statistical distribution of the state of the system.
- ► This corresponds to a family of probability measures Q such that each Q ∈ Q corresponds to an alternative stochastic model (scenario) concering the state of the system.
- ► We restrict to measures Q ~ P such that the Radon-Nikodym derivatives dQ/dP are defined through an exponential martingale of the type employed in Girsanov's theorem,

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_{T}} = \exp\left(\int_{0}^{T}\sum_{n}v_{n}(t)dw_{n}(t) - \frac{1}{2}\int_{0}^{T}\sum_{n}v_{n}^{2}(t)dt\right)$$

where $v = \{v_n\}$, $n \in \mathbb{Z}$ is an ℓ^2 -valued stochastic process which is measurable with respect to the filtration $\{\mathcal{F}_t\}$ satisfying the Novikov condition.

A. N. Yannacopoulos - Department of Statistics

Girsanov's theorem shows that the adoption of the family Q of alternative measures concerning the state of the system, leads to a family of different equations for the state variable

$$dx_n^{u,v} = \left(\sum_m a_{nm}x_m^{u,v} + \sum_m b_{nm}u_m + \sum_m c_{nm}v_m\right)dt + \sum_m c_{nm}d\bar{w}_m$$

where the superscipts u, v in $x^{u,v} := \{x_n^{u,v}\}$ indicates that this is the state of the system when the measure Q corresponding to the "information drift" $v = \{v_n\}$ and the control procedure $u = \{u_n\}$ is adopted.

In compact form this equation becomes the infinite dimensional Ornstein-Uhlenbeck equation

$$dx^{u,v} = (\mathsf{A}x^{u,v} + \mathsf{B}u + \mathsf{C}v)dt + \mathsf{C}d\bar{w}.$$

A. N. Yannacopoulos - Department of Statistics

Workshop on control theory and applications: In honour of Professor Emeritus G. Kalogeropoulos

 For a fixed model v the decision maker solves the control problem

$$\min_{u} \mathbb{E}_{Q} \left[\int_{0}^{\infty} e^{-rt} (\langle \mathsf{P} x^{u,v}(t), x^{u,v}(t) \rangle + \langle \mathsf{Q} u(t), u(t) \rangle) dt \right]$$

subject to the dynamic constraints

$$dx^{u,v} = (Ax^{u,v} + Bu + Cv)dt + Cd\bar{w}, \ x^{u,v}(0) = x_0.$$

where $\langle \cdot, \cdot \rangle$ is the inner product in the Hilbert space ℓ^2 and P, Q : $\ell^2 \rightarrow \ell^2$ are symmetric positive operators, modelling distance from a "target" and cost of control.

This will provide a solution leading to a value function V(x₀; v); corresponding to the minimum deviation obtained for the model Q_v under the minimum possible effort.

AUEB

- ▶ Being uncertain about the true model, the decision maker will opt to choose this strategy that will work in the worst case scenario; this being the one that maximizes $V(x_0; v)$, the minimum over all u having chosen v, over all possible choices for v.
- ► The robust control problem to be solved is of the general form $\min_{u} \max_{v} \mathbb{E}_{Q} \left[\int_{0}^{\infty} e^{-rt} (\langle (\mathsf{P}x^{u,v})(t), x^{u,v}(t) \rangle + \langle (\mathsf{Q}u)(t), u(t) \rangle - \theta \langle (\mathsf{R}v)(t), v(t) \rangle) dt \right],$

subject to the dynamic constraint

$$dx^{u,v} = (Ax^{u,v} + Bu + Cv)dt + Cd\bar{w}, \ x^{u,v}(0) = x_0.$$

where $\theta > 0$ and $R = \{r_{nm}\}$ is a symmetric positive operator.

► The third term corresponds to a quadratic loss function related to the "cost" of model misspecification.

A. N. Yannacopoulos - Department of Statistics

Quadratic loss functions are rather common in statistical decision theory, mainly on account of their connection with the Kullback-Leibler entropy of the two measures.

Proposition

The robust optimization problem is related to a robust control problem with an entropic constraint of the form

$$\begin{split} &\inf_{u} \sup_{Q \in \mathcal{Q}} \mathbb{E}_{Q} \left[\int_{0}^{\infty} e^{-rt} (\langle \mathsf{P} \mathsf{x}(t), \mathsf{x}(t) \rangle + \langle \mathsf{Q} u(t), u(t) \rangle) dt \right], \\ & \text{subject to} \quad \mathcal{H}(P \mid Q) = \int_{\Omega} \ln \left(\frac{dQ}{dP} \right) dQ < H_{0} \end{split}$$

and the dynamic constraint.

 $\boldsymbol{\theta}$ plays the role of the Lagrange multiplier for the entropic constraint.

A. N. Yannacopoulos – Department of Statistics

Workshop on control theory and applications: In honour of Professor Emeritus G. Kalogeropoulos

イロト 不得 トイラト イラト 二日

Connection with stochastic differential games

- One particularly intuitive way of viewing this problem is as a two player game:
- The first player is the decision maker while the second player is an adversarial agent (nature) who has "control" over the uncertainty.
- The first player chooses her actions so as to minimize the distance of the state of the system from a chosen target at the minimum possible cost, whereas the second player is a considered by the first player as a malevolent player who tries to "mess up" the first players efforts.
- This results to the interpretation of solution the robust control problem as a Nash equilibrium of this game.

Hamilton-Jacobi-Bellman-Isaacs equation

- The solution of this stochastic differential game can be obtained using a generalization of the Hamilton-Jacobi-Bellman equation called the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation.
- This equation is a fully nonlinear PDE involving the generator operator which for the Ornstein-Uhlenbeck process is

$$\mathcal{L}V = \langle \mathsf{A}x + \mathsf{B}u + \mathsf{C}v, DV \rangle + Tr(\mathsf{C}\mathsf{C}^*D^2V)$$

▶ Using \mathcal{L} we construct the Hamiltonian $\mathcal{H}: \ell^2 \times \ell^2 \times \ell^2 \to \mathbb{R}$ defined as

$$\mathcal{H}(V; x, u, v) = \mathcal{L}V + \langle \mathsf{P}x, x \rangle + \langle \mathsf{Q}u, u \rangle - \theta \langle \mathsf{R}v, v \rangle$$

A. N. Yannacopoulos - Department of Statistics

Workshop on control theory and applications: In honour of Professor Emeritus G. Kalogeropoulos

▲ロ ▶ ▲ □ ▶ ▲ □ ▶ ▲ □ ▶ ▲ □ ▶ ● ● ● ●

 We need to obtain the upper hamiltonian and lower hamiltonians defined respectively as

$$\bar{H} := \sup_{u} \inf_{v} \mathcal{H}(V; x, u, v), \quad \underline{H} := \inf_{v} \sup_{u} \mathcal{H}(V; x, u, v).$$

The upper solution to the game is the solution of the HJBI equation

$$\frac{\partial V}{\partial t} + \sup_{u} \inf_{v} \mathcal{H}(V; x, u, v) = 0$$

 The lower solution of the game is the solution of the HJBI equation

$$\frac{\partial V}{\partial t} + \inf_{v} \sup_{u} \mathcal{H}(V; x, u, v) = 0$$

A. N. Yannacopoulos - Department of Statistics

Workshop on control theory and applications: In honour of Professor Emeritus G. Kalogeropoulos

< □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > <

A version of the minimax theorem guarantees that:

Theorem

Suppose that

$$H(V; x) := \sup_{u} \inf_{v} \mathcal{H}(V; x, u, v) = \inf_{v} \sup_{u} \mathcal{H}(V; x, u, v)$$

then a Nash equilibrium to this stochastic differential game exists and it is given by the solution of the HJBI equation

$$\frac{\partial V}{\partial t} + H(V; x) = 0$$

The optimal strategies are given by the maximizers and the minimizers of $\sup_{u} \inf_{v} \mathcal{H}(V; x, u, v)$ and are given as feedback laws.

A. N. Yannacopoulos - Department of Statistics

Workshop on control theory and applications: In honour of Professor Emeritus G. Kalogeropoulos

イロト イポト イヨト イヨト

The solution of the linear quadratic robust control problem is given by the following:

Theorem

The robust control problem has a solution for which the optimal controls are of the feedback control form

$$u = -\mathsf{Q}^{-1}\mathsf{B}^*\mathsf{H}^{sym}x, \quad v = \frac{1}{\theta}\mathsf{R}^{-1}\mathsf{C}^*\mathsf{H}^{sym}x,$$

and the optimal state satisfies the Ornstein-Uhlenbeck equation

$$dx = (\mathsf{A} - \mathsf{B}\mathsf{Q}^{-1}\mathsf{B}^*\mathsf{H}^{sym} + rac{1}{ heta}\mathsf{C}\mathsf{R}^{-1}\mathsf{C}^*\mathsf{H}^{sym})x\,dt + \mathsf{C}dW$$

where H^{sym} is the solution of the operator Riccati equation

$$\mathsf{H}^{sym}\mathsf{A} + \mathsf{A}^*\mathsf{H}^{sym} - \mathsf{H}^{sym}\mathsf{E}^{sym}\mathsf{H}^{sym} - r\mathsf{H}^{sym} + \mathsf{P} = \mathsf{O}$$

and $E^{sym} := \frac{1}{2}(E + E^*)$ is the symmetric part of

$$\mathsf{E} := \mathsf{B}\mathsf{Q}^{-1}\mathsf{B}^* - \frac{1}{\theta}\mathsf{C}\mathsf{R}^{-1}\mathsf{C}^*.$$

A. N. Yannacopoulos - Department of Statistics

Workshop on control theory and applications: In honour of Professor Emeritus G. Kalogeropoulos

AUEB

ㅋㅋ ㅋㅋ ㅋㅋ

▲ロト ▲ 課 ト ▲ 臣 ト ▲ 臣 ト ● 回 ● の Q @

AUEB

The solvability and the properties of the solution for the optimal control problem is reduced to the solvability and the properties of the solution of the operator Riccati equation.

Proposition

Let
$$m = ||A||$$
 defined as $m = \{\sup \langle Ax, x \rangle, ||x||_{\ell^2} = 1\}$ and assume that $m < r/2$.
Then, for small enough values of $||E||$ and $||P||$ the operator Riccati equation

 $H^{sym}A + A^*H^{sym} - H^{sym}E^{sym}H^{sym} - rH^{sym} + P = 0$

admits a unique bounded strong solution.

A. N. Yannacopoulos - Department of Statistics

AUEB

Nonlinear systems

Consider now the nonlinear system

$$dx = (Ax + F(x) + Bu)dt + Cdw$$

where A, B : $\mathbb{H} \to \mathbb{H}$ are linear operators and F : $\mathbb{H} \to \mathbb{H}$ is in general a nonlinear operator and C is the covariance operator.

▶ The robust form of the system, using the Girsanov theorem is

$$dx = (Ax + F(x) + Bu + Cv)dt + Cdw.$$

The robust control problem thus becomes

$$\min_{u} \max_{v} \mathbb{E}_{Q} \left[\int_{0}^{\infty} e^{-rt} (\mathsf{U}(x(t)) + \mathsf{K}(u(t)) - \mathsf{T}(v(t))) dt \right]$$

subject to the nonlinear state equation where U, K, T are assumed convex.

A. N. Yannacopoulos - Department of Statistics

Theorem

The Hamilton-Jacobi-Bellman-Isaacs equation associated with the nonlinear robust control problem is the infinite dimensional nonlinear PDE

$$\langle Ax + F(x), DV \rangle + Tr(CC^*D^2V) + U(x) - K^{\diamond}(-B^*DV) + T^{\diamond}(C^*DV) = rV$$

where $K^{\diamondsuit},\,T^{\diamondsuit}$ are the Fenchel-Legendre transforms of K and T^{\diamondsuit} respectively defined by

$$\mathsf{K}^{\diamondsuit}(p) := \sup_{x \in \mathbb{H}} [\langle p, x
angle - \mathsf{K}(x)].$$

Given a solution of this equation $V : \mathbb{H} \to \mathbb{R}$ of sufficient regularity the associated closed loop system is the nonlinear infinite dimensional Ornstein-Uhlenbeck system

 $dx = (\mathsf{A}x + \mathsf{F}(x) - D\mathsf{K}^{\diamond}(\mathsf{B}^*DV(x)) + D\mathsf{T}^{\diamond}(\mathsf{C}^*DV(x)))dt + \mathsf{C}dw$

A. N. Yannacopoulos - Department of Statistics

Workshop on control theory and applications: In honour of Professor Emeritus G. Kalogeropoulos

 $\exists \rightarrow$

The solvability of the infinite dimensional HBJI equation is provided in the next theorem.

Theorem Assume that (i) either A is the generator of an analytic semigroup or that A is the generator of a C_0 semigroup such that $||Q^{-1/2} \exp(tA)|| \le C t^{-\delta}$ for some $\delta \in (0, 1), t \ge 0$. (ii) F is a locally Lipschitz nonlinear operator. Then, there exists a critical discount factor r_{cr} such that for $r > r_{cr}$ the HBJI equation has a unique solution $V \in D(\mathcal{L})$.

Workshop on control theory and applications: In honour of Professor Emeritus G. Kalogeropoulos

イロト 不得 トイラト イラト 二日

What does the model offer?

- A robust control protocol, that may lead to optimal control of spatially extended systems under uncertainty.
- ► This can be useful for a number of possible applications e.g. environmental or urban economics.
- ► Having the feedback laws we may solve

 $dx = (\mathsf{A}x + \mathsf{F}(x) - D\mathsf{K}^{\diamond}(\mathsf{B}^*DV(x)) + D\mathsf{T}^{\diamond}(\mathsf{C}^*DV(x)))dt + \mathsf{C}dw$

to obtain the optimal state.

 A study of this equation may provide us with qualitative information regarding the dynamics of the optimal system such as if there are lattice sites in which we have large deviations from our control objectives – A hot spot.

A. N. Yannacopoulos – Department of Statistics

Breakdown of control: Hot spots

- ► What may be even more important from the conceptual point of view is the failure of the model, rather than its success!
- Regions of important breakdown of the model are called hotspots.
- Hotspots may arise on account of different reasons
 (I) Model misspecification effects are too pronounced at certain units of the system (loss of convexity)
 (II) The deviation of the controlled system from the desired target presents spatial variability

イロト イポト イラト イラト 一日

These hotspots may be characterized in detail using the linearized feedback control system and detailed estimates for the values of the parameters of the model for which such behaviour arise can be found.

Proposition

Assume that V is a C² solution of the HJBI equation and that K^{\diamond} and T^{\diamond} are C². The linearized dynamics are given by

$$dz = (Az + DF(x_0)z - D^2K^{\diamond}B^*D^2V(x_0)z + D^2T^{\diamond}C^*D^2V(x_0)z)dt + Cdw$$

The hot spots correspond to the unstable modes of this equation, *i.e.*, to eigenfunctions of the operator

$$\mathcal{R} := \mathsf{A} + D\mathsf{F}(x_0) - D^2\mathsf{K}^{\Diamond} \mathsf{B}^* D^2 V(x_0) + D^2 \mathsf{T}^{\Diamond} \mathsf{C}^* D^2 V(x_0)$$

with positive eigenvalues.

A. N. Yannacopoulos - Department of Statistics

- The value functions and the Legendre-Fenchel transforms satisfy convexity properties.
- ► This gives important information on the second derivatives D²K[◊], D²V(x₀), D²T[◊] and in particular assuming sufficient regularity they are positive operators.
- ► This property allows us at least to obtain some a priori estimates on the spectrum of the operator *R* and thus provide values on the parameters of the model which allow the generation of hot spot formation.

▲ロト ▲母 ト ▲ヨ ト ▲ヨ ト つのべ

Conclusions

- Robust control is a very important field in stochastic control theory, with interesting applications in economics.
- We have formulated and studied a robust stochastic control problem for a general class of interconnected systems arising in economic modelling and provided solutions in terms of the Hamilton-Jacobi-Bellman-Isaacs equation.
- An interesting phenomenon is the breakdown of control which leads to hot spot formation.

イロト イポト イラト イラト 一日